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SHARP INEQUALITIES FOR FACTORIAL n

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Abstract

Let n be a positive integer. We prove

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\alpha}} \leq n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\beta}}$$

with the best possible constants

$$\alpha = 1 - 2\pi e^{-2} = 0.149663... \text{ and } \beta = 1/6 = 0.166666...$$

This refines and extends a result of Sandor and Debnath, who proved that the double inequality holds with $\alpha = 0$ and $\beta = 1$.

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1. Introduction

Stirling's approximation to $n!$,

$$(1.1) \quad n! \sim n^n e^{-n} \sqrt{2\pi n} = \alpha_n$$

plays a central role in statistical physics and probability theory. Inspired by this formula, many authors have made attempts to find a formula, which has an improvement over (1.1) and as simple as (1.1), to approximate $n!$. Such a typical result is due to Burnside [1] :

$$(1.2) \quad n! \sim \sqrt{2\pi} \left(\frac{n + \frac{1}{2}}{e} \right)^{n + \frac{1}{2}} = \beta_n.$$

It is known that (1.2) has great superiority over (1.1). Formula (1.2) was rediscovered by Y. Weissman [9] and caused a lively debate in the American Journal of Physics in 1983, see [6]. Schuster found some other formulas to approximate $n!$ but they are complicated and not easy to use [8]. In a recently paper Sandor and Debnath [7] found the following inequalities for $n \geq 2$:

$$\frac{n^{n+1} e^{-n} \sqrt{2\pi}}{\sqrt{n}} \leq n! < \frac{n^{n+1} e^{-n} \sqrt{2\pi}}{\sqrt{n-1}}$$

This formula was rediscovered by Guo in very newly paper [2]. In this short note we determine the largest number α and the smallest number β such that the inequalities

$$\frac{n^{n+1} e^{-n} \sqrt{2\pi}}{\sqrt{n-\alpha}} \leq n! < \frac{n^{n+1} e^{-n} \sqrt{2\pi}}{\sqrt{n-\beta}}$$

are valid for all positive integers n . Numerical computations indicate that the approximation

$$(1.3) \quad n! \sim \frac{n^{n+1} e^{-n} \sqrt{2\pi}}{\sqrt{n-1/6}} = \gamma_n$$

gives much more accurate values for $n!$ than α_n and β_n (see the table at the end of the paper). Throughout, we denote the gamma function Γ and its logarithmic derivative, known as psi or digamma function as

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

for positive real numbers x , respectively.

In order to prove our main result we need to present two lemmas.

Lemma 1.1 : For $x \geq 1$ we have

$$\begin{aligned} \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{100}\right)^{\frac{1}{6}} &< \Gamma(x+1) \\ &< \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right). \end{aligned}$$

This result is due to Karatsuba, see [4].

Lemma 1.2 : We have

$$\lim_{x \rightarrow \infty} \left(\frac{2\pi x^{2x} e^{-2x}}{[\Gamma(x)]^2} - x \right) = -\frac{1}{6}.$$

Proof : Applying Stirling's formula, we get after a little simplification

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{2\pi x^{2x} e^{-2x}}{[\Gamma(x)]^2} - x \right) &= \lim_{x \rightarrow \infty} \frac{2\pi x^{2x} e^{-2x}}{[\Gamma(x+1)]^2} \frac{2\pi x^{2x+2} e^{-2x} - x^3 (\Gamma(x))^2}{2\pi x^{2x} e^{-2x}} \\ &= \lim_{x \rightarrow \infty} \frac{2\pi - x^{1-2x} e^{-2x} (\Gamma(x))^2}{2\pi \left(\frac{1}{x}\right)} \end{aligned}$$

By L'Hospital's rule this becomes

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{2\pi x^{2x} e^{-2x}}{[\Gamma(x)]^2} - x \right) &= \\ &= \frac{1}{\pi} \lim_{x \rightarrow \infty} x^{1-2x} e^{2x} (\Gamma(x+1))^2 \left(\psi(x) - \log x + \frac{1}{2x} \right). \end{aligned}$$

