Ordered L-fuzzy $G_\delta$-extremely disconnected spaces and Tietze extension theorem

E. Roja
SRI SARADA COLLEGE FOR WOMEN, INDIA
M. K. Uma
SRI SARADA COLLEGE FOR WOMEN, INDIA
and
G. Balasubramanian
PERIYAR UNIVERSITY, INDIA

Received: October 2007. Accepted: October 2008

Abstract

In this paper we introduce a new class of fuzzy topological spaces called ordered L-fuzzy $G_\delta$-extremely disconnected spaces. Besides giving several characterizations and some interesting properties of these spaces, we also establish Tietze extension theorem.

Keywords. Ordered L-fuzzy $G_\delta$-extremely disconnected space increasing L-fuzzy $G_\delta$-continuous, lower (upper) fuzzy $G_\delta$-continuous, increasing (decreasing) L-fuzzy $\sigma$-closure and increasing (decreasing) L-fuzzy $\sigma$-interior.

1. Introduction and Preliminaries

Ever since the introduction of fuzzy sets by L. A. Zadeh [9], the fuzzy concept has invaded almost all branches of Mathematics. S. E. Rodabaugh [5], discussed normality and the L-fuzzy unit interval. He [6] also studied fuzzy addition in the L-fuzzy real line. Hoehle [3] studied the characterizations of L-topologies by L-valued neighbourhoods. An L-fuzzy normal spaces and Tietze extension theorem were discussed by Tomash Kubiak [8]. The concept of ordered fuzzy topological spaces was introduced by A. K. Katsaras [4]. In this paper we introduce the concept of $G_δ$-extremally disconnectedness in ordered L-fuzzy topology. In Section 2, we discuss the characterization and properties of ordered L-fuzzy $G_δ$-extremally disconnected spaces. In Section 3, we establish Tietze extension theorem for ordered L-fuzzy $G_δ$-extremally disconnected spaces.

Definition 1:
Throughout out this paper $(L, \leq, ')$ stands for an infinitely distributive lattice with an order reversing involution. Such a lattice being complete has a least element 0 and a greatest element 1. Let $X$ be a non-empty set. An L-fuzzy set in $X$ is an element of the set $L^X$ of all functions from $X$ to $L[1]$. 

Definition 2:
The L-fuzzy real line $R(L)$ is the set of all monotone decreasing elements $\lambda \in L^R$ satisfying $\forall \{\lambda(t)/t \in R\} = 1$ and $\wedge\{\lambda(t)/t \in R\} = 0$, after the identification of $\lambda, \mu \in L^R$ iff $\lambda(t-) = \mu(t-)$ and $\lambda(t+) = \mu(t+)$ for all $t \in R$ where $\lambda(t-) = \wedge\{\lambda(s)/s < t\}$ and $\lambda(t+) = \vee\{\lambda(s)/s > t\}$. The natural L-fuzzy topology on $R(L)$ is generated from the subbasis $\{Lt, Rt | t \in R\}$, where $Lt[\lambda] = \lambda(t-)'$ and $Rt[\lambda] = \lambda(t+)$. A partial order on $R(L)$ is defined by $[\lambda] \leq [\mu] \Leftrightarrow \lambda(t-) \leq \mu(t-)$ and $\lambda(t+) \leq \mu(t+)$ for all $t \in R$. The L-fuzzy unit interval $I(L)$ is a subset of $R(L)$ such that $[\lambda] \in I[L]$ if $\lambda(t) = 1$ for $t < 0$ and $\lambda(t) = 0$ for $t > 1$. It is equipped with the subspace L-fuzzy topology [1].

Definition 3:
An L-fuzzy set $\lambda$ in $X$ is called closed if $\lambda'$ is open, where $\lambda'$ is defined as follows. $\lambda'(x) = (\lambda(x))'[1]$. 

