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ORDERED L-FUZZY G_δ -EXTREMALLY DISCONNECTED SPACES AND TIETZE EXTENSION THEOREM

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Abstract

In this paper we introduce a new class of fuzzy topological spaces called ordered L-fuzzy G_δ -extremally disconnected spaces. Besides giving several characterizations and some interesting properties of these spaces, we also establish Tietze extension theorem.

Keywords. *Ordered L-fuzzy G_δ -extremally disconnected space increasing L-fuzzy G_δ -continuous, lower (upper) fuzzy G_δ -continuous, increasing (decreasing) L-fuzzy σ -closure and increasing (decreasing) L-fuzzy σ -interior.*

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1. Introduction and Preliminaries

Ever since the introduction of fuzzy sets by L. A. Zadeh [9], the fuzzy concept has invaded almost all branches of Mathematics. S. E. Rodabaugh [5], discussed normality and the L-fuzzy unit interval. He [6] also studied fuzzy addition in the L-fuzzy real line. Hoehle [3] studied the characterizations of L-topologies by L-valued neighbourhoods. An L-fuzzy normal spaces and Tietze extension theorem were discussed by Tomash Kubiak [8]. The concept of ordered fuzzy topological spaces was introduced by A. K. Katsaras [4]. In this paper we introduce the concept of G_δ -extremally disconnectedness in ordered L-fuzzy topology. In Section 2, we discuss the characterization and properties of ordered L-fuzzy G_δ -extremally disconnected spaces. In Section 3, we establish Tietze extension theorem for ordered L-fuzzy G_δ -extremally disconnected spaces.

Definition - 1 :

Through out this paper $(L, \leq, ')$ stands for an infinitely distributive lattice with an order reversing involution. Such a lattice being complete has a least element 0 and a greatest element 1. Let X be a non-empty set. An L-fuzzy set in X is an element of the set L^X of all functions from X to $L[1]$.

Definition - 2 :

The L-fuzzy real line $R(L)$ is the set of all monotone decreasing elements $\lambda \in L^R$ satisfying $\vee\{\lambda(t)/t \in R\} = 1$ and $\wedge\{\lambda(t)/t \in R\} = 0$, after the identification of $\lambda, \mu \in L^R$ iff $\lambda(t-) = \mu(t-)$ and $\lambda(t+) = \mu(t+)$ for all $t \in R$ where $\lambda(t-) = \wedge\{\lambda(s)/s < t\}$ and $\lambda(t+) = \vee\{\lambda(s)/s > t\}$. The natural L-fuzzy topology on $R(L)$ is generated from the subbasis $\{Lt, Rt \mid t \in R\}$, where $Lt[\lambda] = \lambda(t-)'$ and $Rt[\lambda] = \lambda(t+)$. A partial order on $R(L)$ is defined by $[\lambda] \leq [\mu] \Leftrightarrow \lambda(t-) \leq \mu(t-)$ and $\lambda(t+) \leq \mu(t+)$ for all $t \in R$. The L-fuzzy unit interval $I(L)$ is a subset of $R(L)$ such that $[\lambda] \in I[L]$ if $\lambda(t) = 1$ for $t < 0$ and $\lambda(t) = 0$ for $t > 1$. It is equipped with the subspace L-fuzzy topology [1].

Definition - 3 :

An L-fuzzy set λ in X is called closed if λ' is open, where λ' is defined as follows. $\lambda'(x) = (\lambda(x))'[1]$.

Definition - 4 :

If $A \in L^X$ is crisp, then (A, T_A) is an L-fuzzy topological space, called a crisp subspace of (X, T) , where $T_A = \{\lambda/A \mid \lambda \in T\}$ is called the subspace L-fuzzy topology [1].

Definition - 5 :

An L-fuzzy set λ in a partial ordered set X is called (i) Increasing if $x \leq y \Rightarrow \lambda(x) \leq \lambda(y)$. (ii) Decreasing if $x \leq y \Rightarrow \lambda(x) \geq \lambda(y)$. It is clear that the constant L-fuzzy sets are increasing and decreasing [1].