Using Stirling's formula again we get

$$\lim_{x \rightarrow \infty} \left(\frac{2\pi x^{2x} e^{-2x}}{[\Gamma(x)]^2} - x \right) = \lim_{x \rightarrow \infty} \frac{1 - 2x (\log x - \psi(x))}{\left(\frac{1}{x}\right)}.$$

From [5] we have

$$\log x - \psi(x) = \frac{1}{2x} + \frac{1}{12x^2} + \frac{\theta}{60x^4},$$

where $0 < \theta < 1$. Using this relation we find that

$$\lim_{x \rightarrow \infty} \left(\frac{2\pi x^{2x} e^{-2x}}{[\Gamma(x)]^2} - x \right) = \lim_{x \rightarrow \infty} \frac{1 - 2 \left(\frac{1}{2} + \frac{1}{12x} + \frac{\theta}{60x^4} \right)}{\left(\frac{1}{x}\right)} = -\frac{1}{6}. \quad \square$$

2. Main Result

Our main result is the following theorem.

Theorem 2.1 : For any positive integer n the following double inequality holds

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\alpha}} \leq n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\beta}},$$

where the constants $\alpha = 1 - 2\pi e^{-2} = 0.149663\dots$ and $\beta = \frac{1}{6} = 0.166666\dots$ are best possible.

Proof. : Set

$$h(x) = \frac{2\pi x^{2x}e^{-2x}}{[\Gamma(x)]^2} - x, x > 0.$$

We show that h is strictly decreasing on $(0, \infty)$. Differentiating h , we get

$$h' = \frac{4\pi \left(\frac{x}{e}\right)^{2x} (\log x - \psi(x) - \Gamma(x))^2}{(\Gamma(x))^2}.$$

Hence in order to show that $h'(x) < 0$, it suffices to show that

$$\left(\frac{\Gamma(x+1)}{x^x e^{-x} \sqrt{2\pi x}}\right)^2 - 2x (\log x - \psi(x)) > 0.$$

From the left inequality of Lemma 1.1 we obtain for $x \geq 1$

$$\left(\frac{\Gamma(x+1)}{x^x e^{-x} \sqrt{2\pi x}}\right)^2 > \left(8 + \frac{4}{x} + \frac{1}{x^2} + \frac{1}{100x^3}\right)^{\frac{1}{3}}$$

In [3] it was proved that

$$x (\log x - \psi(x)) < \frac{1}{2} + \frac{1}{12x}.$$

Employing these two inequalities, we find that for $x \geq 1$

$$\left(\frac{\Gamma(x+1)}{x^x e^{-x} \sqrt{2\pi x}}\right)^2 - 2x (\log x - \psi(x)) > \left(8 + \frac{4}{x} + \frac{1}{x^2} + \frac{1}{100x^3}\right)^{\frac{1}{3}} - 1 - \frac{1}{6x} > 0.$$

Hence, h is strictly decreasing on $(1, \infty)$. Using $h(1) = 2\pi e^{-2} - 1$ and $h(\infty) = -\frac{1}{6}$ by Lemma 1.2, we get for any positive integer n

$$-\frac{1}{6} = h(\infty) < h(n) = \frac{2\pi n^{2n+2}e^{-2n}}{(n!)^2} - n < h(1) = 2\pi e^{-2} - 1,$$

for which the proof follows. $\square$

The following table shows that γ_n has great superiority over α_n and β_n , where α_n , β_n and γ_n are as defined by (1.1), (1, 2) and (1, 3).

n	$n!$	α_n	β_n	γ_n
1	1	0.92213	1.02750	1.01015
2	2	1.91900	2.03331	2.00433
3	6	5.83620	6.07151	6.00541
4	24	23.50617	24.22261	24.01174
5	120	118.01916	120.91079	120.03673
6	720	710.07818	724.62384	720.15071
7	5040	4980.39583	5068.04888	5040.76647
8	40320	39902.39545	40517.97261	40324.65478
9	362880	359536.87284	364474.04470	362912.87998

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