Definition 4:
If $A \in L^X$ is crisp, then $(A, T_A)$ is an L-fuzzy topological space, called a crisp subspace of $(X, T)$, where $T_A = \{\lambda/A \mid \lambda \in T\}$ is called the subspace L-fuzzy topology [1].
Definition - 5:
An L-fuzzy set \( \lambda \) in a partial ordered set \( X \) is called (i) Increasing if \( x \leq y \Rightarrow \lambda(x) \leq \lambda(y) \). (ii) Decreasing if \( x \leq y \Rightarrow \lambda(x) \geq \lambda(y) \). It is clear that the constant L-fuzzy sets are increasing and decreasing [1].

2. ORDERED L-FUZZY \( G_\delta \) - EXTREMALLY DISCONNECTED SPACES AND THEIR PROPERTIES

Definition - 6:
Let \((X,T,\leq)\) be an ordered L-fuzzy topological space and let \( \lambda \) be an L-fuzzy set in \( X \), \( \lambda \) is called increasing L-fuzzy \( G_\delta(F_\sigma) \) if \( \lambda = \bigwedge_{i=1}^{\infty} \lambda_i \) (if \( \lambda = \bigvee_{i=1}^{\infty} \lambda_i \)) where each \( \lambda_i \) is increasing L-fuzzy open(closed) in \( X \). The complement of increasing L-fuzzy \( G_\delta(F_\sigma) \)-set is decreasing L-fuzzy \( F_\sigma(G_\delta) \).

Definition - 7:
Let \( \lambda \) be an L-fuzzy set in the ordered L-fuzzy topological space \((X,T,\leq)\). Then we define:
\[
\begin{align*}
IL(\sigma)(\lambda) &= \text{increasing L-fuzzy}\sigma\text{-closure of } \lambda. \\
DL(\sigma)(\lambda) &= \text{decreasing L-fuzzy}\sigma\text{-closure of } \lambda. \\
I^\sigma L(\sigma)(\lambda) &= \text{increasing L-fuzzy}\sigma\text{-interior of } \lambda. \\
D^\sigma L(\sigma)(\lambda) &= \text{decreasing L-fuzzy}\sigma\text{-interior of } \lambda.
\end{align*}
\]

Property - 1. (A) For a fuzzy set \( \lambda \) of an ordered L-fuzzy topological space \((X,T,\leq)\) the following hold.
\[
\begin{align*}
a) (IL(\sigma)(\lambda))' &= DL(\sigma)(\lambda'). \\
b) (DL(\sigma)(\lambda))' &= I^\sigma L(\sigma)(\lambda'). \\
c) (I^\sigma L(\sigma)(\lambda))' &= DL(\sigma)(\lambda'). \\
d) (D^\sigma L(\sigma)(\lambda))' &= IL(\sigma)(\lambda').
\end{align*}
\]

Proof. We shall prove (a) only (b), (c) and (d) can be proved in a similar manner.

Since \( IL(\sigma)(\lambda) \) is an increasing L-fuzzy \( F_\sigma \)-set containing \( \lambda \), \( (IL(\sigma)(\lambda))' \) is a decreasing L-fuzzy \( G_\delta \)-set such that \( (IL(\sigma)(\lambda))' \leq \lambda' \). Let \( \mu \) be another decreasing L-fuzzy \( G_\delta \)-set such that \( \mu \leq \lambda' \). Then \( \mu' \) is an increasing L-fuzzy \( F_\sigma \)-set such that \( \mu' \geq \lambda \). It follows that \( IL(\sigma)(\lambda) \leq \mu' \). That is
\( \mu \leq (IL_{(\sigma)}(\lambda))' \). Thus \((IL_{(\sigma)}(\lambda))'\) is the largest decreasing L-fuzzy \(G_\delta\)-set such that \((IL_{(\sigma)}(\lambda))' \leq \lambda' \). That is \((IL_{(\sigma)}(\lambda))' = D^\phi_{L(\sigma)}(\lambda')\).

