Proyecciones
Vol. 27, N° 3, pp. 289–305, December 2008.
Universidad Católica del Norte
Antofagasta - Chile

POSITIVE MATRICES WITH PRESCRIBED SINGULAR VALUES *

EMEDIN MONTAÑO
UNIVERSIDAD DE MAGALLANES, CHILE

MARIO SALAS
UNIVERSIDAD CATÓLICA DEL NORTE, CHILE
and

RICARDO L. SOTO
UNIVERSIDAD CATÓLICA DEL NORTE, CHILE
Received : June 2008. Accepted : October 2008

Abstract

We consider the problem of constructing positive matrices with prescribed singular values. In particular, we show how to construct an $m \times n$ positive matrix, $m \geq n$, with prescribed singular values $\sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_n$.

AMS classification : 15A18; 15A29.

Keywords : Singular values, positive matrices.

*Supported by Fondecyt 1085125, Chile.

1. Introduction

A *singular value decomposition* of a matrix $A \in \mathbf{C}^{m \times n}$ is a factorization $A = U\Sigma V^*$, where $\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r) \in \mathbf{R}^{m \times n}$, $r = \min\{m, n\}$, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq 0$ and both $U \in \mathbf{C}^{m \times m}$ and $V \in \mathbf{C}^{n \times n}$ are unitary. The diagonal entries of Σ are called *the singular values* of A . The columns $\mathbf{u}_j$ of U are called *left singular vectors* of A and the columns $\mathbf{v}_j$ of V are called *right singular vectors* of A . Every $A \in \mathbf{C}^{m \times n}$ has a *singular value decomposition* $A = U\Sigma V^*$ and the following relations hold: $A\mathbf{v}_j = \sigma_j \mathbf{u}_j$, $A^* \mathbf{u}_j = \sigma_j \mathbf{v}_j$ and $\mathbf{u}_j^* A \mathbf{v}_j = \sigma_j$. If $A \in \mathbf{R}^{m \times n}$, then U and V may be taken to be real (see [3]).

In this paper we consider the following inverse singular value problem:

Problem: Given the real numbers $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$, to construct an $m \times n$, $m \geq n$, entrywise positive matrix A with singular values σ_i , $i = 1, \dots, n$.

In the case $m = n$, the problem has always a solution if $\sigma_1 > \sigma_2$. We construct such a solution from the following two results, due to Fiedler [2], and related with the *nonnegative inverse eigenvalue problem*:

Lemma 1. [2] Let $\alpha > \beta \geq 0$, $\epsilon > 0$. Then there exists a number $c > 0$ such that the matrix

$$\begin{pmatrix} \alpha & c \\ c & \beta \end{pmatrix} \text{ has eigenvalues } \alpha + \epsilon, \beta - \epsilon.$$

Lemma 2. [2] Let A be a symmetric $m \times m$ matrix with eigenvalues $\alpha_1, \dots, \alpha_m$, and let $\mathbf{u}$, $\|\mathbf{u}\| = 1$, be a unit eigenvector corresponding to α_1 . Let B be a symmetric $n \times n$ matrix with eigenvalues $\beta_1, \dots, \beta_n$, and let $\mathbf{v}$, $\|\mathbf{v}\| = 1$, be a unit eigenvector corresponding to β_1 . Then for any ρ , the matrix

$$S = \begin{pmatrix} A & \rho \mathbf{u} \mathbf{v}^T \\ \rho \mathbf{v} \mathbf{u}^T & B \end{pmatrix}$$

has eigenvalues $\alpha_2, \dots, \alpha_m, \beta_2, \dots, \beta_n, \gamma_1, \gamma_2$, where γ_1, γ_2 are eigenvalues of the matrix

$$\hat{S} = \begin{pmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{pmatrix}.$$

To construct a positive matrix, not necessarily symmetric, with prescribed singular values, we first use the below result, Lemma 4, which shows how to construct an $n \times n$ nonnegative semibordered matrix

$$(1.1) \quad C = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix},$$

with prescribed singular values and then, to obtain the required positive matrix, we apply a Brauer [1] type singular value perturbation result given in [4, Corollary 14], which shows how to modify a single singular value of an $n \times n$ matrix without changing any of the remaining singular values.