2. ORDERED L-FUZZY G_δ - EXTREMALLY DISCONNECTED SPACES AND THEIR PROPERTIES

Definition - 6 :

Let $(X, T, \leq)$ be an ordered L-fuzzy topological space and let λ be an L-fuzzy set in X , λ is called increasing L-fuzzy $G_\delta(F_\sigma)$ if $\lambda = \bigwedge_{i=1}^{\infty} \lambda_i$ (if $\lambda = \bigvee_{i=1}^{\infty} \lambda_i$) where each λ_i is increasing L-fuzzy open(closed) in X . The complement of increasing L-fuzzy $G_\delta(F_\sigma)$ -set is decreasing L-fuzzy $F_\sigma(G_\delta)$.

Definition - 7 :

Let λ be an L-fuzzy set in the ordered L-fuzzy topological space $(X, T, \leq)$. Then we define

$$\begin{aligned}
 IL_{(\sigma)}(\lambda) &= \text{increasing L-fuzzy}\sigma\text{-closure of } \lambda. \\
 &= \text{the smallest increasing L-fuzzy } F_\sigma\text{-set containing } \lambda. \\
 DL_{(\sigma)}(\lambda) &= \text{decreasing L-fuzzy}\sigma\text{-closure of } \lambda. \\
 &= \text{the smallest decreasing L-fuzzy } F_\sigma\text{-set containing } \lambda. \\
 I^\circ L_{(\sigma)}(\lambda) &= \text{increasing L-fuzzy}\sigma\text{-interior of } \lambda. \\
 &= \text{the greatest increasing L-fuzzy } G_\delta\text{-set contained in } \lambda. \\
 D^\circ L_{(\sigma)}(\lambda) &= \text{decreasing L-fuzzy}\sigma\text{-interior of } \lambda. \\
 &= \text{the greatest decreasing L-fuzzy } G_\delta\text{-set contained in } \lambda.
 \end{aligned}$$

Property - 1. (A) For a fuzzy set λ of an ordered L-fuzzy topological space $(X, T, \leq)$ the following hold.

$$\begin{aligned}
 a) (IL_{(\sigma)}(\lambda))' &= D_{L(\sigma)}^\circ(\lambda'). \\
 b) (DL_{(\sigma)}(\lambda))' &= I_{L(\sigma)}^\circ(\lambda'). \\
 c) (I^\circ L_{(\sigma)}(\lambda))' &= DL_{(\sigma)}(\lambda'). \\
 d) (D^\circ L_{(\sigma)}(\lambda))' &= IL_{(\sigma)}(\lambda').
 \end{aligned}$$

Proof. We shall prove (a) only (b), (c) and (d) can be proved in a similar manner.

Since $IL_{(\sigma)}(\lambda)$ is an increasing L-fuzzy F_σ -set containing λ , $(IL_{(\sigma)}(\lambda))'$ is a decreasing L-fuzzy G_δ -set such that $(IL_{(\sigma)}(\lambda))' \leq \lambda'$. Let μ be another decreasing L-fuzzy G_δ -set such that $\mu \leq \lambda'$. Then μ' is an increasing L-fuzzy F_σ -set such that $\mu' \geq \lambda$. It follows that $IL_{(\sigma)}(\lambda) \leq \mu'$. That is

$\mu \leq (IL_{(\sigma)}(\lambda))'$. Thus $(IL_{(\sigma)}(\lambda))'$ is the largest decreasing L-fuzzy G_δ -set such that $(IL_{(\sigma)}(\lambda))' \leq \lambda'$. That is $(IL_{(\sigma)}(\lambda))' = D_{L(\sigma)}^o(\lambda')$.

Property - 1 (B) For any two L-fuzzy sets λ and μ of $(X, T, \leq)$ we have,

$$\begin{aligned} a) \lambda \leq \mu & \Rightarrow IL_{(\sigma)}(\lambda) \leq IL_{(\sigma)}(\mu). \\ b) \Pi_{(\sigma)}[IL_{(\sigma)}(\lambda)] &= IL_{(\sigma)}(\lambda). \\ c) \Pi_{(\sigma)}(\lambda \vee \mu) &= IL_{(\sigma)}(\lambda) \vee IL_{(\sigma)}(\mu). \\ d) \wedge_{\alpha \in \Gamma} IL_{(\sigma)}(\lambda_\alpha) &= IL_{(\sigma)}(\wedge_{\alpha \in \Gamma} \lambda_\alpha) \\ e) \vee_{\alpha \in \Gamma} IL_{(\sigma)}^o(\lambda_\alpha) &= IoL_{(\sigma)}(\vee_{\alpha \in \Gamma} \lambda_\alpha) \end{aligned}$$

