ON \( \tilde{g} \)-HOMEOOMORPHISMS IN TOPOLOGICAL SPACES

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Abstract

In this paper, we first introduce a new class of closed map called \( \tilde{g} \)-closed map. Moreover, we introduce a new class of homeomorphism called \( \tilde{g} \)-homeomorphism, which are weaker than homeomorphism. We prove that gc-homeomorphism and \( \tilde{g} \)-homeomorphism are independent. We also introduce \( \tilde{g}^* \)-homeomorphisms and prove that the set of all \( \tilde{g}^* \)-homeomorphisms forms a group under the operation of composition of maps.

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1. Introduction

The notion homeomorphism plays a very important role in topology. By definition, a homeomorphism between two topological spaces $X$ and $Y$ is a bijective map $f : X \rightarrow Y$ when both $f$ and $f^{-1}$ are continuous. It is well known that as Jänich [5], p.13 says correctly: homeomorphisms play the same role in topology that linear isomorphisms play in linear algebra, or that biholomorphic maps play in function theory, or group isomorphisms in group theory, or isometries in Riemannian geometry. In the course of generalizations of the notion of homeomorphism, Maki et al. [9] introduced $g$-homeomorphisms and $gc$-homeomorphisms in topological spaces. Recently, Devi et al. [2] studied semi-generalized homeomorphisms and generalized semi-homeomorphisms.

In this paper, we first introduce $\tilde{g}$-closed maps in topological spaces and then we introduce and study $\tilde{g}$-homeomorphisms, which are weaker than homeomorphisms. We prove that $gc$-homeomorphism and $\tilde{g}$-homeomorphism are independent. We also introduce $\tilde{g}^*$-homeomorphisms. It turns out that the set of all $\tilde{g}^*$-homeomorphisms forms a group under the operation composition of functions.

2. Preliminaries

Throughout this paper $(X, \tau)$ and $(Y, \sigma)$ represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of a space $(X, \tau)$, $\text{cl}(A)$, $\text{int}(A)$ and $A^c$ denote the closure of $A$, the interior of $A$ and the complement of $A$ in $X$, respectively.

We recall the following definitions and some results, which are used in the sequel.

**Definition 1.** A subset $A$ of a space $(X, \tau)$ is called:
(i) semi-open [6] if $A \subset \text{cl}(\text{int}(A))$,
(ii) $\alpha$-open [11] if $A \subset \text{int}(\text{cl}(\text{int}(A)))$,
(iii) regular open [17] if $A = \text{int}(	ext{cl}(A))$,
(iv) pre-closed [10] if $\text{cl}(\text{int}(A)) \subset A$.

The semi-closure [1] (resp. $\alpha$-closure [11]) of a subset $A$ of $X$, denoted by $\text{scl}_X(A)$ (resp. $\text{acl}_X(A)$) briefly $\text{scl}(A)$, (resp. $\text{acl}(A)$) is defined to be the intersection of all semi-closed (resp. $\alpha$-closed) sets containing $A$. 
Definition 2. A subset $A$ of a space $(X, \tau)$ is called a:
(i) generalized closed (briefly $g$-closed) set [7] if $\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is open in $(X, \tau)$.
(ii) generalized $\alpha$-closed (briefly $g\alpha$-closed) set [8] if $\text{ocl}(A) \subset U$ whenever $A \subset U$ and $U$ is $\alpha$-open in $(X, \tau)$.
(iii) $\tilde{g}$-closed set [19] if $\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is semi-open in $(X, \tau)$.
(iv) $^*g$-closed set [20] if $\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is $\tilde{g}$-open in $(X, \tau)$.
(v) $^#g$-semi-closed (briefly $^#gs$-closed) set [21] if $\text{scl}(A) \subset U$ whenever $A \subset U$ and $U$ is $^*g$-open in $(X, \tau)$.
(vi) $\tilde{g}$-closed set [4] if $\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is $^#gs$-open in $(X, \tau)$.

Where $\tilde{g}$-open (resp. $g$-open, $g\alpha$-open, $rg$-open, $^*g$-open, $^#gs$-open, $\tilde{g}$-open), are defined as the complement of $\tilde{g}$-closed (resp. $g$-closed, $g\alpha$-closed, $rg$-closed, $^*g$-closed, $^#gs$-closed, $\tilde{g}$-closed).

Definition 3. A function $f : (X, \tau) \to (Y, \sigma)$ is called:
(i) $g$-continuous [18] if $f^{-1}(V)$ is $g$-closed in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.
(ii) $\tilde{g}$-continuous [19] if $f^{-1}(V)$ is $\tilde{g}$-closed in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.
(iii) $\tilde{g}$-continuous [15] if $f^{-1}(V)$ is $\tilde{g}$-closed in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.
(iv) $\tilde{g}$-irresolute [13] if $f^{-1}(V)$ is $\tilde{g}$-closed in $(X, \tau)$ for every $\tilde{g}$-closed set $V$ in $(Y, \sigma)$.
(v) $gg$-irresolute [18] if $f^{-1}(V)$ is $g$-closed in $(X, \tau)$ for each $g$-closed set $V$ of $(Y, \sigma)$.
(vi) $^#gs$-irresolute [21] if $f^{-1}(V)$ is $^#gs$-closed in $(X, \tau)$ for each $^#gs$-closed set $V$ of $(Y, \sigma)$.
(vii) strongly $\tilde{g}$-continuous [13] if the inverse image of every $\tilde{g}$-open set in $(Y, \sigma)$ is open in $(X, \tau)$.

Definition 4. A function $f : (X, \tau) \to (Y, \sigma)$ is called:
(i) $g$-open [18] if $f(V)$ is $g$-open in $(Y, \sigma)$ for every $g$-open set $V$ in $(X, \tau)$.
(ii) $\tilde{g}$-open [19] if $f(V)$ is $\tilde{g}$-open in $(Y, \sigma)$ for every $\tilde{g}$-open set $V$ in $(X, \tau)$.

Definition 5. A bijective function $f : (X, \tau) \to (Y, \sigma)$ is called a:
(i) generalized homeomorphism (briefly $g$-homeomorphism) [9] if $f$ is both...
g-continuous and g-open,
(ii) gc-homeomorphism \([9]\) if both \(f\) and \(f^{-1}\) are gc-irresolute maps,
(iii) \(\tilde{g}\)-homeomorphism \([19]\) if \(f\) is both \(\tilde{g}\)-continuous and \(\tilde{g}\)-open.

