ON THE LOCAL CONVERGENCE OF A
TWO-STEP STEFFENSEN-TYPE METHOD FOR
SOLVING GENERALIZED EQUATIONS

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Abstract

We use a two—step Steffensen—type method [1], [2], [4], [6], [13]—[16] to solve
a generalized equation in a Banach space setting under Hölder—type conditions
introduced by us in [2], [6] for nonlinear equations. Using some ideas given in
[4], [6] for nonlinear equations, we provide a local convergence analysis with the
following advantages over related [13]—[16]: finer error bounds on the distances
involved, and a larger radius of convergence. An application is also provided.

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continuity, Hölder continuity, radius of convergence, divided difference, set—
valued map.
1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^*$ of the generalized equation

\begin{equation}
0 \in f(x) + G(x),
\end{equation}

where $f$ is a continuous function defined in a neighborhood $V$ of the solution $x^*$ included in a Banach space $X$ with values in itself, and $G$ is a set—valued map from $X$ to its subsets with closed graph.

Many problems in mathematical programming, mathematical economics, variational inequalities and other fields can be formulated as in equation (1.1) \[3\], \[5\], \[6\], \[8\], \[11\], \[12\], \[18\]—\[21\] (see also the application at the end of the study).

We consider the two—step Steffensen—type method \[1\], \[2\], \[4\], \[6\], \[13\]—\[16\] for $x_0 \in V$ being the initial guess and all $k \geq 0$

\begin{equation}
\begin{cases}
0 \in f(x_k) + [g_1(x_k), g_2(x_k); f] (y_k - x_k) + G(y_k) \\
0 \in f(y_k) + [g_1(x_k), g_2(x_k); f] (x_{k+1} - y_k) + G(x_{k+1}),
\end{cases}
\end{equation}

where $g_1$ and $g_2$ are a continuous functions from $V$ into $X$ and $[x, y; f] \in L(X)$ (the space of bounded linear operator on $X$) is a divided difference of order one of $f$ at the points $x$, $y$ satisfying

\begin{equation}
[x, y; f] (y - x) = f(y) - f(x), \text{ for all } x \neq y.
\end{equation}

Note that if $f$ is Fréchet—differentiable at $x$, then $[x, x; f] = \nabla f(x)$.

For $G \equiv 0$, (1.2) reduces to methods studied in \[1\], \[4\], \[6\] for nonlinear equations.

Recently in \[13\] a local convergence analysis was provided for method (1.2) under Hölder—type conditions introduced by us in \[4\], \[6\] to solve nonlinear equations.

Motivated by optimization considerations, and using the ideas from \[4\], \[6\], \[7\] for nonlinear equations we provide under less computational cost a new local convergence analysis for method (1.2) with the following advantages over the corresponding results in \[13\]—\[16\]: finer error bounds on the distances $\| x_k - x^* \| (k \geq 0)$, and a larger radius of convergence leading to fewer steps and a wider choice of initial guesses $x_0$.

This observation is very important in computational mathematics \[1\]—\[22\]. The study ends with an application.
2. Preliminaries and assumptions

In order to make the paper as self-contained as possible we reintroduce some results on fixed point theorem [6]–[9], [13]–[16].

We let $Z$ be a metric space equipped with the metric $\rho$. For $A \subset Z$, we denote by $\text{dist}(x,A) = \inf \{\rho(x,y), \; y \in A\}$ the distance from a point $x$ to $A$. The excess $e$ from $A$ to the set $C \subset Z$ is given by $e(A,C) = \sup \{\text{dist}(x,A), \; x \in C\}$. Let $\Lambda : XY$ be a set-valued map, we denote by $\text{gph} \Lambda = \{(x,y) \in X \times Y, \; y \in \Lambda(x)\}$ and $\Lambda^{-1}(y) = \{x \in X, \; y \in \Lambda(x)\}$ is the inverse of $\Lambda$. We call $B_r(x)$ the closed ball centered at $x$ with radius $r$.