**Property - 1 (B)** For any two L-fuzzy sets \( \lambda \) and \( \mu \) of \((X, T, \leq)\) we have,

a) \( \lambda \leq \mu \Rightarrow IL_{(\sigma)}(\lambda) \leq IL_{(\sigma)}(\mu) \).

b) \( IL_{(\sigma)}[IL_{(\sigma)}(\lambda)] = IL_{(\sigma)}(\lambda) \).

c) \( IL_{(\sigma)}(\lambda \lor \mu) = IL_{(\sigma)}(\lambda) \lor IL_{(\sigma)}(\mu) \).

d) \( \land_{\alpha \in \Gamma} IL_{(\sigma)}(\lambda_\alpha) = IL_{(\sigma)}(\land_{\alpha \in \Gamma} \lambda_\alpha) \).

e) \( \lor_{\alpha \in \Gamma} IL_{(\sigma)}(\lambda_\alpha) = IoL_{(\sigma)}(\lor_{\alpha \in \Gamma} \lambda_\alpha) \).

**Proof.**

a) Let \( \lambda \leq \mu \).

\[
IL_{(\sigma)}(\lambda) = \land\{\gamma \in IX/\gamma \text{is an increasing L-fuzzy } F_{\sigma}\text{-set such that } \gamma \geq \lambda \}
\leq \land\{\gamma \in IX/\gamma \text{is an increasing L-fuzzy } F_{\sigma}\text{-set such that } \gamma \geq \mu \}
= IL_{(\sigma)}(\mu)
\]

b) Since \( IL_{(\sigma)}(\lambda) \) is an increasing L-fuzzy \( F_{\sigma} \)-set, \( IL_{(\sigma)}[IL_{(\sigma)}(\lambda)] = IL_{(\sigma)}(\lambda) \).

c) Since \( \lambda \leq \lambda \lor \mu \) and \( \mu \leq \lambda \lor \mu \) by (i)

\[
IL_{(\sigma)}(\lambda) \leq IL_{(\sigma)}(\lambda \lor \mu) \text{ and } IL_{(\sigma)}(\mu) \leq IL_{(\sigma)}(\lambda \lor \mu).
\]

Therefore

\[ (2.1) \quad IL_{(\sigma)}(\lambda) \lor IL_{(\sigma)}(\mu) \leq IL_{(\sigma)}(\lambda \lor \mu) \]

Now \( \lambda \leq IL_{(\sigma)}(\lambda) \) and \( \mu \leq IL_{(\sigma)}(\mu) \)

\[ \lambda \lor \mu \leq IL_{(\sigma)}(\lambda) \lor IL_{(\sigma)}(\mu), \]

\[ (2.2) \quad IL_{(\sigma)}(\lambda \lor \mu) \leq IL_{(\sigma)}(\lambda) \lor IL_{(\sigma)}(\mu) \]

From (2.1) and (2.2) we get

\[ IL_{(\sigma)}(\lambda \lor \mu) = IL_{(\sigma)}(\lambda) \lor IL_{(\sigma)}(\mu). \]

\[ \land_{\alpha \in \Gamma} IL_{(\sigma)}(\lambda_\alpha) = IL_{(\sigma)}(\land_{\alpha \in \Gamma} \lambda_\alpha) \]

\[ \land_{\alpha \in \Gamma} IL_{(\sigma)}(\lambda_\alpha) = \land_{\alpha \in \Gamma}[\land\{\lambda/\lambda \text{is an increasing L-fuzzy } F_{\sigma}\text{-set such that } \lambda \geq \lambda_\alpha \}] \]

\[ = \land[\{\lambda/\lambda \text{is an increasing L-fuzzy } F_{\sigma}\text{-set such that } \lambda \geq \land_{\alpha \in \Gamma} \lambda_\alpha \}] \]

\[ = IL_{(\sigma)}(\land_{\alpha \in \Gamma} \lambda_\alpha) \]
\( \lor I^a L(\sigma)(\lambda_\alpha) = I^a L(\sigma)(\lor \lambda_\alpha) \)
\( \lor Io L(\sigma)(\lambda_\alpha) = \lor (DL(\sigma)(\lambda_\alpha))' \)
\( = (\land DL(\sigma)(\lambda_\alpha))' \)
\( = (DL(\sigma)(\land \lambda_\alpha))' \)
\( = (DL(\sigma)(\lor \lambda_\alpha))' \)
\( = Io L(\sigma)(\lor \lambda_\alpha) \)

Similar properties can be discussed for other cases also.