**Definition 6.** \([14]\) Let \((X, \tau)\) be a topological space and \(E \subset X\). We define the \(\tilde{g}\)-closure of \(E\) (briefly \(\tilde{g}\)-cl(E)) to be the intersection of all \(\tilde{g}\)-closed sets containing \(E\), i.e.,

\[
\tilde{g}\text{-cl}(E) = \bigcap\{A : E \subset A \text{ and } A \in \tilde{g}GC(X, \tau)\}.
\]

**Proposition 2.1.** \([13]\) If \(f : (X, \tau) \rightarrow (Y, \sigma)\) is \(\tilde{g}\)-irresolute, then it is \(\tilde{g}\)-continuous.

**Proposition 2.2.** \([14]\) Let \((X, \tau)\) be a topological space and \(E \subset X\). The following properties are satisfied:
(i) \(\tilde{g}\)-cl(E)) is the smallest \(\tilde{g}\)-closed set containing \(E\) and
(ii) \(E\) is \(\tilde{g}\)-closed if and only if \(\tilde{g}\)-cl(E)) = \(E\).

**Proposition 2.3.** \([14]\) For any two subsets \(A\) and \(B\) of \((X, \tau)\),
(i) If \(A \subset B\), then \(\tilde{g}\)-cl(A)) \(\subset\) \(\tilde{g}\)-cl(B))
(ii) \(\tilde{g}\)-cl(A \cap B))) \(\subset\) \(\tilde{g}\)-cl(A)) \(\cap\) \(\tilde{g}\)-cl(B)).

**Proposition 2.4.** \([4]\) Every \(\tilde{g}\)-closed set is \(g\alpha\)-closed and hence pre-closed.

**Proof.** Let \(A\) be \(\tilde{g}\)-closed in \((X, \tau)\) and \(U\) be any \(\alpha\)-open set containing \(A\). Since every \(\alpha\)-open set is \(#gs\)-open \([21]\) and since \(\alpha cl(A) \subset cl(A)\) for every subset \(A\) of \(X\), we have by hypothesis, \(\alpha cl(A) \subset cl(A) \subset U\) and so \(A\) is \(g\alpha\)-closed in \((X, \tau)\).

In \([8]\), it has been proved that every \(g\alpha\)-closed set is pre-closed. Therefore, every \(\tilde{g}\)-closed set is pre-closed.

**Theorem 2.5.** \([4]\) Suppose that \(B \subset A \subset X, B\) is a \(\tilde{g}\)-closed set relative to \(A\) and that \(A\) is open and \(\tilde{g}\)-closed in \((X, \tau)\). Then \(B\) is \(\tilde{g}\)-closed in \((X, \tau)\).

**Corollary 2.6.** \([4]\) If \(A\) is a \(\tilde{g}\)-closed set and \(F\) is a closed set, then \(A \cap F\) is a \(\tilde{g}\)-closed set.

**Theorem 2.7.** \([4]\) A set \(A\) is \(\tilde{g}\)-open in \((X, \tau)\) if and only if \(F \subset int(A)\) whenever \(F\) is \(#gs\)-closed in \((X, \tau)\) and \(F \subset A\).

**Definition 7.** \([14]\) Let \((X, \tau)\) be a topological space and \(E \subset X\). We define the \(\tilde{g}\)-interior of \(E\) (briefly \(\tilde{g}\)-int(E)) to be the union of all \(\tilde{g}\)-open sets contained in \(E\).
Lemma 2.8. [14] For any $E \subseteq X$, $\text{int}(E) \subseteq \tilde{g} \text{-int}(E) \subseteq E$.

Proof. Since every open set is $\tilde{g}$-open, the proof follows immediately.

Definition 8. A topological space $(X, \tau)$ is a,
(i) $T_{1/2}$ space [7] if every $g$-closed subset of $(X, \tau)$ is closed in $(X, \tau)$,
(ii) $T_{2}$ space [16] if every $\tilde{g}$-closed subset of $(X, \tau)$ is closed in $(X, \tau)$,
(iii) semi-$T_{1/2}$ space [19] if every $\tilde{g}$-closed subset of $(X, \tau)$ is closed in $(X, \tau)$.

3. $\tilde{g}$-closed maps

Malghan [12] introduced the concept of generalized closed maps in topological spaces. In this section, we introduce $\tilde{g}$-closed maps, $\tilde{g}$-open maps, $\tilde{g}^*$-closed maps, $\tilde{g}^*$-open maps and obtain certain characterizations of these maps.

Definition 9. A map $f : (X, \tau) \to (Y, \sigma)$ is said to be $\tilde{g}$-closed if the image of every closed set in $(X, \tau)$ is $\tilde{g}$-closed in $(Y, \sigma)$.

Example 3.1. (a) Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$ and $\sigma = \{\emptyset, Y, \{b\}\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then $f$ is a $\tilde{g}$-closed map.

(b) Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\emptyset, Y, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then $f$ is not a $\tilde{g}$-closed map.

Proposition 3.2. A mapping $f : (X, \tau) \to (Y, \sigma)$ is $\tilde{g}$-closed if and only if $\tilde{g}\text{-cl}(f(A)) \subseteq f(\text{cl}(A))$ for every subset $A$ of $(X, \tau)$.

Proof. Suppose that $f$ is $\tilde{g}$-closed and $A \subseteq X$. Then $f(\text{cl}(A))$ is $\tilde{g}$-closed in $(Y, \sigma)$. We have $f(A) \subseteq f(\text{cl}(A))$ and by Propositions 2.2 and 2.3, $\tilde{g}\text{-cl}(f(A)) \subseteq \tilde{g}\text{-cl}(f(\text{cl}(A))) = f(\text{cl}(A))$.

Conversely, let $A$ be any closed set in $(X, \tau)$. By hypothesis and Proposition 2.2, we have $A = \text{cl}(A)$ and so $f(A) = f(\text{cl}(A)) \supseteq \tilde{g}\text{-cl}(f(A))$. Therefore, $f(A) = \tilde{g}\text{-cl}(f(A))$. i.e., $f(A)$ is $\tilde{g}$-closed and hence $f$ is $\tilde{g}$-closed.

Proposition 3.3. If $f : (X, \tau) \to (Y, \sigma)$ is a $\tilde{g}$-closed mapping, then for each subset $A$ of $(X, \tau)$, $\text{cl}(\text{int}(f(A))) \subseteq f(\text{cl}(A))$.

Proof. Let $f$ be a $\tilde{g}$-closed map and $A \subseteq X$. Then since $\text{cl}(A)$ is a closed set in $(X, \tau)$, we have $f(\text{cl}(A))$ is $\tilde{g}$-closed and hence pre-closed by Proposition 2.4. Therefore, $\text{cl}(\text{int}(f(\text{cl}(A)))) \subseteq f(\text{cl}(A))$. i.e., $\text{cl}(\text{int}(f(A))) \subseteq
Let Example 3.7. for each subset is closed set. For the converse, let \( g\)-closed subset of \((Y,\sigma)\) be the identity map. Then for each subset \( A \subset X\), we have \( cl(int(f(A))) \subset f(cl(A))\), but \( f\) is not a \( g\)-closed map.