**Definition 2.1.** (see [8], [17], [20])

A set-valued $\Lambda$ is said to be pseudo-Lipschitz around $(x_0,y_0) \in \text{gph} \Lambda$ with modulus $M$ if there exist constants $a$ and $b$ such that

\[
(2.1) \quad e(\Lambda(y'), \Lambda(y'')) \leq M \| y' - y'' \|, \quad \text{for all } y' \text{ and } y'' \text{ in } B_b(x_0).
\]

**Definition 2.2.** ([6])

Let $\Omega$ be open subset of $X$, we say that the operator $[.,.; f]$ is $(\nu_0, \nu, p)$-Hölder continuous in $\Omega$ where $\nu_0 \geq 0$, $\nu \geq 0$ and $p \in [0,1]$ if the following inequalities hold

\[
(2.2) \quad \| [x,x^*;f] - [y,u;f] \| \leq \nu_0(\| x - y \|^p + \| x^* - u \|^p),
\]

\[
(2.3) \quad \| [x,y;f] - [u,v;f] \| \leq \nu(\| x - u \|^p + \| y - v \|^p),
\]

for all $x,y,u,v \in \Omega$.

We need the following fixed point theorems.

**Lemma 2.3.** (see [9]) Let $(Z, \| . \|)$ be a Banach space, let $\phi$ a set-valued map from $Z$ into the closed subsets of $Z$, let $\eta_0 \in Z$ and let $r$ and $\lambda$ be such that $0 \leq \lambda < 1$ and

(a) $\text{dist}(\eta_0, \phi(\eta_0)) \leq r(1 - \lambda),$

(b) $e(\phi(x_1) \cap B_r(\eta_0), \phi(x_2)) \leq \lambda \| x_1 - x_2 \|, \; \forall x_1, x_2 \in B_r(\eta_0),$

then $\phi$ has a fixed-point in $B_r(\eta_0)$. That is, there exists $x \in B_r(\eta_0)$ such that $x \in \phi(x)$. If $\phi$ is single-valued, then $x$ is the unique fixed point of $\phi$ in $B_r(\eta_0)$.

We suppose that, for every distinct points $x$ and $y$ in a open neighborhood $V$ of $x^*$, there exists a first order divided difference of $f$ at these points. We will make the following assumptions:
For $i = 1, 2$; the function $g_i$ is $\alpha_i$–center–Lipschitz from $V$ into $V$, with $g_i(x^*) = x^*$, and $\alpha_i \in [0, 1)$. That is

\[
(2.4) ||g_1(x) - g_1(x^*)|| \leq \alpha_1 ||x - x^*|| \quad \text{and} \quad ||g_2(x) - g_2(x^*)|| \leq \alpha_2 ||x - x^*||, 
\]

for all $x \in V$;

\[ (H) \quad [[.,.; f]] \text{ is } (\nu_0, \nu, p)-\text{Hölder continuous in } V. \]

\[ (H) \quad \text{The set–valued map } (f(x^*) + G)^{-1} \text{ is } M-\text{pseudo–Lipschitz around } (0, x^*). \]

\[ (H) \quad \text{For all } x, y \in V, \text{ we have } ||[x, y; f]|| \leq d \text{ with } M d < 1, \text{ and } ||f(x) - f(x^*)|| \leq d_0 ||x - x^*||. \]

Before stating the main result on this study, we need to introduce some notations. First, for $k \in \mathbb{N}$ and $(y_k)$, $(x_k)$ defined in (1.2), let us define the set–valued mappings $Q, \psi_k, \phi_k : XX$ by the following

\[
(2.5) \quad Q(.) := f(x^*) + G(.); \quad \psi_k(.) := Q^{-1}(Z_k(.)); \quad \phi_k(.) := Q^{-1}(W_k(.))
\]

where $Z_k$ and $W_k$ are defined from $X$ to $X$ by

\[
(2.6) \quad Z_k(x) := f(x^*) - f(y_k) - [g_1(x_k), g_2(x_k); f](x - y_k) \\
W_k(x) := f(x^*) - f(x_k) - [g_1(x_k), g_2(x_k); f](x - x_k)
\]

3. Local convergence analysis for method (1.2)

We show the main local convergence result for method (1.2):