**Definition 8**: Let \((X, T, \leq)\) be an ordered \(L\)-fuzzy topological space. Let \(\lambda\) be any increasing \(L\)-fuzzy \(G_\delta\) - set in \((X, T, \leq)\). If \(IL(\sigma)(\lambda)\) is increasing \(L\)-fuzzy \(G_\delta\) - in \((X, T, \leq)\) then \((X, T, \leq)\) is said to be upper \(L\)-fuzzy \(G_\delta\)-extremally disconnected. Similarly we can define lower \(L\)-fuzzy \(G_\delta\)-extremally disconnected space. A \(L\)-fuzzy topological space \((X, T, \leq)\) is said to be an ordered \(L\)-fuzzy \(G_\delta\) - extremally disconnected if it is both upper and lower \(L\)-fuzzy \(G_\delta\) - extremally disconnected.

**Example - 1**: Let \(X = \{a, b, c\}\), \(T = \{0, 1, \lambda_1, \lambda_2\}\) where \(\lambda_1: X \to IL(L)\) is such that \(\lambda_1(a) = 0, \lambda_1(b) = 1/4, \lambda_1(c) = 3/4\) and \(\lambda_2: X \to IL(L)\) is such that \(\lambda_2(a) = 1, \lambda_2(b) = 3/4, \lambda_2(c) = 3/4\). The partial order ‘\(\leq\)’ is defined as \(a \leq b, b \leq c\). Then \((X, T, \leq)\) is an ordered \(L\)-fuzzy topological space. It is clear that \(IL(\sigma)(\lambda_1) = 1, DL(\sigma)(\lambda_1) = 1, IL(\sigma)(\lambda_2) = 1\) and \(DL(\sigma)(\lambda_2) = 1\). Hence \((X, T, \leq)\) is an ordered \(L\)-fuzzy \(G_\delta\)-extremally disconnected space.

**Property - 2**: For an ordered \(L\)-fuzzy topological space \((X, T, \leq)\) the following are equivalent.

a) \((X, T, \leq)\) is an upper \(L\)-fuzzy \(G_\delta\)-extremally disconnected space.

b) For each decreasing \(L\)-fuzzy \(F_\sigma\)-set \(\lambda\), \(DL(\sigma)(\lambda)\) is decreasing \(L\)-fuzzy \(F_\sigma\)-set.

c) For decreasing \(L\)-fuzzy \(G_\delta\)-set \(\lambda\) and decreasing \(L\)-fuzzy \(F_\sigma\)-set \(\mu\) such that \(\lambda \leq \mu\), we have \(DL(\sigma)(\lambda) \leq DL(\sigma)(\mu)\).

**Proof.**

\(a \Rightarrow b\) Let \(\lambda\) be a decreasing \(L\)-fuzzy \(F_\sigma\)-set. We claim \(DL(\sigma)(\lambda)\) is a decreasing \(L\)-fuzzy \(F_\sigma\)-set. Now \(\lambda'\) is an increasing \(L\)-fuzzy \(G_\delta\)-set and so by assumption (a), \(IL(\sigma)(\lambda')\) is an increasing \(L\)-fuzzy \(G_\delta\)-set. That is \(DL(\sigma)(\lambda)\) is a decreasing \(L\)-fuzzy \(F_\sigma\)-set.