**Theorem 3.5.** A map \( f : (X,\tau) \to (Y,\sigma)\) is \( g\)-closed if and only if for each subset \( S \subset (Y,\sigma)\) and for each open set \( U\) containing \( f^{-1}(S)\) there is a \( g\)-open set \( V\) of \((Y,\sigma)\) such that \( S \subset V\) and \( f^{-1}(V) \subset U\).

**Proof.** Suppose that \( f\) is a \( g\)-closed map. Let \( S \subset Y\) and \( U\) be an open subset of \((X,\tau)\) such that \( f^{-1}(S) \subset U\). Then \( V = (f(U^c))^c\) is a \( g\)-open set containing \( S\) such that \( f^{-1}(V) \subset U\).

For the converse, let \( S\) be a closed set of \((X,\tau)\). Then \( f^{-1}((f(S))^c) \subset S^c\) and \( S^c\) is open. By assumption, there exists a \( g\)-open set \( V\) of \((Y,\sigma)\) such that \((f(S))^c \subset V\) and \( f^{-1}(V) \subset S^c\) and so \( S \subset (f^{-1}(V))^c\). Hence \( V^c \subset f(S) \subset f((f^{-1}(V))^c) \subset V^c\) which implies \( f(S) = V^c\). Since \( V^c\) is \( g\)-closed, \( f(S)\) is \( g\)-closed and therefore \( f\) is \( g\)-closed.

**Proposition 3.6.** If \( f : (X,\tau) \to (Y,\sigma)\) is \#gs-irresolute \( g\)-closed and \( A\) is a \( g\)-closed subset of \((X,\tau)\), then \( f(A)\) is \( g\)-closed.

**Proof.** Let \( U\) be a \#gs-open set in \((Y,\sigma)\) such that \( f(A) \subset U\). Since \( f\) is \#gs-irresolute, \( f^{-1}(U)\) is a \#gs-open set containing \( A\). Hence \( cl(A) \subset f^{-1}(U)\) as \( A\) is \( g\)-closed in \((X,\tau)\). Since \( f\) is \( g\)-closed, \( f(cl(A))\) is a \( g\)-closed set contained in the \#gs-open set \( U\), which implies that \( cl(f(cl(A))) \subset U\) and hence \( cl(f(A)) \subset U\). Therefore, \( f(A)\) is a \( g\)-closed set.

The following example shows that the composition of two \( g\)-closed maps need not be \( g\)-closed.

**Example 3.7.** Let \( X = Y = Z = \{a,b,c\}\), \( \tau = \{\emptyset,X,\{a\},\{b,c\}\}\), \( \sigma = \{\emptyset,Y,\{a,c\}\}\) and \( \gamma = \{\emptyset,Z,\{b\},\{a,c\}\}\). Define a map \( f : (X,\tau) \to (Y,\sigma)\) by \( f(a) = f(b) = b\) and \( f(c) = a\) and a map \( g : (Y,\sigma) \to (Z,\gamma)\) by \( g(a) = c\), \( g(b) = b\) and \( g(c) = a\). Then both \( f\) and \( g\) are \( g\)-closed maps but their composition \( g \circ f : (X,\tau) \to (Z,\gamma)\) is not a \( g\)-closed map, since for the closed set \( \{b,c\}\) in \((X,\tau)\), \((g \circ f)(\{b,c\}) = \{a,b\}\), which is not a \( g\)-closed set in \((Z,\gamma)\).
Corollary 3.8. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\tilde{g}$-closed map and $g : (Y, \sigma) \to (Z, \gamma)$ be $\tilde{g}$-closed and $\#gs$-irresolute map, then their composition $g \circ f : (X, \tau) \to (Z, \gamma)$ is $\tilde{g}$-closed.

Proof. Let $A$ be a closed set of $(X, \tau)$. Then by hypothesis $f(A)$ is a $\tilde{g}$-closed set in $(Y, \sigma)$. Since $g$ is both $\tilde{g}$-closed and $\#gs$-irresolute by Proposition 3.6, $g(f(A)) = (g \circ f)(A)$ is $\tilde{g}$-closed in $(Z, \gamma)$ and therefore $g \circ f$ is $\tilde{g}$-closed.

Proposition 3.9. If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \gamma)$ are $\tilde{g}$-closed maps and $(Y, \sigma)$ is a $T_\tilde{g}$ space, then their composition $g \circ f : (X, \tau) \to (Z, \gamma)$ is a $\tilde{g}$-closed map.

Proof. Let $A$ be a closed set of $(X, \tau)$. Then by assumption $f(A)$ is $\tilde{g}$-closed in $(Y, \sigma)$. Since $(Y, \sigma)$ is a $T_\tilde{g}$ space, $f(A)$ is closed in $(Y, \sigma)$ and again by assumption $g(f(A))$ is $\tilde{g}$-closed in $(Z, \gamma)$. i.e., $(g \circ f)(A)$ is $\tilde{g}$-closed in $(Z, \gamma)$ and so $g \circ f$ is $\tilde{g}$-closed.

Proposition 3.10. If $f : (X, \tau) \to (Y, \sigma)$ is $\tilde{g}$-closed, $g : (Y, \sigma) \to (Z, \gamma)$ is $g$-closed and $(Y, \sigma)$ is a $T_\tilde{g}$ space, then their composition $g \circ f : (X, \tau) \to (Z, \gamma)$ is $g$-closed.

Proof. Similar to Proposition 3.9.

Proposition 3.11. Let $f : (X, \tau) \to (Y, \sigma)$ be a closed map and $g : (Y, \sigma) \to (Z, \gamma)$ be a $\tilde{g}$-closed map, then their composition $g \circ f : (X, \tau) \to (Z, \gamma)$ is $\tilde{g}$-closed.

Proof. Similar to Proposition 3.9.

Remark 3.12. If $f : (X, \tau) \to (Y, \sigma)$ is $\tilde{g}$-closed and $g : (Y, \sigma) \to (Z, \gamma)$ is closed, then their composition need not be a $\tilde{g}$-closed map as seen from the following example.

Example 3.13. Let $X = Y = Z = \{a, b, c\}$, $\tau = \emptyset, X, \{a\}, \{b, c\}$, $\sigma = \emptyset, Y, \{a, c\}$ and $\gamma = \emptyset, Z, \{b\}, \{a, c\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = f(b) = b$ and $f(c) = a$ and $g : (Y, \sigma) \to (Z, \gamma)$ be the identity map. Then $f$ is a $\tilde{g}$-closed map and $g$ is a closed map. But their composition $g \circ f : (X, \tau) \to (Z, \gamma)$ is not a $\tilde{g}$-closed map, since for the closed set $\{b, c\}$ in $(X, \tau)$, $(g \circ f)(\{b, c\}) = \{a, b\}$, which is not a $\tilde{g}$-closed set in $(Z, \gamma)$. 
Theorem 3.14. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \gamma)$ be two mappings such that their composition $g \circ f : (X, \tau) \rightarrow (Z, \gamma)$ be a $\tilde{g}$-closed mapping. Then the following statements are true if:

(i) $f$ is continuous and surjective, then $g$ is $\tilde{g}$-closed.