**Theorem 3.1.** We suppose that assumptions $(H)$–$(H \exists)$ are satisfied. For every constant $C > C_0 = \frac{M \nu_0 (1 + \alpha_1^p + \alpha_2^p)}{1 - M d}$, there exist $\delta > 0$ such that for every starting point $x_0$ in $B_\delta(x^*)$ ($x_0$ and $x^*$ distinct), and a sequence $(x_k)$ defined by (1.2) which satisfies

\[
(3.1) \quad ||x_{k+1} - x^*|| \leq C ||x_k - x^*||^{p+1}.
\]

The proof of Theorem 3.1 is by induction on $k$. We need to give two lemmas. In the first, we prove the existence of starting point $y_0$ for $x_0$ in $V$. In the second, we state a result which the starting point $(x_0, y_0)$.

Let us mention that $y_0$ and $x_1$ are a fixed points of $\phi_0$ and $\psi_0$ respectively if and only if $0 \in f(x_0) + [g_1(x_0), g_2(x_0); f](y_0 - x_0) + G(y_0)$ and $0 \in f(y_0) + [g_1(x_0), g_2(x_0); f](x_1 - y_0) + G(x_1)$ respectively.
Proposition 3.2. Under the assumptions of Theorem 3.1, there exists $\delta > 0$ such that for every starting point $x_0$ in $B_\delta(x^*)$ ($x_0$ and $x^*$ distincts), the set–valued map $\phi_0$ has a fixed point $y_0$ in $B_\delta(x^*)$, and satisfying

\begin{equation}
\| y_0 - x^* \| \leq C \| x_0 - x^* \|^{p+1}.
\end{equation}

Proof of the Proposition 3.2.

By hypothesis ($\mathcal{H} \in \mathcal{H}$) there exist positive numbers $M$, $a$ and $b$ such that

\begin{equation}
e(Q^{-1}(y') \cap B_\delta(x^*), Q^{-1}(y'')) \leq M \| y' - y'' \|, \forall y', y'' \in B_\delta(0).
\end{equation}

Fix $\delta > 0$ such that

\begin{equation}
\delta < \delta_0 = \min \left\{ a ; \frac{b}{\sqrt{4 \nu \left( [1 + \alpha_1]^p + ([1 + \alpha_2]^p \right) \frac{1}{\sqrt{C}} ; \frac{b}{2 d_0} \right)}
\end{equation}

The main idea of the proof of Proposition 3.2 is to show that both assertions (a) and (b) of Lemma 2.3 hold; where $\eta_0 := x^*$, $\phi$ is the function $\phi_0$ defined in (2.5) and where $r$ and $\lambda$ are numbers to be set. According to the definition of the excess $e$, we have

\begin{equation}
\text{dist} (x^*, \phi_0(x^*)) \leq e \left( Q^{-1}(0) \cap B_\delta(x^*), \phi_0(x^*) \right).
\end{equation}

Moreover, for all point $x_0$ in $B_\delta(x^*)$ ($x_0$ and $x^*$ distincts) we have

\begin{equation}
\| W_0(x^*) \| = \| f(x^*) - f(x_0) - [g_1(x_0), g_2(x_0); f](x^* - x_0) \|.
\end{equation}

Note that for $x \in B_\delta(x^*)$ we get (since $\alpha_i \in [0, 1])$

\begin{equation}
\| g_i(x) - x^* \| \leq \| g_i(x) - g_i(x^*) \| \leq \| x - x^* \| \leq \delta,
\end{equation}

which implies that $g_i(x) \in B_\delta(x^*)$.

In view of assumptions ($\mathcal{H}l$–$\mathcal{H}\infty$) we obtain

\begin{equation}
\| W_0(x^*) \| = \| \left( [x_0, x^*; f] - [g_1(x_0), g_2(x_0); f] \right)(x^* - x_0) \|
\leq \nu_0 \left( \| x_0 - g_1(x_0) \|^{p} + \| x^* - g_2(x_0) \|^{p} \right) \| x^* - x_0 \|
\leq \nu_0 \left( [1 + \alpha_1]^p + \alpha_2^p \right) \| x^* - x_0 \|^{p+1}
\end{equation}
Then (3.4) yields, $W_0(x^*) \in B_0(0)$.