Proof.

a) Let $\lambda \leq \mu$.

$$\begin{aligned} IL_{(\sigma)}(\lambda) &= \wedge \{ \gamma \in IX / \gamma \text{ is an increasing L-fuzzy } F_\sigma\text{-set such that } \gamma \geq \lambda \\ &\leq \wedge \{ \gamma \in IX / \gamma \text{ is an increasing L-fuzzy } F_\sigma\text{-set such that } \gamma \geq \mu \} \\ &= IL_{(\sigma)}(\mu) \end{aligned}$$

b) Since $IL_{(\sigma)}(\lambda)$ is an increasing L-fuzzy F_σ - set, $\Pi_{(\sigma)}[IL_{(\sigma)}(\lambda)] = IL_{(\sigma)}(\lambda)$

c) Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$ by (i)

$$IL_{(\sigma)}(\lambda) \leq IL_{(\sigma)}(\lambda \vee \mu) \text{ and } IL_{(\sigma)}(\mu) \leq IL_{(\sigma)}(\lambda \vee \mu).$$

Therefore

$$(2.1) \quad IL_{(\sigma)}(\lambda) \vee IL_{(\sigma)}(\mu) \leq IL_{(\sigma)}(\lambda \vee \mu)$$

$$\begin{aligned} \text{Now } \lambda &\leq IL_{(\sigma)}(\lambda) \text{ and } \mu \leq IL_{(\sigma)}(\mu) \\ \lambda \vee \mu &\leq IL_{(\sigma)}(\lambda) \vee IL_{(\sigma)}(\mu), \end{aligned}$$

$$(2.2) \quad IL_{(\sigma)}(\lambda \vee \mu) \leq IL_{(\sigma)}(\lambda) \vee IL_{(\sigma)}(\mu)$$

From (2.1) and (2.2) we get

$$\Pi_{(\sigma)}(\lambda \vee \mu) = IL_{(\sigma)}(\lambda) \vee IL_{(\sigma)}(\mu).$$

$$\wedge_{\alpha \in \Gamma} IL_{(\sigma)}(\lambda_\alpha) = IL_{(\sigma)}(\wedge_{\alpha \in \Gamma} \lambda_\alpha)$$

$$\begin{aligned} \wedge_{\alpha \in \Gamma} IL_{(\sigma)}(\lambda_\alpha) &= \wedge_{\alpha \in \Gamma} [\wedge \{ \lambda / \lambda \text{ is an increasing L-fuzzy } F_\sigma\text{-set} \\ &\text{such that } \lambda \geq \lambda_\alpha \}] \\ &= \wedge [\{ \lambda / \lambda \text{ is an increasing L-fuzzy } F_\sigma\text{-set} \\ &\text{such that } \lambda \geq \wedge_{\alpha \in \Gamma} \lambda_\alpha \}] \\ &= IL_{(\sigma)}(\wedge_{\alpha \in \Gamma} \lambda_\alpha) \end{aligned}$$

$$\begin{aligned}
\vee I^o L_{(\sigma)}(\lambda_\alpha) &= I^o L_{(\sigma)}(\vee \lambda_\alpha) \\
\vee I o L_{(\sigma)}(\lambda_\alpha) &= \vee (DL_{(\sigma)}(\lambda'_\alpha))' \\
&= (\wedge DL_{(\sigma)}(\lambda'_\alpha))' \\
&= (DL_{(\sigma)}(\wedge \lambda'_\alpha))' \\
&= (DL_{(\sigma)}(\vee \lambda_\alpha))' \\
&= I o L_{(\sigma)}(\vee \lambda_\alpha)
\end{aligned}$$

Similar properties can be discussed for other cases also.