Lemma 3. *The characteristic polynomial of the arrow matrix*

$$B = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ b_2 & a_2 & & \\ \vdots & & \ddots & \\ b_n & & & a_n \end{pmatrix}$$

is

$$p(\lambda) = (b_1 - \lambda) \prod_{i=2}^n (a_i - \lambda) - \sum_{i=2}^n b_i^2 \prod_{\substack{k=2 \\ k \neq i}}^n (a_k - \lambda).$$

Proof. By expanding

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \begin{vmatrix} b_1 - \lambda & b_2 & b_3 & \cdots & b_n \\ b_2 & a_2 - \lambda & & & \\ b_3 & & a_3 - \lambda & & \\ \vdots & & & \ddots & \\ b_n & & & & a_n - \lambda \end{vmatrix} \\ &= (b_1 - \lambda) \prod_{i=2}^n (a_i - \lambda) - \sum_{i=2}^n b_i^2 \prod_{\substack{k=2 \\ k \neq i}}^n (a_k - \lambda). \end{aligned}$$

□

Lemma 4. Given the real numbers $\sigma_1 > \sigma_2 \geq \dots \geq \sigma_n > 0$, then there exists an $n \times n$ nonnegative matrix C of the form (1.1) with singular values $\sigma_1, \sigma_2, \dots, \sigma_n$.

Proof. Let $A = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_n \end{pmatrix}$.

Then

$$\begin{aligned} B &= AA^T = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_n \end{pmatrix} \begin{pmatrix} b_1 & & & & \\ b_2 & a_2 & & & \\ b_3 & & a_3 & & \\ \vdots & & & \ddots & \\ b_n & & & & a_n \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^n b_i^2 & a_2 b_2 & a_3 b_3 & \cdots & a_n b_n \\ a_2 b_2 & a_2^2 & & & \\ a_3 b_3 & & a_3^2 & & \\ \vdots & & & \ddots & \\ a_n b_n & & & & a_n^2 \end{pmatrix} \end{aligned}$$

is an $n \times n$ arrow symmetric matrix with characteristic polynomial

$$(1.2) \quad p(\lambda) = \left(\sum_{i=1}^n b_i^2 - \lambda \right) \prod_{i=2}^n (a_i^2 - \lambda) - \sum_{i=2}^n (a_i b_i)^2 \prod_{j=2, j \neq i}^n (a_j^2 - \lambda)$$

and eigenvalues $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$. We define

$$a_k = \sqrt{\frac{\sigma_{k-1}^2 + \sigma_k^2}{2}}, \quad k = 2, 3, \dots, n.$$

Then, by taking $\lambda = \frac{\sigma_{k-1}^2 + \sigma_k^2}{2}$, $k = 2, 3, \dots, n$, in (1.2), we have

$$\prod_{i=2}^n \left(a_i^2 - \frac{\sigma_{k-1}^2 + \sigma_k^2}{2} \right) = 0$$

and

$$-(a_k b_k)^2 \prod_{\substack{j=2 \\ j \neq k}}^n \left(a_j^2 - \frac{\sigma_{k-1}^2 + \sigma_k^2}{2} \right) = \prod_{i=1}^n \left(\frac{\sigma_{k-1}^2 + \sigma_k^2}{2} - \sigma_i^2 \right).$$

Hence,

$$(1.3) \quad b_k = \sqrt{-\frac{\prod_{i=1}^n (a_k^2 - \sigma_i^2)}{a_k^2 \prod_{\substack{i=2 \\ i \neq k}}^n (a_i^2 - a_k^2)}}, \quad k = 2, 3, \dots, n.$$

For b_1 we take $\lambda = \sigma_1^2$ in (1.2). Then

$$\left(b_1^2 + \sum_{i=2}^n b_i^2 - \sigma_1^2 \right) \prod_{i=2}^n (a_i^2 - \sigma_1^2) - \sum_{i=2}^n (a_i b_i)^2 \prod_{\substack{j=2 \\ j \neq i}}^n (a_j^2 - \sigma_1^2) = p(\sigma_1^2) = 0$$

and

$$\begin{aligned} b_1 &= \sqrt{\frac{1}{\prod_{i=2}^n (a_i^2 - \sigma_1^2)} \left(\sum_{i=2}^n a_i^2 b_i^2 \prod_{\substack{j=2 \\ j \neq i}}^n (a_j^2 - \sigma_1^2) + \sigma_1^2 \prod_{i=2}^n (a_i^2 - \sigma_1^2) \right) - \sum_{i=2}^n b_i^2} \\ &= \sqrt{\frac{\sum_{i=2}^n a_i^2 b_i^2 \prod_{\substack{j=2 \\ j \neq i}}^n (a_j^2 - \sigma_1^2)}{\prod_{i=2}^n (a_i^2 - \sigma_1^2)} + \sigma_1^2 - \sum_{i=2}^n b_i^2} \\ &= \sqrt{\sum_{i=2}^n \frac{a_i^2 b_i^2}{a_i^2 - \sigma_1^2} - \sum_{i=2}^n b_i^2 + \sigma_1^2} \end{aligned}$$

Thus,

$$(1.4) \quad b_1 = \sqrt{\left[\sum_{i=2}^n \left(\frac{b_i^2}{a_i^2 - \sigma_1^2} \right) \right] \sigma_1^2 + \sigma_1^2}$$