\(b \Rightarrow c\) Let \(\lambda\) be a decreasing \(L\)-fuzzy \(G_\delta\)-set, \(\mu\) be a decreasing \(L\)-fuzzy \(F_\sigma\)-set such that \(\lambda \leq \mu\). Then by (b), \(DL(\sigma)(\mu)\) is decreasing \(L\)-fuzzy.
Let $\gamma$ sets and $X$. Hence $(\text{D}_0(\sigma)) = (\text{I}_L(\sigma))$. Again, since $\text{D}_0(\sigma)$ is decreasing L-fuzzy $F_{\sigma}$. Therefore $(\text{D}_0(\sigma)^\prime)$ is increasing L-fuzzy $F_{\sigma}$-set.

\textbf{Property - 3}

Let $(X, T, \leq)$ be an ordered L-fuzzy $G_{\delta}$ - extremally disconnected space. Let $\gamma_i, i \in N$ be a collection such that $\gamma_i$ 's are decreasing L-fuzzy $G_{\delta}$-sets and $\mu_i$'s are decreasing L-fuzzy $F_{\sigma}$-sets. Let $\gamma, \mu$ be decreasing L-fuzzy $G_{\delta}$-set and increasing L-fuzzy $G_{\delta}$-set respectively. If $\gamma_i \leq \gamma \leq \mu$ and $\gamma_i \leq \mu_j$ for all $i, j \in N$, then there exists a decreasing L-fuzzy $G_{\delta}F_{\sigma}$-set $\gamma$ such that $D_L(\sigma)(\gamma_i) \leq \gamma \leq D_L(\sigma)(\mu_j)$ for all $i, j \in N$. By Property 2, $D_L(\sigma)(\gamma_i) \leq D_L(\sigma)(\gamma) \land DOL(\sigma)(\mu_j) \leq DOL(\sigma)(\mu_j)$ $(i, j \in N)$. Put $\gamma = D_L(\sigma)(\gamma) \land D_L(\sigma)(\mu)$. Now $\gamma$ satisfies our required condition.

\textbf{Proof.} Let us arrange into sequence $\{\gamma_i\}$ of rational numbers without repetitions. For every $n \geq 2$, we shall define inductively a collection \{\(\gamma_i : 1 \leq i \leq n\}\} \subset L^X$ such that...
\[ D_{L(\sigma)}(\lambda_q) \leq \gamma q_i \quad \text{if} \quad q < q_i \]
\[ \gamma q_i \leq D_{L(\sigma)}^\sigma(\mu_q) \quad \text{if} \quad q_i < q \]

for all \( i < n \).

By Property 2, the family \( \{D_{L(\sigma)}(\lambda_q)\} \) and \( \{D_{L(\sigma)}(\mu_q)\} \) satisfying
\[ DL(\sigma)(\lambda_{q_1}) \leq D_L^\sigma(\sigma)(\mu_{q_2}) \text{ if } q_1 < q_2. \]
By Remark, there exists decreasing L-fuzzy \( G_\delta F_\sigma \)-set \( \delta_1 \) such that \( DL(\sigma)(\lambda_{q_1}) \leq \delta_1 \leq DOL(\sigma)(\mu_{q_2}) \).

Setting \( \gamma q_i = \delta_1 \) we get \( S_2 \). Assume that L-fuzzy sets \( \gamma q_i \) are already defined for \( i < n \) and satisfy \( S_n \). Define \( \Sigma = \bigvee\{\gamma q_i : i < n, q_i < q_n\} \lor \lambda q_n \) and \( \phi = \land\{r_{q_j} : j < n, q_j > q_n\} \lor \mu q_n \). Then we have that
\[ DL(\sigma)(\gamma q_i) \leq DL(\sigma)(\Sigma) \leq DOL(\sigma)(\gamma q_j) \quad \text{and} \quad DL(\sigma)(\gamma q_i) \leq D^\sigma L(\sigma)(\phi) \leq D^\sigma L(\sigma)(\gamma q_j) \]
whenever \( q_i < q_n < q_j \) (i, j < n) as well as \( \lambda q \leq D_L(\sigma)(\Sigma) \leq \mu q \).