(ii) $g$ is $\tilde{g}$-irresolute and injective, then $f$ is $\tilde{g}$-closed.

(iii) $f$ is $g$-continuous, surjective and $(X, \tau)$ is a $T_{1/2}$ space, then $g$ is $\tilde{g}$-closed.

(iv) $f$ is $\tilde{g}$-continuous, surjective and $(X, \tau)$ is a semi-$T_{1/2}$ space, then $g$ is $\tilde{g}$-closed.

(v) $g$ is strongly $\tilde{g}$-continuous and injective, then $f$ is closed.

Proof. (i) Let $A$ be a closed set of $(Y, \sigma)$. Since $f$ is continuous, $f^{-1}(A)$ is closed in $(X, \tau)$ and since $g \circ f$ is $\tilde{g}$-closed, $(g \circ f)(f^{-1}(A))$ is $\tilde{g}$-closed in $(Z, \gamma)$. i.e., $g(A)$ is $\tilde{g}$-closed in $(Z, \gamma)$, since $f$ is surjective. Therefore, $g$ is a $\tilde{g}$-closed map.

(ii) Let $B$ be a closed set of $(X, \tau)$. Since $g \circ f$ is $\tilde{g}$-closed, $(g \circ f)(B)$ is $\tilde{g}$-closed in $(Z, \gamma)$. Since $g$ is $\tilde{g}$-irresolute, $g^{-1}((g \circ f)(B))$ is $\tilde{g}$-closed in $(Y, \sigma)$, i.e., $f(B)$ is $\tilde{g}$-closed in $(Y, \sigma)$, since $g$ is injective. Thus, $f$ is a $\tilde{g}$-closed map.

(iii) Let $A$ be a closed set of $(Y, \sigma)$. Since $f$ is $g$-continuous, $f^{-1}(A)$ is $g$-closed in $(X, \tau)$. Since $(X, \tau)$ is a $T_{1/2}$ space, $f^{-1}(A)$ is closed in $(X, \tau)$ and so as in (i), $g$ is a $\tilde{g}$-closed map.

(iv) Let $A$ be a closed set of $(Y, \sigma)$. Since $f$ is $\tilde{g}$-continuous, $f^{-1}(A)$ is $\tilde{g}$-closed in $(X, \tau)$. Since $(X, \tau)$ is a semi-$T_{1/2}$ space, $f^{-1}(A)$ is closed in $(X, \tau)$ and so as in (i), $g$ is a $g$-closed map.

(v) Let $D$ be a closed set of $(X, \tau)$. Since $g \circ f$ is $\tilde{g}$-closed, $(g \circ f)(D)$ is $\tilde{g}$-closed in $(Z, \gamma)$. Since $g$ is strongly $\tilde{g}$-continuous, $g^{-1}((g \circ f)(D))$ is closed in $(Y, \sigma)$. i.e., $f(D)$ is closed in $(Y, \sigma)$, since $g$ is injective. Therefore, $f$ is a closed map.

As for the restriction $f_A$ of a map $f : (X, \tau) \rightarrow (Y, \sigma)$ to a subset $A$ of $(X, \tau)$, we have the following:

Theorem 3.15. Let $(X, \tau)$ and $(Y, \sigma)$ be any topological spaces. Then if:

(i) $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\tilde{g}$-closed and $A$ is a closed subset of $(X, \tau)$, then $f_A : (A, \tau_A) \rightarrow (Y, \sigma)$ is $\tilde{g}$-closed.

(ii) $f : (X, \tau) \rightarrow (Y, \sigma)$ is #gs-irresolute and $\tilde{g}$-closed and $A$ is an open subset of $(X, \tau)$, then $f_A : (A, \tau_A) \rightarrow (Y, \sigma)$ is $\tilde{g}$-closed.

(iii) $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\tilde{g}$-closed (resp. closed) and $A = f^{-1}(B)$ for
some closed (resp. $\tilde{g}$-closed) set $B$ of $(Y, \sigma)$, then $f_A : (A, \tau_A) \to (Y, \sigma)$ is $\tilde{g}$-closed.

**Proof.** (i) Let $B$ be a closed set of $A$. Then $B = A \cap F$ for some closed set $F$ of $(X, \tau)$ and so $B$ is closed in $(X, \tau)$. By hypothesis, $f(B)$ is $\tilde{g}$-closed in $(Y, \sigma)$. Therefore, $f(A) = f_A(B)$ is $\tilde{g}$-closed.

(ii) Let $C$ be a closed set of $A$. Then $C$ is $g$-closed in $A$. Since $A$ is both open and $\tilde{g}$-closed, $C$ is $\tilde{g}$-closed, by Theorem 2.5. Since $f$ is both $\#gs$-irresolute and $\tilde{g}$-closed, $f(C)$ is $\tilde{g}$-closed in $(Y, \sigma)$, by Proposition 3.6. Since $f(C) = f_A(C)$, $f_A$ is a $\tilde{g}$-closed map.

(iii) Let $D$ be a closed set of $A$. Then $D = A \cap H$ for some closed set $H$ of $(X, \tau)$. Now $f_A(D) = f(H) = f(A \cap H) = f(f^1(B) \cap H) = B \cap f(H)$. Since $f$ is $\tilde{g}$-closed, $f(H)$ is $\tilde{g}$-closed and so $B \cap f(H)$ is $\tilde{g}$-closed in $(Y, \sigma)$ by Corollary 2.6. Therefore, $f_A$ is a $\tilde{g}$-closed map.

The next theorem shows that normality is preserved under continuous $\tilde{g}$-closed maps.

**Theorem 3.16.** If $f : (X, \tau) \to (Y, \sigma)$ is a continuous, $\tilde{g}$-closed map from a normal space $(X, \tau)$ onto a space $(Y, \sigma)$ then $(Y, \sigma)$ is normal.

**Proof.** It follows from Theorem 1.12 of [12] and the fact that every $\tilde{g}$-closed map is $g$-closed.

Analogous to a $\tilde{g}$-closed map, we define a $\tilde{g}$-open map as follows:

**Definition 10.** A map $f : (X, \tau) \to (Y, \sigma)$ is said to be a $\tilde{g}$-open map if the image $f(A)$ is $\tilde{g}$-open in $(Y, \sigma)$ for each open set $A$ in $(X, \tau)$.