Definition 8 : Let $(X, T, \leq)$ be an ordered L-fuzzy topological space. Let λ be any increasing L-fuzzy G_δ -set in $(X, T, \leq)$. If $IL(\sigma)(\lambda)$ is increasing L-fuzzy G_δ -set in $(X, T, \leq)$ then $(X, T, \leq)$ is said to be upper L-fuzzy G_δ -extremally disconnected. Similarly we can define lower L-fuzzy G_δ -extremally disconnected space. A L-fuzzy topological space $(X, T, \leq)$ is said to be an ordered L-fuzzy G_δ -extremally disconnected if it is both upper and lower L-fuzzy G_δ -extremally disconnected.

Example - 1 : Let $X = \{a, b, c\}$, $T = \{0, 1, \lambda_1, \lambda_2\}$ where $\lambda_1 : X \rightarrow I[L]$ is such that $\lambda_1(a) = 0$, $\lambda_1(b) = 1/4$, $\lambda_1(c) = 3/4$ and $\lambda_2 : X \rightarrow I[L]$ is such that $\lambda_2(a) = 1$, $\lambda_2(b) = 3/4$, $\lambda_2(c) = 3/4$. The partial order ' $\leq$ ' is defined as $a \leq b$, $b \leq c$. Then $(X, T, \leq)$ is an ordered L-fuzzy topological space. It is clear that $IL(\sigma)(\lambda_1) = 1$, $DL(\sigma)(\lambda_1) = 1$, $IL(\sigma)(\lambda_2) = 1$ and $DL(\sigma)(\lambda_2) = 1$. Hence $(X, T, \leq)$ is an ordered L-fuzzy G_δ -extremally disconnected space.

Property - 2 : For an ordered L-fuzzy topological space $(X, T, \leq)$ the following are equivalent.

- a) $(X, T, \leq)$ is an upper L-fuzzy G_δ -extremally disconnected space.
- b) For each decreasing L-fuzzy F_σ -set λ , $D_{L(\sigma)}^o(\lambda)$ is decreasing L-fuzzy F_σ -set.
- c) For decreasing L-fuzzy G_δ -set λ and decreasing L-fuzzy F_σ -set μ such that $\lambda \leq \mu$, we have $D_{L(\sigma)}(\lambda) \leq D_{L(\sigma)}^o(\mu)$.

Proof.

- $a \Rightarrow b$ Let λ be a decreasing L-fuzzy F_σ -set. We claim $D_{L(\sigma)}^o(\lambda)$ is a decreasing L-fuzzy F_σ -set. Now λ' is an increasing L-fuzzy G_δ -set and so by assumption (a), $IL(\sigma)(\lambda')$ is an increasing L-fuzzy G_δ -set. That is $D_{L(\sigma)}^o(\lambda)$ is a decreasing L-fuzzy F_σ -set.
- $b \Rightarrow c$ Let λ be a decreasing L-fuzzy G_δ -set, μ be a decreasing L-fuzzy F_σ -set such that $\lambda \leq \mu$. Then by (b), $D_{L(\sigma)}^o(\mu)$ is decreasing L-fuzzy

F_σ . Also, since λ is decreasing L-fuzzy G_δ and $\lambda \leq \mu$, it follows that $\lambda \leq DOL(\sigma)(\mu)$. Again, since $D_{L(\sigma)}^o(\mu)$ is decreasing L-fuzzy F_σ , it follows that $DL(\sigma)(\lambda) \leq D_{L(\sigma)}^o(\mu)$.

$c \Rightarrow a$ Let λ be an increasing L-fuzzy G_δ -set. Then λ' is a decreasing L-fuzzy F_σ -set. By definition $D_{L(\sigma)}^o(\lambda')$ is decreasing L-fuzzy G_δ and it is also clear that $D_{L(\sigma)}^o(\lambda') \leq \lambda'$. Then by (c),

$$DL(\sigma)(DOL(\sigma)(\lambda')) \leq DOL(\sigma)(\lambda') \dots (1)$$

We know that $DOL(\sigma)(\lambda') \leq DL(\sigma)(DOL(\sigma)(\lambda')) \dots (2)$.