Setting \( \tau q_n = \delta_n \) we obtain the L-fuzzy sets \( \gamma q_1, \gamma q_2, \ldots, \gamma q_n \) that satisfy \( S_{n+1} \). Therefore the collection \( \{\gamma q_i : i = 1, 2, \ldots\} \) has the required property. This completes the proof.

**Definition - 9:** Let \( (X, T, \leq) \) and \( (Y, S, \leq) \) be ordered L-fuzzy topological spaces. A mapping \( f : (X, T, \leq) \to (Y, S, \leq) \) is called increasing L-fuzzy \( G_\delta \)-continuous if \( f^{-1}(\lambda) \) is increasing L-fuzzy \( G_\delta \) (resp. \( F_\sigma \)) set of \( X \) for every L-fuzzy \( G_\delta \) (resp. \( F_\sigma \)) set \( \lambda \) of \( Y \).

**Definition - 10:** Let \( (X, T, \leq) \) be an ordered L-fuzzy topological space. A function \( f : X \to R(L) \) is called lower L-fuzzy \( G_\delta \)-continuous if \( f^{-1}(R_t) \) is increasing or decreasing L-fuzzy \( G_\delta \) for each \( t \in R \) and upper L-fuzzy \( G_\delta \)-continuous if \( f^{-1}(L_t) \) is increasing or decreasing L-fuzzy \( G_\delta \) for each \( t \in R \).

**Lemma 1:** Let \( (X, T, \leq) \) be an ordered L-fuzzy topological space, let \( \lambda \in L^X \), and let
\[ f : X \to R(L) \text{ be such that } f(x)(t) = \begin{cases} 1 & \text{if } t < 0 \\ \lambda(x) & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t > 1, \end{cases} \]
for all \( x \in X \). Then \( f \) is lower (resp. upper) fuzzy \( G_\delta \)-continuous iff \( \lambda \) is fuzzy increasing or decreasing \( G_\delta \) (resp. \( F_\sigma \))-set.
Property - 4

Let \((X, T, \leq)\) be an ordered L-fuzzy topological space. Then the following statements are equivalent.

a) \((X, T, \leq)\) is ordered L-fuzzy \(G_\delta\)-extremally disconnected.

b) If \(g, h : X \to R[L]\), \(g\) is lower L-fuzzy \(G_\delta\)-continuous, \(h\) is upper L-fuzzy \(G_\delta\)-continuous and \(g \leq h\), then there exists an increasing L-fuzzy \(G_\delta\)-continuous function \(f : (X, T, \leq) \to R[L]\) such that \(g \leq f \leq h\).

c) If \(\lambda\) is increasing L-fuzzy \(G_\delta\), \(\mu\) is decreasing L-fuzzy \(G_\delta\) and \(\mu \leq \lambda\), then there exists an increasing L-fuzzy \(G_\delta\)-continuous function \(f : (X, T, \leq) \to I(L)\) such that \(\mu \leq (L)f \leq Rof \leq \lambda\).

Proof.

\(a \Rightarrow b\) Define \(Hr = Lrh\) and \(Gr = Rr'g\), \(r \in Q\). Thus we have two monotone increasing families of respectively decreasing L-fuzzy \(G_\delta\)-sets and decreasing L-fuzzy \(F_\sigma\)-sets of \(X\). Moreover \(Hr \leq Gs\) if \(r < s\). By Property 3, there exists a monotone increasing family \(\{Fr\}_{r \in Q}\) of decreasing L-fuzzy \(G_\delta F_\sigma\)-sets of \(X\) such that 
\[
DL(\sigma)(Hr) \leq Fs \quad \text{and} \quad Fr \leq DL(\sigma)(Gs) \quad \text{whenever} \quad r < s.
\]
Letting \(V_t = \land_{r < t} Fr'\) for all \(t \in R\), we define a monotone decreasing family \(\{V_t : t \in R\} \subset L^X\). Moreover we have \(IL(\sigma)(V_t) \leq IL(\sigma)(Vs)\), whenever \(s < t\). We have