□

Lemma 5. [4] Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq 0$, $r = \min\{m, n\}$. Let $\mathbf{u}_i$, $\mathbf{v}_i$, respectively, the left and right singular vectors corresponding to σ_i , $i = 1, \dots, r$. Let $\alpha \in \mathbf{R}$ such that $\alpha + \sigma_i \geq 0$. Then $A + \alpha \mathbf{u}_i \mathbf{v}_i^T$ has singular values

$$\sigma_1, \dots, \sigma_{i-1}, \sigma_i + \alpha, \sigma_{i+1}, \dots, \sigma_r.$$

The paper is organized as follows: In section 2 we construct an $n \times n$ positive matrix with prescribed singular values by using results related with the *symmetric nonnegative inverse eigenvalue problem*. In section 3 we construct a positive matrix, not necessarily symmetric, from Lemmas 4 and 5. In section 4 we consider the rectangular case $m \times n$, $m > n$. Finally, in section 5 we give some examples to illustrate the results.

2. Positive matrices with prescribed singular values I

In this section we construct a positive symmetric matrix with prescribed nonnegative eigenvalues, which in this case, are also its singular values.

Theorem 1. Given the real numbers $\sigma_1 > \sigma_2 \geq \dots \geq \sigma_n \geq 0$, there exists a positive matrix A with singular values $\sigma_1, \sigma_2, \dots, \sigma_n$.

Proof. Let

$$B_k = \begin{pmatrix} \frac{\sigma_k + \sigma_{n-k+1}}{2} & \frac{\sigma_k - \sigma_{n-k+1}}{2} \\ \frac{\sigma_k - \sigma_{n-k+1}}{2} & \frac{\sigma_k + \sigma_{n-k+1}}{2} \end{pmatrix}, \quad k = 1, 2, \dots, \frac{n}{2}, \text{ even } n,$$

with $B_{\frac{n+1}{2}} = \left(\sigma_{\frac{n+1}{2}} \right)$ for odd n . Thus, B_k is a positive symmetric matrix with singular values σ_k and σ_{n-k+1} (eigenvalues σ_k and σ_{n-k+1}). Then

$$B = \begin{pmatrix} B_1 & 0 & \ddots & 0 \\ 0 & B_2 & \ddots & \ddots \\ \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 0 & B_{\frac{n}{2}} \end{pmatrix},$$

with $B_{\frac{n+1}{2}}$ in the last diagonal position if n is odd, is an $n \times n$ nonnegative symmetric matrix with singular values $\sigma_1, \sigma_2, \dots, \sigma_n$. In order to obtain a

positive matrix with prescribed singular values let us consider $\mu_1 > \mu_2 \geq \dots \geq \mu_n \geq 0$, where

$$\begin{aligned}\mu_1 &= \frac{\sigma_1 + \sigma_2}{2}, \\ \mu_i &= \sigma_i + \epsilon_i \leq \mu_{i-1}, \quad i = 2, \dots, \frac{n}{2} \left(\frac{n+1}{2} \text{ for odd } n \right), \\ \mu_i &= \sigma_i, \quad i = \frac{n+2}{2}, \dots, n \left(\frac{n+3}{2} \text{ for odd } n \right)\end{aligned}$$

with

$$\epsilon = \frac{\sigma_1 - \sigma_2}{2} \text{ and } \epsilon_i = \frac{2\epsilon}{n-2} = \frac{\sigma_1 - \sigma_2}{n-2}$$

Now we apply the Lemma 2, $\frac{n-2}{2}$ times if n is even ($\frac{n-1}{2}$ times if n is odd). In the first step of this process we apply Lemma 2 to the matrices

$$B_1 = \begin{pmatrix} \frac{\mu_1 + \sigma_n}{2} & \frac{\mu_1 - \sigma_n}{2} \\ \frac{\mu_1 - \sigma_n}{2} & \frac{\mu_1 + \sigma_n}{2} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \frac{\mu_2 + \sigma_{n-1}}{2} & \frac{\mu_2 - \sigma_{n-1}}{2} \\ \frac{\mu_2 - \sigma_{n-1}}{2} & \frac{\mu_2 + \sigma_{n-1}}{2} \end{pmatrix},$$

with singular values μ_1, σ_n and μ_2, σ_{n-1} , respectively. From Lemma 1, the matrix

$$\begin{pmatrix} \mu_1 & \rho \\ \rho & \mu_2 \end{pmatrix}, \quad \text{where } \rho = (\epsilon_2(\mu_1 - \mu_2 + \epsilon_2))^{\frac{1}{2}},$$

has eigenvalues $\mu_1 + \epsilon_2$ and $\mu_2 - \epsilon_2 = \sigma_2$. The unitary vectors

$$\mathbf{u} = \mathbf{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T$$

are the Perron eigenvectors of B_1 and B_2 , respectively. Then the 4×4 matrix

$$\mathcal{B}_1 = \begin{pmatrix} B_1 & \rho \mathbf{u} \mathbf{v}^T \\ \rho \mathbf{v} \mathbf{u}^T & B_2 \end{pmatrix}$$