From (1) and (2) it follows that $D_{L(\sigma)}^o(\lambda') = DL(\sigma)(D_{L(\sigma)}^o(\lambda'))$. Therefore $D_{L(\sigma)}^o(\lambda')$ is decreasing L-fuzzy F_σ -set. But

$D_{L(\sigma)}^o(\lambda') = (IL(\sigma)(\lambda))'$. Therefore $IL(\sigma)(\lambda)$ is increasing L-fuzzy G_δ -set. Hence $(X, T, \leq)$ is ordered L-fuzzy G_δ -extremally disconnected.

Similar property holds for all other cases.

Notation :

L-fuzzy $G_\delta F_\sigma$ denotes the set, which is both L-fuzzy G_δ and L-fuzzy F_σ .

Remark :

Let $(X, T, \leq)$ be an ordered L-fuzzy G_δ -extremally disconnected space. Let $\lambda_i, \mu'_i/i \in N$ be a collection such that λ_i 's are decreasing L-fuzzy G_δ -sets and μ'_i s are decreasing L-fuzzy F_σ -sets. Let λ, μ' be decreasing L-fuzzy G_δ -set and increasing L-fuzzy G_δ -set respectively. If $\lambda_i \leq \lambda \leq \mu_j$ and $\lambda_i \leq \mu \leq \mu_j$ for all $i, j \in N$, then there exists a decreasing L-fuzzy $G_\delta F_\sigma$ -set γ such that $DL(\sigma)(\lambda_i) \leq \gamma \leq D_{L(\sigma)}^o(\mu_j)$ for all $i, j \in N$. By Property 2, $DL(\sigma)(\lambda_i) \leq DL(\sigma)(\lambda) \wedge DOL(\sigma)(\mu) \leq DOL(\sigma)(\mu_j)$ ($i, j \in N$). Put $\gamma = DL(\sigma)(\lambda) \wedge D_{L(\sigma)}^o(\mu)$. Now γ satisfies our required condition.

Property - 3

Let $(X, T, \leq)$ be an ordered L-fuzzy G_δ - extremally disconnected space. Let $\{\lambda_q\}_{q \in Q}$ and $\{\mu_q\}_{q \in Q}$ be monotone increasing collections of decreasing L-fuzzy G_δ -sets and decreasing L-fuzzy F_σ -sets of X and suppose that $\lambda_{q1} \leq \mu_{q2}$ whenever $q_1 < q_2$ (Q is the set of rational numbers). Then there exists a monotone increasing collection $\{\gamma_q\}_{q \in Q}$ of decreasing L-fuzzy $G_\delta F_\sigma$ - sets of X such that $DL(\sigma)(\lambda_{q1}) \leq \gamma_{q2}$ and $\gamma_{q1} \leq D_{L(\sigma)}^o(\mu_{q2})$ whenever $q_1 < q_2$.

Proof. Let us arrange into sequence $\{q_n\}$ of rational numbers without repetitions. For every $n \geq 2$, we shall define inductively a collection $\{\gamma_{qi} : 1 \leq i \leq n\} \subset L^X$ such that

$$\left. \begin{array}{ll} D_{L(\sigma)}(\lambda_q) \leq \gamma_{q_i} & \text{if } q < q_i \\ \gamma_{q_i} \leq D_{L(\sigma)}^o(\mu_q) & \text{if } q_i < q \end{array} \right\} S_n$$

for all $i < n$.