**Proposition 3.17.** For any bijection $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:

(i) $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $\tilde{g}$-continuous,

(ii) $f$ is a $\tilde{g}$-open map and

(iii) $f$ is a $\tilde{g}$-closed map.

**Proof.** (i) $\Rightarrow$ (ii): Let $U$ be an open set of $(X, \tau)$. By assumption $(f^{-1})^{-1}(U) = f(U)$ is $\tilde{g}$-open in $(Y, \sigma)$ and so $f$ is $\tilde{g}$-open.

(ii) $\Rightarrow$ (iii): Let $F$ be a closed set of $(X, \tau)$. Then $F^c$ is open in $(X, \tau)$. By assumption, $f(F^c)$ is $\tilde{g}$-open in $(Y, \sigma)$ i.e., $f(F)^c = \tilde{g}$-open in $(Y, \sigma)$ and therefore $f(F)$ is $\tilde{g}$-closed in $(Y, \sigma)$. Hence $f$ is $\tilde{g}$-closed.

(iii) $\Rightarrow$ (i): Let $F$ be a closed set in $(X, \tau)$. By assumption $f(F)$ is $\tilde{g}$-closed in $(Y, \sigma)$. But $f(F) = (f^{-1})^{-1}(F)$ and therefore $f^{-1}$ is $\tilde{g}$-continuous on $Y$. 

**Definition 11.** Let $x$ be a point of $(X,\tau)$ and $V$ be a subset of $X$. Then $V$ is called a $\bar{g}$-neighbourhood of $x$ in $(X,\tau)$ if there exists a $\bar{g}$-open set $U$ of $(X,\tau)$ such that $x \in U \subset V$.

In the next two theorems, we obtain various characterizations of $\bar{g}$-open maps.

**Theorem 3.18.** Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be a mapping. Then the following statements are equivalent:

(i) $f$ is a $\bar{g}$-open mapping.

(ii) For a subset $A$ of $(X,\tau)$, $f(\text{int}(A)) \subset \bar{g}$-$\text{int}(f(A))$.

(iii) For each $x \in X$ and for each neighborhood $U$ of $x$ in $(X,\tau)$, there exists a $\bar{g}$-neighborhood $W$ of $f(x)$ in $(Y,\sigma)$ such that $W \subset f(U)$.

*Proof.* (i) $\Rightarrow$ (ii): Suppose $f$ is $\bar{g}$-open. Let $A \subset X$. Since $\text{int}(A)$ is open in $(X,\tau)$, $f(\text{int}(A))$ is $\bar{g}$-open in $(Y,\sigma)$. Hence $f(\text{int}(A)) \subset f(A)$ and we have, $f(\text{int}(A)) \subset \bar{g}$-$\text{int}(f(A))$.

(ii) $\Rightarrow$ (iii): Suppose (ii) holds. Let $x \in X$ and $U$ be an arbitrary neighborhood of $x$ in $(X,\tau)$. Then there exists an open set $G$ such that $x \in G \subset U$. By assumption, $f(G) = f(\text{int}(G)) \subset \bar{g}$-$\text{int}(f(G))$. This implies $f(G) = \bar{g}$-$\text{int}(f(G))$. Therefore, $f(G)$ is $\bar{g}$-open in $(Y,\sigma)$. Further, $f(x) \in f(G) \subset f(U)$ and so (iii) holds, by taking $W = f(G)$.

(iii) $\Rightarrow$ (i): Suppose (iii) holds. Let $U$ be any open set in $(X,\tau)$, $x \in U$ and $f(x) = y$. Then $x \in U$ and for each $y \in f(U)$, by assumption there exists a $\bar{g}$-neighborhood $W_y$ of $y$ in $(Y,\sigma)$ such that $W_y \subset f(U)$. Since $W_y$ is a $\bar{g}$-neighborhood of $y$, there exists a $\bar{g}$-open set $V_y$ in $(Y,\sigma)$ such that $y \in V_y \subset W_y$. Therefore, $f(U) = \cup \{V_y : y \in f(U)\}$. Since the arbitrary union of $\bar{g}$-open sets is $\bar{g}$-open, $f(U)$ is a $\bar{g}$-open set of $(Y,\sigma)$. Thus, $f$ is a $\bar{g}$-open mapping.

**Theorem 3.19.** A function $f : (X,\tau) \rightarrow (Y,\sigma)$ is $\bar{g}$-open if and only if for any subset $B$ of $(Y,\sigma)$ and for any closed set $S$ containing $f^{-1}(B)$, there exists a $\bar{g}$-closed set $A$ of $(Y,\sigma)$ containing $B$ such that $f^{-1}(A) \subset S$.

*Proof.* Similar to Theorem 3.5.

**Corollary 3.20.** A function $f : (X,\tau) \rightarrow (Y,\sigma)$ is $\bar{g}$-open if and only if $f^{-1}(\bar{g}$-$\text{cl}(B)) \subset \text{cl}(f^{-1}(B))$ for every subset $B$ of $(Y,\sigma)$.

*Proof.* Suppose that $f$ is $\bar{g}$-open. Then for any $B \subset Y$, $f^{-1}(B) \subset \text{cl}(f^{-1}(B))$. By Theorem 3.19, there exists a $\bar{g}$-closed set $A$ of $(Y,\sigma)$
such that $B \subset A$ and $f^{-1}(A) \subset \text{cl}(f^{-1}(B))$. Therefore, $f^{-1}(\tilde{g}\text{-cl}(B)) \subset f^{-1}(A) \subset \text{cl}(f^{-1}(B))$, since $A$ is a $\tilde{g}$-closed set in $(Y, \sigma)$.

Conversely, let $S$ be any subset of $(Y, \sigma)$ and $F$ be any closed set containing $f^{-1}(S)$. Put $A = \tilde{g}\text{-cl}(S)$. Then $A$ is a $\tilde{g}$-closed set and $S \subset A$. By assumption, $f^{-1}(A) = f^{-1}(\tilde{g}\text{-cl}(S)) \subset \text{cl}(f^{-1}(S)) \subset A$ and therefore by Theorem 3.19, $f$ is $\tilde{g}$-open.

Finally in this section, we define another new class of maps called $\tilde{g}^*$-closed maps which are stronger than $\tilde{g}$-closed maps.

**Definition 12.** A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be a $\tilde{g}^*$-closed map if the image $f(A)$ is $\tilde{g}$-closed in $(Y, \sigma)$ for every $\tilde{g}$-closed set $A$ in $(X, \tau)$.

For instance the map $f$ in Example 3.1 is a $\tilde{g}^*$-closed map

**Remark 3.21.** Since every closed set is a $\tilde{g}$-closed set we have every $\tilde{g}^*$-closed map is a $\tilde{g}$-closed map. The converse is not true in general as seen from the following example.