is positive symmetric with singular values (eigenvalues) $\mu_1 + \epsilon_2, \sigma_2, \sigma_{n-1}, \sigma_n$. In the second step of the process we again apply Lemma 2 to the matrices $\mathcal{B}_1$ and

$$B_3 = \begin{pmatrix} \frac{\mu_3 + \sigma_{n-2}}{2} & \frac{\mu_3 - \sigma_{n-2}}{2} \\ \frac{\mu_3 - \sigma_{n-2}}{2} & \frac{\mu_3 + \sigma_{n-2}}{2} \end{pmatrix},$$

which has singular values μ_3 and σ_{n-2} . The matrix

$$\begin{pmatrix} \mu_1 + \epsilon_2 & \rho \\ \rho & \mu_3 \end{pmatrix}, \text{ where } \rho = (\epsilon_3(\mu_1 + \epsilon_2 - \mu_3 + \epsilon_3))^{\frac{1}{2}}$$

has eigenvalues $\mu_1 + \epsilon_2 + \epsilon_3$ and $\mu_3 - \epsilon_3 = \sigma_3$. Now let $\mathcal{B}_1 \mathbf{u} = (\mu_1 + \epsilon_2) \mathbf{u}$, $\|\mathbf{u}\| = 1$ and $B_3 \mathbf{v} = \mu_3 \mathbf{v}$, $\mathbf{v} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$. Then the 6×6 matrix

$$\mathcal{B}_2 = \begin{pmatrix} \mathcal{B}_1 & \rho \mathbf{u} \mathbf{v}^T \\ \rho \mathbf{v} \mathbf{u}^T & B_3 \end{pmatrix},$$

is positive symmetric with singular values $\mu_1 + \epsilon_2 + \epsilon_3, \sigma_2, \sigma_3, \sigma_{n-2}, \sigma_{n-1}, \sigma_n$. The process is continued until in the last step we apply Lemma 2 to the matrices $\mathcal{B}_{\frac{n-4}{2}}$ ($\mathcal{B}_{\frac{n-3}{2}}$ for odd n), of order $n-2$ ($n-1$ for odd n) and $B_{\frac{n}{2}}$ ($B_{\frac{n+1}{2}}$ for odd n), which has the singular values $\mu_{\frac{n}{2}}$ and $\mu_{\frac{n+2}{2}}$ ($\mu_{\frac{n+1}{2}}$ for odd n). The matrix

$$\begin{pmatrix} \mu_1 + (\epsilon - \epsilon_{\frac{n}{2}}) & \rho \\ \rho & \mu_{\frac{n}{2}} \end{pmatrix}, \text{ where } \rho = (\epsilon_{\frac{n}{2}}(\mu_1 + (\epsilon - \epsilon_{\frac{n}{2}}) - \mu_{\frac{n}{2}} + \epsilon_{\frac{n}{2}}))^{\frac{1}{2}},$$

has eigenvalues $\mu_1 + \epsilon$ and $\mu_{\frac{n}{2}} - \epsilon_{\frac{n}{2}} = \sigma_{\frac{n}{2}}$. Then the $n \times n$ matrix

$$A = \mathcal{B}_{\frac{n-2}{2}} = \begin{pmatrix} \mathcal{B}_{\frac{n-4}{2}} & \rho \mathbf{u} \mathbf{v}^T \\ \rho \mathbf{v} \mathbf{u}^T & B_{\frac{n}{2}} \end{pmatrix} \text{ for even } n$$

or

$$A = \mathcal{B}_{\frac{n-1}{2}} = \begin{pmatrix} \mathcal{B}_{\frac{n-4}{2}} & \rho \mathbf{u} \mathbf{v}^T \\ \rho \mathbf{v} \mathbf{u}^T & B_{\frac{n+1}{2}} \end{pmatrix} \text{ for odd } n,$$

where $\mathbf{u}$ and $\mathbf{v}$ are the unitary eigenvectors of $\mathcal{B}_{\frac{n-2}{2}}$ ($\mathcal{B}_{\frac{n-1}{2}}$ for odd n) and $B_{\frac{n}{2}}$ ($B_{\frac{n+1}{2}}$ for odd n), respectively, is positive and since

$$\mu_1 + \epsilon_2 + \cdots + \epsilon_{\frac{n}{2}} = \mu_1 + \epsilon = \sigma_1,$$

A has the prescribed singular values $\sigma_1, \sigma_2, \dots, \sigma_n$. $\square$

3. Positive matrices with prescribed singular values II

In this section we construct an $n \times n$ positive matrix A , no necessarily symmetric, with prescribed singular values $\sigma_1, \sigma_2, \dots, \sigma_n$. First, by applying Lemma 4 we construct a nonnegative semibordered diagonal matrix C of the form

$$(3.1) \quad C = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_n \end{pmatrix}, \quad b_j > 0, \quad j = 1, \dots, n.$$

and then, by applying Lemma 5, we add to the matrix C the positive matrix $\alpha \mathbf{u} \mathbf{v}^T$, where $\alpha > 0$ and $\mathbf{u}$ and $\mathbf{v}$ are the left and right singular vectors of C corresponding to its maximal singular value.