By Property 2, the family $\{D_{L(\sigma)}(\lambda_q)\}$ and $\{D_{L(\sigma)}^o(\mu_q)\}$ satisfying $DL(\sigma)(\lambda_{q_1}) \leq D_{L(\sigma)}^o(\mu_{q_2})$ if $q_1 < q_2$. By Remark, there exists decreasing L-fuzzy $G_\delta F_\sigma$ -set δ_1 such that $DL(\sigma)(\lambda_{q_1}) \leq \delta_1 \leq DOL(\sigma)(\mu_{q_2})$. Setting $\gamma_{q_1} = \delta_1$ we get S_2 . Assume that L-fuzzy sets γ_{q_i} are already defined for $i < n$ and satisfy S_n . Define $\sum = \vee\{\gamma_{q_i} : i < n, q_i < q_n\} \vee \lambda_{q_n}$ and $\phi = \wedge\{r_{q_j} : j < n, q_j > q_n\} \wedge \mu_{q_n}$. Then we have that $DL(\sigma)(\gamma_{q_i}) \leq DL(\sigma)(\sum) \leq DOL(\sigma)(\gamma_{q_j})$ and $DL(\sigma)(\gamma_{q_i}) \leq D^o L(\sigma)(\phi) \leq D_{L(\sigma)}^o(\gamma_{q_j})$ whenever $q_i < q_n < q_j$ ($i, j < n$) as well as $\lambda_q \leq D_{L(\sigma)}(\sum) \leq \mu'_q$ and $\lambda_q \leq D_{L(\sigma)}^o(\phi) \leq \mu'_q$ whenever $q < q_n < q'$. This shows that the countable collection $\{\gamma_{q_i} : i < n, q_i < q_n\} \cup \{\lambda_q : q < q_n\}$ and $\{\gamma_{q_j} : j < n, q_j > q_n\} \cup \{\mu_q : q > q_n\}$ together with $\sum$ and ϕ fulfill all conditions of the said Remark. Hence there exists a decreasing L-fuzzy $G_\delta F_\sigma$ -set δ_n such that $D_{L(\sigma)}(\delta_n) \leq \mu_q$ if $q_n < q$, $\lambda_q \leq D_{L(\sigma)}^o(\delta_n)$ if $q < q_n$, $D_{L(\sigma)}(\gamma_{q_i}) \leq D_{L(\sigma)}^o(\delta_n)$ if $q_i < q_n$, $DL(\sigma)(\delta_n) \leq D_{L(\sigma)}^o(\gamma_{q_j})$ if $q_n < q_j$, where $1 \leq i, j \leq n-1$. Now setting $\gamma_{q_n} = \delta_n$ we obtain the L-fuzzy sets $\gamma_{q_1}, \gamma_{q_2}, \dots, \gamma_{q_n}$ that satisfy S_{n+1} . Therefore the collection $\{\gamma_{q_i} : i = 1, 2, \dots\}$ has the required property. This completes the proof.

Definition - 9 : Let $(X, T, \leq)$ and $(Y, S, \leq)$ be ordered L-fuzzy topological spaces. A mapping $f : (X, T, \leq) \rightarrow (Y, S, \leq)$ is called increasing L-fuzzy G_δ -continuous if $f^{-1}(\lambda)$ is increasing L-fuzzy G_δ (resp. F_σ) set of X for every L-fuzzy G_δ (resp. F_σ) set λ of Y .

Definition - 10: Let $(X, T, \leq)$ be an ordered L-fuzzy topological space. A function $f : X \rightarrow R(L)$ is called lower L-fuzzy G_δ -continuous if $f^{-1}(R_t)$ is increasing or decreasing L-fuzzy G_δ for each $t \in R$ and upper L-fuzzy G_δ -continuous if $f^{-1}(L_t)$ is increasing or decreasing L-fuzzy G_δ for each $t \in R$.

Lemma 1 : Let $(X, T, \leq)$ be an ordered L-fuzzy topological space, let $\lambda \in L^X$, and let

$$f : X \rightarrow R(L) \text{ be such that } f(x)(t) = \begin{cases} 1 & \text{if } t < 0 \\ \lambda(x) & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t > 1, \end{cases}$$

for all $x \in X$. Then f is lower (resp. upper) fuzzy G_δ -continuous iff λ is fuzzy increasing or decreasing G_δ (resp. F_σ)-set.

Property - 4

Let $(X, T, \leq)$ be an ordered L-fuzzy topological space. Then the following statements are equivalent.

- a) $(X, T, \leq)$ is ordered L-fuzzy G_δ -extremally disconnected.
- b) If $g, h : X \rightarrow R[L]$, g is lower L-fuzzy G_δ -continuous, h is upper L-fuzzy G_δ -continuous and $g \leq h$, then there exists an increasing L-fuzzy G_δ -continuous function $f : (X, T, \leq) \rightarrow R[L]$ such that $g \leq f \leq h$.
- c) If λ' is increasing L-fuzzy G_δ , μ is decreasing L-fuzzy G_δ and $\mu \leq \lambda$, then there exists an increasing L-fuzzy G_δ -continuous function $f : (X, T, \leq) \rightarrow I(L)$ such that $\mu \leq (L)f \leq Rof \leq \lambda$.