\[
\forall_{t \in R} V_t = \forall_{t \in R} \land_{r < t} Fr' \\
\geq \forall_{t \in R} \land_{r < t} Gr' \\
= \forall_{t \in R} \land_{r < t} g^{-1}(hr') \\
= \forall_{t \in R} g^{-1}(Lt') \\
= g^{-1}(\forall_{t \in R} L_t) = 1. \quad \text{Similarly} \quad \forall_{t \in R} V_t = 0.
\]

We now define a function \(f : (X, T, \leq) \to R(L)\) satisfying the required properties. Let \(f(x)(t) = Vt(x)\) for all \(x \in X\) and \(t \in R\). By the above discussion it follows that \(f\) is well defined. To prove \(f\) is L-fuzzy increasing \(G_\delta\)-continuous, we observe that

\[
\forall_{s \geq t} Vs = \forall_{s \geq t} IO(\sigma)(Vs) \\
\land_{s < t} Vs = \land_{s < t} IL(\sigma)(Vs)
\]
Then $f^{-1}(R_t) = \vee_{s > t} V_s = \vee_{s > t} I^a L(\sigma)(V_s)$ is increasing L-fuzzy $G_\delta$ (by Property 1), $f^{-1}(L t') = \wedge_{s < t} V_s = \wedge_{s < t} I L(\sigma)(V_s)$ is increasing L-fuzzy $F_\eta$ (by Property 1), so that $f$ is increasing L-fuzzy $G_\delta$-continuous. To conclude the proof it remains to show that $g \leq f \leq h$, that is $g^{-1}(L t') \leq f^{-1}(L t') \leq h^{-1}(L t')$ and $g^{-1}(R t) \leq f^{-1}(R_t) \leq h^{-1}(R_t)$ for each $t \in R$.

We have

\[
g^{-1}(L_t') = \wedge_{s < t} g^{-1}(L_s') = \wedge_{s < t} \wedge_{r < s} g^{-1}(L_r') = \wedge_{s < t} \wedge_{r < s} G_r' \leq \wedge_{s < t} \wedge_{r < s} F_r' = \wedge_{s < t} V_s = f^{-1}(L_t') \quad \text{and}
\]

\[
f^{-1}(L_t') = \wedge_{s < t} V_s = \wedge_{s < t} \wedge_{r < s} F_r t \leq \wedge_{s < t} \wedge_{r < s} H r' = \wedge_{s < t} \wedge_{r < s} h^{-1}(R_r) = \wedge_{s < t} h^{-1}(L s') = h^{-1}(L_t'). \quad \text{Similarly we obtain}
\]

\[
g^{-1}(R_t) = \vee_{s > t} g^{-1}(R_s) = \vee_{s > t} \vee_{r > s} g^{-1}(L_r') = \vee_{s > t} \vee_{r > s} G_r' \leq \vee_{s > t} \vee_{r > s} F_r' = \vee_{s > t} V_s = f^{-1}(R_t) \quad \text{and}
\]

\[
f^{-1}(R_t) = \vee_{s > t} V_s = \vee_{s > t} \wedge_{r < s} F_r' \leq \vee_{s > t} \vee_{r > s} H r' = \vee_{s > t} \vee_{r > s} h^{-1}(R_r) = \vee_{s > t} h^{-1}(R_s) = h^{-1}(R_t). \quad \text{Thusa} \Rightarrow b \text{ is proved.}
\]