Theorem 2. Let the nonnegative real numbers $\sigma_1 > \sigma_2 \geq \cdots \geq \sigma_n > 0$ be given. Then there exists an $n \times n$ positive matrix A with singular values $\sigma_1, \sigma_2, \dots, \sigma_n$.

Proof. Let $\alpha > 0$ such that $\tilde{\sigma}_1 = \sigma_1 - \alpha > \sigma_2$ and consider $\tilde{\sigma}_1 > \sigma_2 \geq \cdots \geq \sigma_n$. Then, from Lemma 4 there exists an $n \times n$ nonnegative matrix C of the form (3.1), with singular values $\tilde{\sigma}_1, \sigma_2, \dots, \sigma_n$. The arrow matrix CC^T is nonnegative, irreducible, with Perron eigenvector equal to $\mathbf{u}_1$ corresponding to the Perron root $\tilde{\sigma}_1^2$ (left singular vector of C corresponding to the singular value $\tilde{\sigma}_1$). Then $\mathbf{u}_1$ is an entrywise positive vector and so is the right singular vector $\mathbf{v}_1$. Therefore the matrix $\alpha \mathbf{u}_1 \mathbf{v}_1^T$ is positive and from Lemma 5, $A = C + \alpha \mathbf{u}_1 \mathbf{v}_1^T$ is positive with the prescribed singular values $\sigma_1, \sigma_2, \dots, \sigma_n$. $\square$

Lemma 4 and the form of the matrix in (3.1) allow us to construct a nonnegative matrix, say C , whose exponential matrix e^C has prescribed singular values: To show this, we consider the matrix C in (3.1). Then a straight forward calculation shows that

$$C^m = \begin{pmatrix} b_1^m & b_2 \sum_{k=0}^{m-1} b_1^{m-1-k} a_2^k & \cdots & \cdots & b_n \sum_{k=0}^{m-1} b_1^{m-1-k} a_n^k \\ & a_2^m & & & \\ & & a_3^m & & \\ & & & \ddots & \\ & & & & a_n^m \end{pmatrix}$$

and

$$e^C = \sum_{j=0}^{\infty} \frac{1}{j!} \begin{pmatrix} b_1^j & b_2 \sum_{k=0}^{j-1} b_1^{j-1-k} a_2^k & \cdots & \cdots & b_n \sum_{k=0}^{j-1} b_1^{j-1-k} a_n^k \\ & a_2^j & & & \\ & & a_3^j & & \\ & & & \ddots & \\ & & & & a_n^j \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=0}^{\infty} \frac{1}{j!} b_1^j & b_2 \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{k=0}^{j-1} b_1^{j-1-k} a_2^k & \cdots & \cdots & b_n \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{k=0}^{j-1} b_1^{j-1-k} a_n^k \\ & \sum_{j=0}^{\infty} \frac{1}{j!} a_2^j & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \sum_{j=0}^{\infty} \frac{1}{j!} a_n^j \end{pmatrix}$$

is also a semibordered diagonal matrix. Now, for $i = 2, 3, \dots, n$, we have

$$\begin{aligned} (e^C)_{1i} &= b_i \sum_{j=0}^{\infty} \left(\frac{1}{j!} \sum_{k=0}^{j-1} b_1^{j-1-k} a_i^k \right) \\ &= b_i \sum_{j=0}^{\infty} \left(\frac{1}{j!} \sum_{k=0}^{j-1} b_1^{j-1} \left(\frac{a_i}{b_1} \right)^k \right) \\ &= b_i \sum_{j=0}^{\infty} \left(\frac{b_1^{j-1}}{j!} \sum_{k=0}^{j-1} \left(\frac{a_i}{b_1} \right)^k \right). \end{aligned}$$

Since

$$\sum_{k=0}^{j-1} \left(\frac{a_i}{b_1} \right)^k = \frac{1 - \left(\frac{a_i}{b_1} \right)^j}{1 - \left(\frac{a_i}{b_1} \right)} = \frac{b_1^j - a_i^j}{b_1^{j-1} (b_1 - a_i)}$$

then

$$\begin{aligned}
(e^C)_{1i} &= \frac{b_i}{(b_1 - a_i)} \sum_{j=0}^{\infty} \left(\frac{b_1^j - a_i^j}{j!} \right) = \frac{b_i}{(b_1 - a_i)} \sum_{j=0}^{\infty} \left(\frac{b_1^j}{j!} - \frac{a_i^j}{j!} \right) \\
&= \frac{b_i (e^{b_1} - e^{a_i})}{b_1 - a_i}, \quad i = 2, \dots, n.
\end{aligned}$$