**Example 3.22.** Let $X = Y = \{a, b, c\}$, $\tau = \emptyset, X, \{a, b\}$, $\sigma = \emptyset, Y, \{a\}, \{a, b\}$ and $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then $f$ is a $\tilde{g}$-closed map but not a $\tilde{g}^*$-closed map, since $\{a, c\}$ is a $\tilde{g}$-closed set in $(X, \tau)$, but its image under $f$ is $\{a, c\}$, which is not $\tilde{g}$-closed in $(Y, \sigma)$.

**Proposition 3.23.** A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\tilde{g}^*$-closed if and only if $\tilde{g}\text{-cl}(f(A)) \subset f(\tilde{g}\text{-cl}(A))$ for every subset $A$ of $(X, \tau)$.

**Proof.** Similar to Proposition 3.2.

Analogous to $\tilde{g}^*$-closed map we can also define $\tilde{g}^*$-open map.

**Proposition 3.24.** For any bijection $f : (X, \tau) \rightarrow (Y, \sigma)$, the following are equivalent:

(i) $f^{-1} : (Y, \sigma) \rightarrow (X, \tau)$ is $\tilde{g}$-irresolute,

(ii) $f$ is a $\tilde{g}^*$-open and

(iii) $f$ is a $\tilde{g}^*$-closed map.

**Proof.** Similar to Proposition 3.17.

**Proposition 3.25.** If $f : (X, \tau) \rightarrow (Y, \sigma)$ is $\#gs$-irresolute and $\tilde{g}$-closed, then it is a $\tilde{g}^*$-closed map.
Theorem 3.27. Let $X$ be a subset of $X$. Then $p \in \overline{g}(A)$ if and only if for any $\overline{g}$-neighborhood $N$ of $p$ in $X$, $A \cap N \neq \emptyset$.

Definition 13. Let $A$ be a subset of $X$. A mapping $r : X \to A$ is called a $\overline{g}$-continuous retraction if $r$ is $\overline{g}$-continuous and the restriction $r_A$ is the identity mapping on $A$.

Definition 14. A topological space $(X, \tau)$ is called a $\overline{g}$-Hausdorff if for each pair $x, y$ of distinct points of $X$, there exists $\overline{g}$-neighborhoods $U_1$ and $U_2$ of $x$ and $y$, respectively, that are disjoint.

Theorem 3.27. Let $A$ be a subset of $X$ and $r : X \to A$ be a $\overline{g}$-continuous retraction. If $X$ is $\overline{g}$-Hausdorff, then $A$ is a $\overline{g}$-closed set of $X$.

Proof. Suppose that $A$ is not $\overline{g}$-closed. Then there exists a point $x$ in $X$ such that $x \in \overline{g}(A)$ but $x \notin A$. It follows that $r(x) \neq x$ because $r$ is $\overline{g}$-continuous retraction. Since $X$ is $\overline{g}$-Hausdorff, there exists disjoint $\overline{g}$-open sets $U$ and $V$ in $X$ such that $x \in U$ and $r(x) \in V$. Now let $W$ be an arbitrary $\overline{g}$-neighborhood of $x$. Then $W \cap U$ is a $\overline{g}$-neighborhood of $x$. Since $x \in \overline{g}(A)$, by Lemma 3.26, we have $(W \cap U) \cap A \neq \emptyset$. Therefore there exists a point $y$ in $W \cap U \cap A$. Since $y \in A$, we have $r(y) = y \in U$ and hence $r(y) \notin V$. This implies that $r(W) \notin V$ because $y \in W$. This is contrary to the $\overline{g}$-continuity of $r$. Consequently, $A$ is a $\overline{g}$-closed set of $X$.

Theorem 3.28. Let $\{X_i : i \in I\}$ be any family of topological spaces. If $f : X \to \prod X_i$ is a $\overline{g}$-continuous mapping, then $Pr_i \circ f : X \to X_i$ is $\overline{g}$-continuous for each $i \in I$, where $Pr_i$ is the projection of $\prod X_j$ onto $X_i$.

Proof. We shall consider a fixed $i \in I$. Suppose $U_i$ is an arbitrary open set in $X$. Then $Pr_i^{-1}(U_i)$ is open in $\prod X_i$. Since $f$ is $\overline{g}$-continuous, we have, $f^{-1}(Pr_i^{-1}(U_i)) = (Pr_i \circ f)^{-1}(U_i)$ $\overline{g}$-open in $X$. Therefore $Pr_i \circ f$ is $\overline{g}$-continuous.

4. $\overline{g}$-Homeomorphisms

In this section we introduce and study two new homeomorphisms namely $\overline{g}$-homeomorphism and $\overline{g}^*$-homeomorphism. We prove that $gc$-homeomorphism and $\overline{g}$-homeomorphism are independent and $\overline{g}^*$-homeomorphism is an equivalence relation between topological spaces.
Definition 15. A bijection \( f : (X, \tau) \to (Y, \sigma) \) is called \( \tilde{g} \)-homeomorphism if \( f \) is both \( \tilde{g} \)-continuous and \( \tilde{g} \)-open.

Proposition 4.1. Every homeomorphism is a \( \tilde{g} \)-homeomorphism but not conversely.

Proof. It follows from definitions.

The converse of the above Proposition 4.1 need not be true as seen from the following example.

Example 4.2. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, X, \{a, b\}\} \) and \( \sigma = \{\emptyset, Y, \{a\}, \{a, b\}\} \). The identity map \( f \) on \( X \) is \( \tilde{g} \)-homeomorphism but not a homeomorphism, because it is not continuous.

Thus, the class of \( \tilde{g} \)-homeomorphisms properly contains the class of homeomorphisms. Next we show that the class of \( g \)-homeomorphisms properly contains the class of \( \tilde{g} \)-homeomorphisms.

Proposition 4.3. Every \( \tilde{g} \)-homeomorphism is a \( g \)-homeomorphism (resp. \( \tilde{g} \)-homeomorphism) but not conversely.

Proof. Since every \( \tilde{g} \)-continuous map is \( g \)-continuous (resp. \( \tilde{g} \)-continuous) and every \( \tilde{g} \)-open map is \( g \)-open (resp. \( \tilde{g} \)-open), the proof follows.

The converses of the above Proposition 4.3 need not be true as seen from the following examples.

Example 4.4. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, X, \{a\}\} \) and \( \sigma = \{\emptyset, Y, \{b\}\} \). Define a map \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = c \), \( f(b) = a \) and \( f(c) = b \). Then \( f \) is a \( g \)-homeomorphism. However, \( f \) is not a \( \tilde{g} \)-homeomorphism.