Using (3.3) we have

$$e \left( Q^{-1}(0) \cap B_\delta(x^*), \phi_0(x^*) \right) = e \left( Q^{-1}(0) \cap B_\delta(x^*), Q^{-1}[W_0(x^*)] \right)$$

$$\leq M \nu_0 \left( [1 + \alpha_1]^p + \alpha_2^p \right) \| x^* - x_0 \|^{p+1}$$

(3.7)

By the inequality (3.5), we get

$$\text{dist} \left( x^*, \phi_0(x^*) \right) \leq M \nu_0 \left( [1 + \alpha_1]^p + \alpha_2^p \right) \| x^* - x_0 \|^{p+1}.$$ (3.8)

Since $C(1 - M d) > M \nu_0 \left( [1 + \alpha_1]^p + \alpha_2^p \right)$, there exists $\lambda \in [M d, 1[$ such that $C(1 - \lambda) \geq M \nu_0 \left( [1 + \alpha_1]^p + \alpha_2^p \right)$ and

$$\text{dist} \left( x^*, \phi_0(x^*) \right) \leq C \left( 1 - \lambda \right) \| x_0 - x^* \|^{p+1}.$$ (3.9)

By setting $r := r_0 = C \| x_0 - x^* \|^{p+1}$ we can deduce from the inequality (3.9) that the assertion (a) in Lemma 2.3 is satisfied.

Now, we show that condition (b) of Lemma 2.3 is satisfied. By (3.4) we have $r_0 \leq \delta \leq \alpha$ and moreover for $x \in B_\delta(x^*)$ we have

$$\| W_0(x) \| = \| f(x^*) - f(x_0) - g_1(x_0), g_2(x_0); f \| (x - x_0) \|$$

$$\leq \| f(x^*) - f(x) \| + \| f(x) - f(x_0) - g_1(x_0), g_2(x_0); f \| (x - x_0) \|$$

$$\leq \| f(x^*) - f(x) \| + \| [x_0, x; f] - [g_1(x_0), g_2(x_0); f] \| (x - x_0) \|$$

(3.10)

Using assumptions ($\mathcal{H}$t)–($\mathcal{H}$∞), and ($\mathcal{H}$∋), we get

$$\| W_0(x) \| \leq d_0 \| x^* - x \| + \nu \left( \| x_0 - g_1(x_0) \|^p + \| x - g_2(x_0) \|^p \right) \| x - x_0 \|$$

$$\leq d_0 \| x^* - x \| + \nu \left( \| x_0 - x^* \| + \| x^* - g_1(x_0) \|^{p+1} \right)$$

$$\leq d_0 \delta + \nu ([1 + \alpha_1]^p + ([1 + \alpha_2]^p \delta^p (2\delta))$$

$$= d_0 \delta + 2 \nu ([1 + \alpha_1]^p + ([1 + \alpha_2]^p \delta^{p+1})$$

(3.11)

Then by (3.4) we deduce that for all $x \in B_\delta(x^*)$ we have $W_0(x) \in B_0(0)$. Then it follows that for all $x', x'' \in B_{r_0}(x^*)$, we have
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\[ e(\psi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) \leq e(\phi_0(x') \cap B_\delta(x^*), \phi_0(x'')) , \]

which yields by (3.3)

\[
\begin{align*}
 e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) & \leq M \| W_0(x') - W_0(x'') \| \\
 & \leq M \| [g_1(x_0), g_2(x_0); f] \| \| x'' - x' \| 
\end{align*}
\]

(3.12)

Using (H3) and the fact that \( \lambda \geq M d \), we obtain

\[
(3.13) \quad e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) \leq M d \| x'' - x' \| \leq \lambda \| x'' - x' \|
\]

and thus condition (b) of Lemma 2.3 is satisfied. Since both conditions of Lemma 2.3 are fulfilled, we can deduce the existence of a fixed point \( y_0 \in B_{r_0}(x^*) \) for the map \( \phi_0 \). This finishes the proof of Proposition 3.2.