Moreover,

$$(e^C)_{11} = e^{b_1} \quad \text{and} \quad (e^C)_{ii} = e^{a_i}, \quad i = 2, \dots, n.$$

Hence, given the real positive numbers $\sigma_1 > \sigma_2 \geq \dots \geq \sigma_n > 0$, from Lemma 4, we may construct a matrix $\tilde{C}$ of the form (3.1), with singular values $\sigma_1, \dots, \sigma_n$, where the $\tilde{b}_i$ are given by (1.3) and (1.4) and can be chosen as positive. Clearly, the $\tilde{a}_i$ can also be chosen positive. Then, if $e^C = \tilde{C}$, we have

$$(e^C)_{1i} = \frac{b_i (e^{b_1} - e^{a_i})}{b_1 - a_i} = \tilde{b}_i, \quad i = 2, \dots, n$$

and

$$(3.2) \quad b_i = \frac{(b_1 - a_i)}{(e^{b_1} - e^{a_i})} \tilde{b}_i \quad i = 2, \dots, n.$$

In the same way,

$$(e^C)_{11} = e^{b_1} = \tilde{b}_1$$

and

$$(3.3) \quad b_1 = \frac{1}{2} \ln \tilde{b}_1$$

From (1.3) we see that

$$\sum_{i=2}^n \left(\frac{\tilde{b}_i^2}{\tilde{a}_i^2 - \sigma_1^2} \right) > 0.$$

Finally, from

$$(e^C)_{ii} = e^{a_i} = \tilde{a}_i = \sqrt{\frac{\sigma_{i-1} + \sigma_i}{2}}, \quad i = 2, \dots, n,$$

we have

$$(3.4) \quad a_i = \frac{1}{2} \ln \left(\frac{\sigma_{i-1} + \sigma_i}{2} \right), \quad i = 2, \dots, n.$$

Observe that if $\sigma_n > 1$, then $\frac{\sigma_i + \sigma_{i-1}}{2} > 1$ and the a_i are all positive. Thus, if $\sigma_1 > \sigma_2 \geq \dots \geq \sigma_n > 1$, we can construct a nonnegative matrix C of the form (3.1), in such a way that e^C has prescribed singular values $\sigma_1, \sigma_2, \dots, \sigma_n$. To illustrate this construction, let us consider the numbers $\sigma_i : 7, 5, 3, 2$. Then, from (3.2), (3.3) and (3.4) we compute the matrix C such that

$$e^C = \begin{bmatrix} 3.2845 & 2.428 & 2.7309 & 1.5365 \\ 0 & 6.083 & 0 & 0 \\ 0 & 0 & 4.1231 & 0 \\ 0 & 0 & 0 & 2.5495 \end{bmatrix}$$

has the prescribed singular values 7, 5, 3, 2.

4. Positive matrices with prescribed singular values III

In this section we consider the construction of a rectangular positive matrix with prescribed singular values. First we construct an $m \times n$ nonnegative matrix B , $m > n$, of the form

$$(4.1) \quad B = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \\ a_{n+1} & & & \\ \vdots & & & \\ a_m & & & \end{pmatrix}, \quad m > n$$

and then we apply the perturbation result given by Lemma 5 to fill out B with positive entries.

Theorem 3. Given the numbers $\sigma_1 > \sigma_2 > \dots > \sigma_n > 0$, there exists an $m \times n$ nonnegative matrix, $m > n$, with singular values $\sigma_1, \sigma_2, \dots, \sigma_n$.

Proof. We define

$$a_i = \frac{\sigma_{i-1} + \sigma_i}{2}, \quad i = 2, \dots, n,$$

and we choose $a_{n+1}, a_{n+2}, \dots, a_m$ in such a way that

$$a_1^2 = \sum_{n+1}^m a_j^2 < \sigma_n.$$

Since we want to construct the matrix B in (4.1), then we observe that

$$B^T B = \begin{pmatrix} b_1^2 + \sum_{j=n+1}^m a_j^2 & b_1 b_2 & b_1 b_3 & \cdots & b_1 b_n \\ b_1 b_2 & b_2^2 + a_2^2 & b_2 b_3 & \ddots & b_2 b_n \\ b_1 b_3 & b_2 b_3 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ b_1 b_n & b_2 b_n & \cdots & b_{n-1} b_n & b_n^2 + a_n^2 \end{pmatrix} = D^2 + \mathbf{b}\mathbf{b}^T,$$

where $D = \text{diag}\{a_1, \dots, a_n\}$ and $\mathbf{b} = (b_1, \dots, b_n)^T$. Then