\[
\Rightarrow c \quad \text{Suppose} \lambda' \text{ is increasing L-fuzzy } G_\delta \text{ and } \mu \text{ is decreasing L-fuzzy } G_\delta \text{ such that } \mu \leq \lambda. \text{ Then } \chi \mu \leq \chi \lambda [7], \text{ and } \chi \mu \text{ and } \chi \lambda \text{ are lower and upper L-fuzzy } G_\delta \text{-continuous functions respectively. Hence by (b), there exists an increasing L-fuzzy } G_\delta \text{-continuous function } f : (X, T, \leq) \rightarrow R(L) \text{ such that } \chi \mu \leq f \leq \chi \lambda. \text{ Clearly } f(x) \in I(L) \text{ for all } x \in X \text{ and } \mu = (L_t') \chi \mu \leq (L_t') f \leq R_0 f \leq
\]
This follows from Property 2, and the fact that \((L^1')f\) and \(Rof\) are decreasing L-fuzzy \(F\sigma\) and decreasing L-fuzzy \(G_\delta\)-sets respectively. Hence the result.

**TIETZE EXTENSION THEOREM FOR ORDERED L-FUZZY \(G_\delta\)-EXTREMALLY DISCONNECTED SPACES**

**Tietze Extension Theorem**

Let \((X, T, \leq)\) be an ordered L-fuzzy \(G_\delta\)-extremally disconnected space. Let \(A_0 \in T\) be crisp and let \(f : (A, T/A) \to I(L)\) be an increasing L-fuzzy \(G_\delta\)-continuous and isotone function. Then \(f\) admits an extension \(F : (X, T, \leq) \to I(L)\) with all its properties preserved if \(f\) satisfies the following \# property

\[
\#[\lambda] < [\mu] \Rightarrow f^{-1}\{\chi[[0], [\lambda]]\} < f^{-1}\{\chi[[\mu], [1]]\},
\]

where

\[
\delta < \theta \iff D_{L(\sigma)}(\delta) \land I_{L(\sigma)}(\theta) = 0
\]

and \([\lambda_1], [\lambda_2] = \{[\mu] \in I(L) : [\lambda_1] \leq [\mu] \leq [\lambda_2]\}\).

**Proof.** Define two functions \(g, h : X \to I(L)\) by

\[
\begin{align*}
g(x) &= f(x) & \text{if} \ x \in A \\
     &= [\lambda_0] & \text{if} \ x \notin A, \ & \text{and} \\
h(x) &= f(x) & \text{if} \ x \in A \\
     &= [\lambda_1] & \text{if} \ x \notin A, \ & \text{where} \ [\lambda_0] \ & \text{and} \ [\lambda_1] \ & \text{are equivalence classes}[2] \\
& & \text{determined by} \lambda_0, \lambda_1 : R \to IL \ & \text{such that} \\
\lambda_0(t) &= 1, t < 0 \\
& = 0, \ & \text{and} \\
\lambda_1(t) &= 1, t < 1 \\
& = 0, t > 1, \ & g \ & \text{and} \ h
\end{align*}
\]

are respectively lower and upper L-fuzzy \(G_\delta\)-continuous functions and \(g \leq h\). Now by Property 4, there exists an increasing L-fuzzy \(G_\delta\)-continuous
function $F : (X, T, \leq)(L)$ such that $g(x) \leq F(x) \leq h(x)$ for all $x \in X$. Hence for all $x \in A$, we get $f(x) \leq F(x) \leq f(x)$, so that $F$ is the required extension of $f$ over $(X, T, \leq)$. Also $F$ is isotone as $f$ satisfies $\#$ property. Hence the theorem.

References


E. ROJA
Department of Mathematics
Sri Sarada College for Women
Salem – 636 016
Tamil Nadu
India
e-mail: ar.udhay@yahoo.co.in

M. K. UMA
Department of Mathematics
Sri Sarada College for Women
Salem – 636 016
Tamil Nadu
India
e-mail: ar.udhay@yahoo.co.in

and

G. BALASUBRAMANIAN
Department of Mathematics
Periyar University
Salem – 636 011
Tamil Nadu
India
e-mail: ar.udhay@yahoo.co.in