Proof.

$a \Rightarrow b$ Define $Hr = Lrh$ and $Gr = Rr'g$, $r \in Q$. Thus we have two monotone increasing families of respectively decreasing L-fuzzy G_δ -sets and decreasing L-fuzzy $F\sigma$ -sets of X . Moreover $Hr \leq Gs$ if $r < s$. By Property 3, there exists a monotone increasing family $\{Fr\}_{r \in Q}$ of decreasing L-fuzzy $G_\delta F\sigma$ -sets of X such that $DL_{(\sigma)}(Hr) \leq F_s$ and $Fr \leq D_{L(\sigma)}^o(Gs)$ whenever $r < s$. Letting $V_t = \bigwedge_{r < t} Fr'$ for all $t \in R$, we define a monotone decreasing family $\{V_t : t \in R\} \subset L^X$. Moreover we have $I_{L(\sigma)}(V_t) \leq I_{L(\sigma)}^o(V_s)$, whenever $s < t$. We have

$$\begin{aligned}
 \bigvee_{t \in R} V_t &= \bigvee_{t \in R} \bigwedge_{r < t} Fr' \\
 &\geq \bigvee_{t \in R} \bigwedge_{r < t} Gr' \\
 &= \bigvee_{t \in R} \bigwedge_{r < t} g^{-1}(hr') \\
 &= \bigvee_{t \in R} g^{-1}(Lt') \\
 &= g^{-1}(\bigvee_{t \in R} Lt') = 1. \text{ Similarly } \bigwedge_{t \in R} V_t = 0.
 \end{aligned}$$

We now define a function $f : (X, T, \leq) \rightarrow R(L)$ satisfying the required properties. Let $f(x)(t) = Vt(x)$ for all $x \in X$ and $t \in R$. By the above discussion it follows that f is well defined. To prove f is L-fuzzy increasing G_δ -continuous, we observe that

$$\begin{aligned}
 \bigvee_{s > t} Vs &= \bigvee_{s > t} IoL_{(\sigma)}(V_s) \\
 \bigwedge_{s < t} Vs &= \bigwedge_{s < t} IL_{(\sigma)}(V_s)
 \end{aligned}$$

Then $f^{-1}(R_t) = \vee_{s>t} V_s = \vee_{s>t} I^o L_{(\sigma)}(V_s)$ is increasing L-fuzzy G_δ (by Property 1), $f^{-1}(L_t') = \wedge_{s<t} V_s = \wedge_{s<t} I L_{(\sigma)}(V_s)$ is increasing L-fuzzy F_σ (by Property 1), so that f is increasing L-fuzzy G_δ -continuous. To conclude the proof it remains to show that $g \leq f \leq h$, that is $g^{-1}(L_t') \leq f^{-1}(L_t') \leq h^{-1}(L_t')$ and $g^{-1}(R_t) \leq f^{-1}(R_t) \leq h^{-1}(R_t)$ for each $t \in R$.

We have

$$\begin{aligned} g^{-1}(L_t') &= \wedge_{s<t} g^{-1}(Ls') \\ &= \wedge_{s<t} \wedge_{r<s} g^{-1}(Lr') \\ &= \wedge_{s<t} \wedge_{r<s} Gr' \\ &\leq \wedge_{s<t} \wedge_{r<s} Fr' \\ &= \wedge_{s<t} V_s = f^{-1}(L_t') \text{ and} \end{aligned}$$

$$\begin{aligned} f^{-1}(L_t') &= \wedge_{s<t} V_s \\ &= \wedge_{s<t} \wedge_{r<s} Fr' \\ &\leq \wedge_{s<t} \wedge_{r<s} Hr' \\ &= \wedge_{s<t} \wedge_{r<s} h^{-1}(R_r) \\ &= \wedge_{s<t} h^{-1}(Ls') \\ &= h^{-1}(L_t'). \text{ Similarly we obtain} \end{aligned}$$