$$\begin{aligned} \prod_{k=1}^n (\sigma_k^2 - \sigma) &= \det(B^T B - \sigma I) = \det(D^2 + \mathbf{b}\mathbf{b}^T - \sigma I) \\ &= \det(D^2 - \sigma I) \det(I + (D^2 - \sigma I)^{-1} \mathbf{b}\mathbf{b}^T) \end{aligned}$$

and from the well known formula $\det(I + \mathbf{xy}^T) = 1 + \mathbf{y}^T \mathbf{x}$, we have

$$\begin{aligned} \prod_{k=1}^n (\sigma_k^2 - \sigma) &= \left(\sum_{j=n+1}^m a_j^2 - \sigma \right) \prod_{k=2}^n (a_k^2 - \sigma) \left(1 + \frac{b_1^2}{\sum_{j=n+1}^m a_j^2 - \sigma} + \sum_{k=2}^n \frac{b_k^2}{a_k^2 - \sigma} \right) \\ &= (a_1^2 - \sigma) \prod_{k=2}^n (a_k^2 - \sigma) + b_1^2 \prod_{k=2}^n (a_k^2 - \sigma) + \\ &\quad + (a_1^2 - \sigma) \prod_{k=2}^n (a_k^2 - \sigma) \sum_{k=2}^n \frac{b_k^2}{a_k^2 - \sigma} \\ &= (a_1^2 - \sigma) \prod_{k=2}^n (a_k^2 - \sigma) + b_1^2 \prod_{k=2}^n (a_k^2 - \sigma) + \\ &\quad + (a_1^2 - \sigma) \prod_{k=2}^n (a_k^2 - \sigma) \sum_{\substack{k=2 \\ k \neq i}}^n \frac{b_k^2}{a_k^2 - \sigma} + b_i^2 \prod_{\substack{k=1 \\ k \neq i}}^n (a_k^2 - \sigma). \end{aligned}$$

Thus, for $\sigma = a_i^2$, $i = 1, \dots, n$, we have

$$(4.2) \quad b_i^2 = \frac{\prod_{k=1}^n (\sigma_k^2 - a_i^2)}{\prod_{\substack{k=1 \\ k \neq i}}^n (a_k^2 - a_i^2)}, \quad i = 1, \dots, n.$$

Since $\sigma_1 > a_2 > \sigma_2 > a_3 > \dots > \sigma_n > a_1$, then the right side in (4.2) is positive and we may choose b_i , $i = 1, \dots, n$, as positive. Hence, we may construct the nonnegative matrix B in (4.1), with the prescribed singular values $\sigma_1, \sigma_2, \dots, \sigma_n$. $\square$

Corollary 1. *Given the positive numbers $\sigma_1 > \sigma_2 > \dots > \sigma_n > 0$, there exists an $m \times n$ positive matrix A , $m > n$, with singular values $\sigma_1, \sigma_2, \dots, \sigma_n$.*

Proof. Let $\alpha > 0$ such that $\tilde{\sigma}_1 = \sigma_1 - \alpha > \sigma_2$. Then from Theorem ?? there exists an $m \times n$ nonnegative matrix B of the form (4.1) with singular values $\tilde{\sigma}_1, \sigma_2, \dots, \sigma_n$. The matrix BB^T is irreducible nonnegative and therefore its Perron eigenvector $\mathbf{u}_1$ (left singular vector of B corresponding to $\tilde{\sigma}_1$) is positive and so is the right singular vector $\mathbf{v}_1$. Hence, from Lemma 5, $A = B + \alpha \mathbf{u}_1 \mathbf{v}_1^T$ is an $m \times n$ positive matrix with singular values $\sigma_1, \sigma_2, \dots, \sigma_n$. $\square$

5. Examples

Example 1. *This example illustrates the construction of the matrix A of Theorem 1: Let $S = \{24, 16, 16, 16, 16, 2\}$ be given. Since $\epsilon = \frac{\sigma_1 - \sigma_2}{2} = 4$, then $\epsilon_i = 2$, $i = 1, 2$. Thus, instead S we consider $\Gamma = \{20, 18, 18, 16, 16, 2\}$ as the prescribed set of singular values. Then*

$$B_1 = \begin{pmatrix} 11 & 9 \\ 9 & 11 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 17 & 1 \\ 1 & 17 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 17 & 1 \\ 1 & 17 \end{pmatrix}.$$

We apply Lemma 2 to the matrices B_1 and B_2 , with $\rho = 2\sqrt{2}$, $\mathbf{u} = \mathbf{v} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$ to obtain

$$\mathcal{B}_1 = \begin{pmatrix} B_1 & \rho \mathbf{u} \mathbf{v}^T \\ \rho \mathbf{v} \mathbf{u}^T & B_2 \end{pmatrix} = \begin{pmatrix} 11 & 9 & \sqrt{2} & \sqrt{2} \\ 9 & 11 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 17 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 17 \end{pmatrix},$$

with eigenvalues 22, 16, 16, 2. Next we again apply Lemma 2 to the matrices $\mathcal{B}_1$ and $\mathcal{B}_3$, with