Example 4.5. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, X, \{a\}, \{b, c\}\} \) and \( \sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{b, c\}\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be the identity map. Then \( f \) is \( \tilde{g} \)-homeomorphism. However, \( f \) is not a \( \tilde{g} \)-homeomorphism.

Proposition 4.6. Let \( f : (X, \tau) \to (Y, \sigma) \) be a bijection \( \tilde{g} \)-continuous map.
Then the following statements are equivalent:
(i) \( f \) is a \( \tilde{g} \)-open map.
(ii) \( f \) is a \( g \)-homeomorphism.
(iii) \( f \) is a \( \tilde{g} \)-closed map.
Proof. It follows from Proposition 3.17.

The composition of two $g$-homeomorphism maps need not be a $g$-homeomorphism as can be seen from the following example.

**Example 4.7.** Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{\emptyset, Y, \{a, b\}\}$ and $\gamma = \{\emptyset, Z, \{a\}, \{a, b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \gamma)$ be identity maps. Then both $f$ and $g$ are $g$-homeomorphisms but their composition $g \circ f : (X, \tau) \rightarrow (Z, \gamma)$ is not a $g$-homeomorphism, because for the open set $\{b\}$ in $(X, \tau)$, $(g \circ f)(\{b\}) = \{b\}$, which is not a $g$-open set in $(Z, \gamma)$. Therefore $g \circ f$ is not a $g$-open map and so $g \circ f$ is not a $g$-homeomorphism.

We next introduce a new class of maps called $g^*$-homeomorphisms which forms a sub class of $g$-homeomorphisms. This class of maps is closed under composition of maps.

**Definition 16.** A bijection $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $g^*$-homeomorphism if both $f$ and $f^{-1}$ are $g$-irresolute.

We denote the family of all $g$-homeomorphisms (resp. $g^*$-homeomorphism and homeomorphisms) of a topological space $(X, \tau)$ onto itself by $g\text{-}h(X, \tau)$ (resp. $g^*\text{-}h(X, \tau)$ and $h(X, \tau)$).

**Proposition 4.8.** Every $g^*$-homeomorphism is a $g$-homeomorphism but not conversely. i.e., for any space $(X, \tau)$, $g^*\text{-}h(X, \tau) \subset g\text{-}h(X, \tau)$.

Proof. It follows from Proposition 2.1 and the fact that every $g^*$-open map is $g$-open.

The function $g$ in Example 4.7 is a $g$-homeomorphism but not a $g^*$-homeomorphism, since for the $g$-closed set $\{a, c\}$ in $(Y, \sigma)$, $(g^{-1})^{-1}(\{a, c\}) = g(\{a, c\}) = \{a, c\}$ which is not $g$-closed in $(Z, \gamma)$. Therefore, $g^{-1}$ is not $g$-irresolute and so $g$ is not a $g^*$-homeomorphism.

**Proposition 4.9.** Every $g^*$-homeomorphism is a $g$-homeomorphism but not conversely.

Proof. Follows from Propositions 4.8 and 4.3.

The map $f$ in Example 4.4 is a $g$-homeomorphism but not a $g^*$-homeomorphism.
Remark 4.10. $\bar{g}$-homeomorphism and gc-homeomorphism are independent as can be seen from the following examples.

Example 4.11. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\sigma = \{\emptyset, Y, \{b\}\{c, b\}\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then $f$ is a gc-homeomorphism but not $\bar{g}$-homeomorphism, because $f$ is not a $\bar{g}$-continuous map.

The map $f$ in Example 4.4 is a $\bar{g}$-homeomorphism but not a gc-homeomorphism, because $f^{-1}$ is not a gc-irresolute map.

Proposition 4.12. If $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \gamma)$ are $\bar{g}*$-homeomorphisms, then their composition $g \circ f : (X, \tau) \rightarrow (Z, \gamma)$ is also $\bar{g}*$-homeomorphism.

Proof. Let $U$ be a $\bar{g}$-open in $(Z, \gamma)$. Now, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$, where $V = g^{-1}(U)$. By hypothesis, $V$ is $\bar{g}$-open in $(Y, \sigma)$ and so again by hypothesis, $f^{-1}(V)$ is $\bar{g}$-open in $(X, \tau)$. Therefore, $g \circ f$ is $\bar{g}$-irresolute. Also for a $\bar{g}$-open set $G$ in $(X, \tau)$, we have $(g \circ f)(G) = g(f(G)) = g(W)$, where $W = f(G)$. By hypothesis $f(G)$ is $\bar{g}$-open in $(Y, \sigma)$ and so again by hypothesis, $g(f(G))$ is $\bar{g}$-open in $(Z, \gamma)$. i.e., $(g \circ f)(G)$ is $\bar{g}$-open in $(Z, \gamma)$ and therefore $(g \circ f)^{-1}$ is $\bar{g}$-irresolute. Hence $g \circ f$ is a $\bar{g}$*-homeomorphism.

Theorem 4.13. The set $\bar{g}^*-h(X, \tau)$ is a group under the composition of maps.

Proof. Define a binary operation $*: \bar{g}^*-h(X, \tau) \times \bar{g}^*-h(X, \tau) \rightarrow \bar{g}^*-h(X, \tau)$ by $f * g = g \circ f$ for all $f, g \in \bar{g}^*-h(X, \tau)$ and $\circ$ is the usual operation of composition of maps. Then by Proposition 4.12, $g \circ f \in \bar{g}^*-h(X, \tau)$. We know that the composition of maps is associative and the identity map $I : (X, \tau) \rightarrow (X, \tau)$ belonging to $\bar{g}^*-h(X, \tau)$ servers as the identity element. If $f \in \bar{g}^*-h(X, \tau)$, then $f^{-1} \in \bar{g}^*-h(X, \tau)$ such that $f \circ f^{-1} = f^{-1} \circ f = I$ and so inverse exists for each element of $\bar{g}^*-h(X, \tau)$. Therefore, $(\bar{g}^*-h(X, \tau), \circ)$ is a group under the operation of composition of maps.

Theorem 4.14. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $(\bar{g}^*)$-homeomorphism. Then $f$ induces an isomorphism from the group $\bar{g}^*-h(X, \tau)$ onto the group $\bar{g}^*-h(Y, \sigma)$. 
Theorem 4.15. $\tilde{g}^*$-homeomorphism is an equivalence relation in the collection of all topological spaces.

**Proof.** Reflexivity and symmetry are immediate and transitivity follows from Proposition 4.12.

Theorem 4.16. If $f : (X, \tau) \to (Y, \sigma)$ is a $\tilde{g}^*$-homeomorphism, then $\tilde{g}^*\text{-cl}(f^{-1}(B)) = f^{-1}(\tilde{g}^*\text{-cl}(B))$ for all $B \subset Y$.