**Proposition 3.3.** Under the assumptions of Theorem 3.1, there exist \( \delta > 0 \) such that for every starting point \( x_0 \) in \( B_\delta(x^*) \) and \( y_0 \) given by Proposition 3.2 (\( x_0 \) and \( x^* \) distincts), and the set—valued map \( \psi_0 \) has a fixed point \( x_1 \) in \( B_\delta(x^*) \) satisfying

\[
(3.14) \quad \| x_1 - x^* \| \leq C \| x_0 - x^* \|^{p+1} .
\]

**Idea of the proof of Proposition 3.3.**

The proof of Proposition 3.3 is the same one as that of Proposition 3.2. The choice of \( \delta \) is the same one given by (3.4).

The inequality (3.5) is valid if we replace \( \phi_0 \) by \( \psi_0 \).

Moreover, for all point \( x_0 \) in \( B_\delta(x^*) \) (\( x_0 \) and \( x^* \) distincts), we have

\[
\| Z_0(x^*) \| = \| f(x^*) - f(y_0) - [g_1(x_0), g_2(x_0); f](x^* - y_0) \| .
\]

In view of assumptions (H1)–(H\( \infty \)) we get

\[
(3.15) \quad \| Z_0(x^*) \| = \| [y_0, x^*; f] - [g_1(x_0), g_2(x_0); f] \| (x^* - y_0) \| \\
\leq \| [y_0, x^*; f] - [g_1(x_0), g_2(x_0); f] \| \| x^* - y_0 \| \\
\leq \nu_0 \| y_0 - g_1(x_0) \|^p + \| x^* - g_2(x_0) \|^p \| x^* - y_0 \|
\]

By Proposition 3.2 and (3.4) we have
\[ \| Z_0(x^*) \| \leq C \nu_0 \left( (C \| x_0 - x^* \|^{p+1} + \alpha_1 \| x_0 - x^* \|^{p}) \| x^* - x_0 \|^{p+1} \right. \]
\[ \leq \left. \nu_0 \left( [1 + \alpha_1]^p + \alpha_2^p \right) \| x^* - x_0 \|^{p+1} \right. \]
\[ (3.16) \]

Then (3.4) yields, \( Z_0(x^*) \in B_{0}(0) \).

Setting \( r := r_0 = C \| x_0 - x^* \|^{p+1} \), we can deduce from the assertion (a) in Lemma 2.3 is satisfied.

By (3.4) we have
\[ \| Z_0(x) \| = \| f(x^*) - f(y_0) - [g_1(x_0), g_2(x_0); f](x - y_0) \| \]
\[ \leq \| f(x^*) - f(x) \| + \| f(x) - f(y_0) - [g_1(x_0), g_2(x_0); f](x - y_0) \| \]
\[ \leq \| f(x^*) - f(x) \| + \| y_0, x; f \| - [g_1(x_0), g_2(x_0); f] \| \| x - y_0 \| \]
\[ (3.17) \]

Using the assumptions (\(\mathcal{H}l\))- (\(\mathcal{H}\infty\)) and (\(\mathcal{H}\exists\)), Proposition 3.2 and (3.4) we obtain
\[ \| Z_0(x) \| \leq d_0 \delta + 2 \nu \left( [1 + \alpha_1]^p + [1 + \alpha_2]^p \right) \delta^{p+1} \]
\[ (3.18) \]

A slight change in the end of proof of Proposition 3.2 shows that the condition (b) of Lemma 2.3 is satisfied. The existence of a fixed point \( x_1 \in B_{0}(x^*) \) for the map \( \psi_0 \) is ensured. This finishes the proof of Proposition 3.3.

**Proof of Theorem 3.1.**

Keeping \( \eta_0 = x^* \) and setting \( r := r_k = C \| x^* - x_k \|^{p+1} \), the application of Proposition 3.2 and Proposition 3.3 to the map \( \phi_k \) and \( \psi_k \) respectively gives the existence of a fixed points \( y_k \) and \( x_{k+1} \) for \( \phi_k \) and \( \psi_k \) respectively which is an elements of \( B_{r_k}(x^*) \). This last fact implies the inequality (3.1), which is the desired conclusion.