$\rho = 2\sqrt{3}$, $\hat{\mathbf{u}} = \left(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}, \frac{1}{6}\sqrt{6}, \frac{1}{6}\sqrt{6}\right)^T$, $\mathbf{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T$,
to obtain the positive matrix

$$A = \mathcal{B}_2 = \begin{pmatrix} \mathcal{B}_1 & \rho \hat{\mathbf{u}} \mathbf{v}^T \\ \rho \mathbf{v} \hat{\mathbf{u}}^T & B_3 \end{pmatrix} = \begin{pmatrix} 11 & 9 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 9 & 11 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 17 & 1 & 1 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 17 & 1 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 1 & 17 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 1 & 1 & 17 \end{pmatrix}$$

with the prescribed singular values: 24, 16, 16, 16, 16, 2.

Example 2. This example illustrates the construction of a 4×4 positive nonsymmetric matrix with the prescribed singular values $3, \sqrt{5}, \sqrt{3}, 1$. We consider the list $\sqrt{7}, \sqrt{5}, \sqrt{3}, 1$. Then from Lemma 4 we construct the non-negative matrix

$$C = \begin{pmatrix} \frac{\sqrt{35}}{4} & \frac{\sqrt{5}}{4} & \frac{3}{4} & \frac{\sqrt{15}}{4} \\ 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}$$

with singular values $\sqrt{7}, \sqrt{5}, \sqrt{3}, 1$. Next, we apply Lemma 5 to obtain the positive matrix

$$\begin{aligned} A &= C + (3 - \sqrt{7}) \mathbf{u}_1 \mathbf{v}_1^T \\ &= \begin{pmatrix} 1.5409 & 0.72275 & 0.82322 & 1.0250 \\ 8.4740 \times 10^{-2} & 2.6737 & 0.10026 & 7.7665 \times 10^{-2} \\ 3.0943 \times 10^{-2} & 8.1867 \times 10^{-2} & 2.0366 & 2.8359 \times 10^{-2} \\ 1.6948 \times 10^{-2} & 4.4837 \times 10^{-2} & 2.0052 \times 10^{-2} & 1.4297 \end{pmatrix} \end{aligned}$$

with the prescribed singular values $3, \sqrt{5}, \sqrt{3}, 1$.

Example 3. This example illustrates the construction of a 7×4 positive matrix with the prescribed singular values $3, \sqrt{5}, \sqrt{3}, 1$. Let $\alpha = 3 - \sqrt{7}$ and let us consider the numbers $\sqrt{7}, \sqrt{5}, \sqrt{3}, 1$, as in the previous example. From Theorem ??, we compute the nonnegative matrix

$$B = \begin{pmatrix} 0.546\,23 & 0.592\,52 & 0.841\,92 & 1.401\,6 \\ 0 & 2.440\,9 & 0 & 0 \\ 0 & 0 & 1.984\,1 & 0 \\ 0 & 0 & 0 & 1.366 \\ 0.707\,11 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0.408\,25 & 0 & 0 & 0 \end{pmatrix},$$

with singular values $\sqrt{7}, \sqrt{5}, \sqrt{3}, 1$. Since $B^T B$ is irreducible nonnegative, its Perron eigenvector, $\mathbf{u}_1$, corresponding to the Perron root 7 (left singular vector of B corresponding to the singular value $\sqrt{7}$), is entrywise positive and so is the right singular vector $\mathbf{v}_1$. Therefore the matrix $\alpha \mathbf{u}_1 \mathbf{v}_1^T$ is positive and from Lemma 5, $A = B + \alpha \mathbf{u}_1 \mathbf{v}_1^T$ is positive with the prescribed singular values $3, \sqrt{5}, \sqrt{3}, 1$.

References

- [1] A. Brauer, Limits for the characteristic roots of a matrix IV: Applications to stochastic matrices, *Duke Math. J.* 19, pp. 75-91, (1952).
- [2] M. Fiedler, Eigenvalues of nonnegative symmetric matrices, *Linear Algebra Appl.* 9, pp. 119-142, (1974).
- [3] R. A. Horn, C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, (1991).
- [4] E. Montaña, M. Salas, R. L. Soto, Nonnegative matrices with prescribed extremal singular values, *Computer and Mathematics with Applications*, doi:10.1016/j.camwa.2007.11.030, (2007).

Emedin Montaña

Departamento de Matemática y Física

Universidad de Magallanes

e-mail : emedin.montano@umag.cl

Mario Salas

Departamento de Matemáticas
Universidad Católica del Norte
Casilla 1280, Antofagasta
Chile
e-mail : msalas@ucn.cl

and

Ricardo L. Soto

Departamento de Matemáticas
Universidad Católica del Norte
Casilla 1280, Antofagasta
Chile
e-mail : rsoto@ucn.cl