$$\begin{aligned} g^{-1}(R_t) &= \vee_{s>t} g^{-1}(R_s) \\ &= \vee_{s>t} \vee_{r>s} g^{-1}(Lr') \\ &= \vee_{s>t} \vee_{r>s} Gr' \\ &\leq \vee_{s>t} \vee_{r>s} Fr' \\ &= \vee_{s>t} V_s = f^{-1}(R_t) \text{ and} \end{aligned}$$

$$\begin{aligned} f^{-1}(R_t) &= \vee_{s>t} V_s \\ &= \vee_{s>t} \wedge_{r<s} Fr' \\ &\leq \vee_{s>t} \vee_{r>s} Hr' \\ &= \vee_{s>t} \vee_{r>s} h^{-1}(R_r) \\ &= \vee_{s>t} h^{-1}(R_s) = h^{-1}(R_t). \text{ Thus } a \Rightarrow b \text{ is proved.} \end{aligned}$$

$b \Rightarrow c$ Suppose λ' is increasing L-fuzzy G_δ and μ is decreasing L-fuzzy G_δ such that $\mu \leq \lambda$. Then $\chi\mu \leq \chi\lambda$ [7], and $\chi\mu$ and $\chi\lambda$ are lower and upper L-fuzzy G_δ -continuous functions respectively. Hence by (b), there exists an increasing L-fuzzy G_δ -continuous function $f : (X, T, \leq) \rightarrow R(L)$ such that $\chi\mu \leq f \leq \chi\lambda$. Clearly $f(x) \in I(L)$ for all $x \in X$ and $\mu = (L'_1)\chi\mu \leq (L'_1)f \leq R_0f \leq$

$$R_{0\chi\lambda=\lambda}.$$

$c \Rightarrow a$ This follows from Property 2, and the fact that $(L1')f$ and Rof are decreasing L-fuzzy $F\sigma$ and decreasing L-fuzzy G_δ -sets respectively. Hence the result.

TIETZE EXTENSION THEOREM FOR ORDERED L-FUZZY G_δ -EXTREMALLY DISCONNECTED SPACES

Tietze Extension Theorem

Let $(X, T, \leq)$ be an ordered L-fuzzy G_δ -extremally disconnected space. Let $A' \in T$ be crisp and let $f : (A, T/A) \rightarrow I(L)$ be an increasing L-fuzzy G_δ -continuous and isotone function. Then f admits an extension $F : (X, T, \leq) \rightarrow I(L)$ with all its properties preserved if f satisfies the following # property

$$\#[\lambda] < [\mu] \Rightarrow f^{-1}\{\chi[[0], [\lambda]]\} < f^{-1}\{\chi[[\mu], [1]]\},$$

where

$$\delta < \theta \Leftrightarrow D_{L(\sigma)}(\delta) \wedge I_{L(\sigma)}(\theta) = 0$$

and $[[\lambda_1], [\lambda_2]] = \{[\mu] \in I(L) : [\lambda_1] \leq [\mu] \leq [\lambda_2]\}$.

Proof. Define two functions $g, h : X \rightarrow I(L)$ by

$$\begin{aligned} g(x) &= f(x) \text{ if } x \in A \\ &= [\lambda_o] \text{ if } x \notin A, \text{ and} \\ h(x) &= f(x) \text{ if } x \in A \\ &= [\lambda_1] \text{ if } x \notin A, \text{ where } [\lambda_o] \text{ and } [\lambda_1] \text{ are equivalence classes [2]} \\ &\quad \text{determined by } \lambda_o, \lambda_1 : R \rightarrow IL \text{ such that} \\ \lambda_o(t) &= 1, t < 0 \\ &= 0, t > 0 \text{ and} \\ \lambda_1(t) &= 1, t < 1 \\ &= 0, t > 1, \text{ g and h} \end{aligned}$$

are respectively lower and upper L-fuzzy G_δ -continuous functions and $g \leq h$. Now by Property 4, there exists an increasing L-fuzzy G_δ -continuous

function $F : (X, T, \leq)(L)$ such that $g(x) \leq F(x) \leq h(x)$ for all $x \in X$. Hence for all $x \in A$, we get $f(x) \leq F(x) \leq f(x)$, so that F is the required extension of f over $(X, T, \leq)$. Also F is isotone as f satisfies $\#$ property. Hence the theorem.

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