**Remark 3.4.** The sequence \( (y_n) \) given by algorithm (1.2) is also super–linearly convergent to a solution \( x^* \) of (1.1).

**Remark 3.5.** In order for us to compare our results with corresponding ones in [13], let us introduce assumptions:

(\(\mathcal{H}l\)' For \( i = 1, 2 \); there exist parameters \( \alpha_3, \alpha_4 \in [0, 1) \) such that
\[ (3.19) \]
\[ \| g_1(x) - g_1(y) \| \leq \alpha_3 \| x - y \|, \]
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\[ \| g_2(x) - g_2(y) \| \leq \alpha_4 \| x - y \|, \]
for all \( x, y \in V \),

and

\[ g_i(x^*) = x^*. \]

\((\mathcal{H}_\infty)\) \(\nu, p\)–Hölder continuous in \( V \).

\((\mathcal{H}_\exists)\) For all \( x, y \in V \), we have \( ||[x, y; f]|| \leq d \), and \( M d < 1 \).

Using \((\mathcal{H}_0)^', (\mathcal{H}_1)^', (\mathcal{H}_2), (\mathcal{H}_3)^'\), similar result was shown in [13]. Let us define

\[ C'_0 = \frac{M \nu [(1 + \alpha_3)^2 + \alpha_4^2]}{1 - Md}, \]

and

\[ \delta'_0 = \min \left\{ a ; \frac{b}{\sqrt[\nu]{\frac{4 \nu ((1 + \alpha_3)^p + (1 + \alpha_4)^p)}{\nu C}}} ; \frac{1}{\sqrt{\nu C}} ; \frac{b}{2d} \right\}. \]

Assumption \((\mathcal{H}_0)\) is weaker than \((\mathcal{H}_0)^'\). Note also that in general

\[ \nu_0 \leq \nu, \]

\[ d_0 \leq d, \]

\[ \alpha_1 \leq \alpha_3, \]

and

\[ \alpha_2 \leq \alpha_4 \]

hold, and \( \frac{\nu}{\nu_0} \), \( \frac{d}{d_0} \), \( \frac{\alpha_3}{\alpha_1} \) and \( \frac{\alpha_4}{\alpha_2} \) can be arbitrarily large [4], [6]. Hence, if strict inequality hold in any of (3.23)–(3.26) and \( \delta_0 \) is not equal to \( a \) or \( \frac{1}{\sqrt{\nu C}} \), then we conclude:

\[ C_0 \leq C'_0, \]

and

\[ \delta'_0 \leq \delta_0, \]
which justify the advantages of our analysis over the corresponding ones in [13] 
mentioned in the introduction. Similar improvements can immediately follow the 
same way with the works in [9]—[21].

**Application 3.6.** (see [18]) 
Let $K$ be a convex set in $\mathbb{R}^n$, $P$ is a topological space and $\varphi$ is a function from 
$P \times K$ to $\mathbb{R}^n$, the ”perturbed” variational inequality problem consists of seeking $k_0$ 
in $K$ such that

\[
(\varphi(p,k_0);k-k_0) \geq 0
\]

(3.29) 
where $(\cdot,\cdot)$ is the usual scalar product on $\mathbb{R}^n$ and $p$ is fixed parameter in $P$. Let 
$I_K$ be a convex indicator function of $K$ and $\partial$ denotes the subdifferential operator. 
Then the problem (3.29) is equivalent to problem

\[
0 \in \varphi(p,k_0) + \mathcal{H}(k_0)
\]

(3.30) 
with $\mathcal{H} = \partial I_K$. $\mathcal{H}$ is also called the normal cone of $K$. The ”perturbed” variational 
inequality problem (3.29) is equivalent to (3.30) which is a generalized equation in 
the form (1.1). Consequently, we can approximate the solution $k_0$ of (3.29) using 
our method (1.2).

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