**Proof.** Since $f$ is a $\tilde{g}^*$-homeomorphism, $f$ is $\tilde{g}$-irresolute. Since $\tilde{g}^*\text{-cl}(f(B))$ is a $\tilde{g}$-closed set in $(Y, \sigma)$, $f^{-1}(\tilde{g}^*\text{-cl}(f(B)))$ is $\tilde{g}$-closed in $(X, \tau)$. Now, $f^{-1}(B) \subset f^{-1}(\tilde{g}^*\text{-cl}(B))$ and so by Proposition 2.3, $\tilde{g}^*\text{-cl}(f^{-1}(B)) \subset f^{-1}(\tilde{g}^*\text{-cl}(B))$. Again since $f$ is a $\tilde{g}^*$-homeomorphism, $f^{-1}$ is $\tilde{g}$-irresolute. Since $\tilde{g}^*\text{-cl}(f^{-1}(B))$ is $\tilde{g}$-closed in $(X, \tau)$, $(f^{-1})^{-1}(\tilde{g}^*\text{-cl}(f^{-1}(B))) = f(\tilde{g}^*\text{-cl}(f^{-1}(B)))$ is $\tilde{g}$-closed in $(Y, \sigma)$. Therefore, $f^{-1}(\tilde{g}^*\text{-cl}(B)) \subset f^{-1}(f(\tilde{g}^*\text{-cl}(f^{-1}(B)))) \subset \tilde{g}^*\text{-cl}(f^{-1}(B))$ and hence the equality holds.

Corollary 4.17. If $f : (X, \tau) \to (Y, \sigma)$ is a $\tilde{g}^*$-homeomorphism, then $\tilde{g}^*\text{-cl}(f(B)) = f(\tilde{g}^*\text{-cl}(B))$ for all $B \subset X$.

**Proof.** Since $f : (X, \tau) \to (Y, \sigma)$ is a $\tilde{g}^*$-homeomorphism, $f^{-1} : (Y, \sigma) \to (X, \tau)$ is also a $\tilde{g}^*$-homeomorphism. Therefore, by Theorem 4.16, $\tilde{g}^*\text{-cl}(f^{-1}(B)) = (f^{-1})^{-1}(\tilde{g}^*\text{-cl}(B))$ for all $B \subset X$. i.e., $\tilde{g}^*\text{-cl}(f(B)) = f(\tilde{g}^*\text{-cl}(B))$.

Corollary 4.18. If $f : (X, \tau) \to (Y, \sigma)$ is a $\tilde{g}^*$-homeomorphism, then $f(\tilde{g}^*\text{-int}(B)) = \tilde{g}^*\text{-int}(f(B))$ for all $B \subset X$.

**Proof.** For any set $B \subset X$, $\tilde{g}^*\text{-int}(B) = (\tilde{g}^*\text{-cl}(B^c))^c$. Thus, by utilizing Corollary 4.17, we obtain $f(\tilde{g}^*\text{-int}(B)) = f((\tilde{g}^*\text{-cl}(B^c))^c) = (f(\tilde{g}^*\text{-cl}(B^c)))^c = (\tilde{g}^*\text{-cl}(f(B^c)))^c = (\tilde{g}^*\text{-cl}(f(B)))^c = \tilde{g}^*\text{-int}(f(B))$.

Corollary 4.19. If $f : (X, \tau) \to (Y, \sigma)$ is a $\tilde{g}^*$-homeomorphism, then $f^{-1}(\tilde{g}^*\text{-int}(B)) = \tilde{g}^*\text{-int}(f^{-1}(B))$ for all $B \subset Y$. 

Proof. Using the map $f$, we define a map $\theta_f : \tilde{g}^*\text{-h}(X, \tau) \to \tilde{g}^*\text{-h}(Y, \sigma)$ by $\theta_f(h) = f \circ h \circ f^{-1}$ for every $h \in \tilde{g}^*\text{-h}(X, \tau)$. Then $\theta_f$ is a bijection. Further, for all $h_1, h_2 \in \tilde{g}^*\text{-h}(X, \tau)$, $\theta_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \theta_f(h_1) \circ \theta_f(h_2)$. Therefore, $\theta_f$ is a homeomorphism and so it is an isomorphism induced by $f$.
Proof. Since $f^{-1} : (Y, \sigma) \to (X, \tau)$ is also a $\tilde{g}$*-homeomorphism, the proof follows from Corollary 4.18.

Theorem 4.20. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\tilde{g}$-continuous function and $G_g(f) = \{(x, y) \in X \times Y : y = f(x)\}$, where $X \times Y$ is the product topology and $G_g(f)$ is called the $\tilde{g}$-graph of $f$. Then the following properties are satisfied:

1. $G_g(f)$, as a subspace of $X \times Y$, is $\tilde{g}$-homeomorphic to $X$.
2. If $Y$ is $\tilde{g}$-Hausdorff space, then $G_g(f)$ is $\tilde{g}$-closed in $X \times Y$.

Proof. (1). The function $g : X \to G_g(f)$ is defined by $g(x) = (x, f(x))$ for each $x \in X$ is $\tilde{g}$-continuous and $g^{-1}$ is also $\tilde{g}$-continuous. It is obvious that $g$ is an injective function. Suppose $D$ (resp. $E$) is an arbitrary $\tilde{g}$-neighborhood of $x \in X$ (resp. $g(x) = (x, f(x))$ in $G_g(f)$). So there exist $\tilde{g}$-open sets $U$ and $V$ in $X$ and $Y$ containing $x$ and $f(x)$ respectively for which $(U \times V) \cap G_g(f) \subset E$ and $U \subset D$ and $f(U) \subset V$. Let $M = (U \times V) \cap G_g(f)$. Then $(x, f(x)) \in M$ and $x \in g^{-1}(M) \subset U \subset D$. It implies that $g^{-1}$ is $\tilde{g}$-continuous. Moreover, $g(U) \subset U \times f(U) \subset U \times V_1$ and $g(U) \subset G_g(f)$. Therefore, $g(U) \subset (U \times V) \cap G_g(f) \subset E$. Hence $g$ is $\tilde{g}$-continuous which means that $g$ is a $\tilde{g}$-homeomorphism.

(2). Suppose that $(x, y) \notin G_g(f)$. Then $y_1 = f(x) \neq y$. By hypothesis, there exist disjoint $\tilde{g}$-open sets $V_1$ and $V$ in $Y$ such that $y_1 \in V_1$, $y \in V$. Since $f$ is $\tilde{g}$-continuous, there exists an open set $U$ in $X$ containing $x$ such that $f(U) \subset V_1$. Then $g(U) \subset U \times V_1$. It follows from this and the fact that $V_1 \cap V = \emptyset$ that $(U \times V) \cap G_g(f) = \emptyset$.

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