A NEW APPROACH TO ALMOST FUZZY COMPACTNESS *

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Abstract

A new definition of almost fuzzy compactness is introduced in L-topological spaces by means of open L-sets and their inequality when L is a complete DeMorgan algebra. It can also be characterized by closed L-sets, regularly closed L-sets, regularly open L-sets and their inequalities. When L is a completely distributive DeMorgan algebra, its many characterizations are presented.

Keywords : L-topology, fuzzy compactness, almost fuzzy compactness, almost continuous, weakly continuous

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1. Introduction

Almost compactness has also been generalized to \( L \)-topological spaces by many authors (see [2, 3, 4, 6, 9, 10, 14, 15, 16]). These notions of almost fuzzy compactness rely on the structure of the basis lattice \( L \), where \( L = [0, 1] \) or \( L \) is a completely distributive DeMorgan algebra. In [21], a new definition of fuzzy compactness was presented in \( L \)-fuzzy topological spaces by means of open \( L \)-sets and their inequality.

In this paper, based on [19, 21], we shall introduce a new definition of almost fuzzy compactness in \( L \)-topological spaces. When \( L \) is a completely distributive DeMorgan algebra, its many characterizations are presented. From these characterizations we know that it is a generalization of the notion of almost fuzzy compactness in [3, 9].

2. Preliminaries

Throughout this paper, \( (L, \vee, \wedge', \wedge) \) is a complete DeMorgan algebra and \( X \) is a nonempty set. \( L^X \) is the set of all \( L \)-fuzzy sets (or \( L \)-sets for short) on \( X \). The smallest element and the largest element in \( L^X \) are denoted by \( \underline{0} \) and \( \underline{1} \).

An element \( a \) in \( L \) is called a prime element if \( a \geq b \wedge c \) implies \( a \geq b \) or \( a \geq c \). \( a \) in \( L \) is called a co-prime element if \( a \) is a prime element [7].

The set of non-unit prime elements in \( L \) is denoted by \( P(L) \). The set of non-zero co-prime elements in \( L \) is denoted by \( M(L) \).

The binary relation \( \prec \) in \( L \) is defined as follows: for \( a, b \in L \), \( a \prec b \) if and only if for every subset \( D \subseteq L \), the relation \( b \leq \sup D \) always implies the existence of \( d \in D \) with \( a \leq d \) [5]. In a completely distributive DeMorgan algebra \( L \), each element \( b \) is a supremum of \( \{ a \in L \mid a \prec b \} \). In the sense of [11, 23], \( \{ a \in L \mid a \prec b \} \) is the greatest minimal family of \( b \), denoted by \( \beta(b) \). Moreover for \( b \in L \), define \( \alpha(b) = \{ a \in L \mid a' \prec b' \} \) and \( \alpha^*(b) = \alpha(b) \cap P(L) \).

For \( a \in L \) and \( A \in L^X \), we use the following notations in [20].

\[
A^{(a)} = \{ x \in X \mid A(x) \leq a \}, \quad A_{(a)} = \{ x \in X \mid a \in \beta(A(x)) \}, \\
A_{[a]} = \{ x \in X \mid A(x) \geq a \}.
\]

An \( L \)-topological space (or \( L \)-space for short) is a pair \((X, T)\), where \( T \) is a subfamily of \( L^X \) which contains \( \underline{0}, \underline{1} \) and is closed for any suprema and finite infima. \( T \) is called an \( L \)-topology on \( X \). Each member of \( T \) is called an open \( L \)-set and its quasi-complement is called a closed \( L \)-set.

**Definition 2.1 ([11, 23])**. For a topological space \((X, \tau)\), let \( \omega_L(\tau) \) denote the family of all the lower semi-continuous maps from \((X, \tau)\) to \( L \), i.e.,
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\[ \omega_L(\tau) = \{ A \in L^X \mid A^{(a)} \in \tau, a \in L \}. \] Then \( \omega_L(\tau) \) is an \( L \)-topology on \( X \); in this case, \( (X, \omega_L(\tau)) \) is called topologically generated by \( (X, \tau) \).

**Definition 2.2** ([11, 23]). An \( L \)-space \( (X, T) \) is called weakly induced if \( \forall a \in L, \forall A \in T \), it follows that \( A^{(a)} \in [T] \), where \([T]\) denotes the topology formed by all crisp sets in \( T \).

It is obvious that \( (X, \omega_L(\tau)) \) is weakly induced.

**Lemma 2.3** ([20]). Let \( (X, T) \) be a weakly induced \( L \)-space, \( a \in L, A \in T \). Then \( A^{(a)} \) is an open \( L \)-set in \( [T] \).

For a subfamily \( \Phi \subseteq L^X \), \( 2(\Phi) \) denotes the set of all finite subfamilies of \( \Phi \).

**Definition 2.4** ([19, 21]). Let \( (X, T) \) be an \( L \)-space. \( G \in L^X \) is called fuzzy compact if for every family \( U \subseteq T \), it follows that

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in U} A(x) \right) \leq \bigvee_{y \in 2(U)} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in V} A(x) \right) .
\]

**Lemma 2.5** ([19, 21]). Let \( L \) be a complete Heyting algebra, \( f : X \to Y \) be a map, \( f^{-} : L^X \to L^Y \) is the extension of \( f \), then for any family \( \mathcal{P} \subseteq L^Y \), we have:

\[
\bigvee_{y \in Y} \left( f^{-}(G)(y) \land \bigwedge_{B \in \mathcal{P}} B(y) \right) = \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} f^{-}(B)(x) \right) .
\]

**Definition 2.6** ([1]). Let \( (X, T_1) \) and \( (Y, T_2) \) be two \( L \)-spaces. A map \( f : (X, T_1) \to (Y, T_2) \) is called

1. almost continuous if \( f^{-}(G) \in T_1 \) for all regularly open \( L \)-set \( G \) in \( (Y, T_2) \);
2. weakly continuous if \( f^{-}(G) \leq \text{int}(f^{-}(\text{cl}(G))) \) for every open \( L \)-set \( G \) in \( (Y, T_2) \).

**Lemma 2.7** ([1]). Let \( (X, T_1) \) and \( (Y, T_2) \) be two \( L \)-spaces. A map \( f : (X, T_1) \to (Y, T_2) \) is:

1. almost continuous if and only if \( f^{-}(G) \) is closed in \( (X, T_1) \) for all regularly closed \( L \)-set \( G \) in \( (Y, T_2) \);
2. weakly continuous if and only if \( f^{-}(G) \geq \text{cl}(f^{-}(\text{int}(G))) \) for every closed \( L \)-set \( G \) in \( (Y, T_2) \).
Lemma 2.8 ([1]). The closure of an open $L$-set is regularly closed and the interior of a closed $L$-set is regularly open.

Definition 2.9 ([8]). An $L$-space $(X, T)$ is said to be regular if every open $L$-set $G$ is a supremum of open $L$-sets whose closure is less that $G$.

3. Definition and characterizations of almost fuzzy compactness

Definition 3.1. Let $(X, T)$ be an $L$-space. $G \in L^X$ is called almost fuzzy compact if for every family $\mathcal{U} \subseteq T$, it follows that

$$\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} \text{cl}(A)(x) \right).$$

Definition 3.2. Let $(X, T)$ be an $L$-space. $G \in L^X$ is called almost countably fuzzy compact if for every countable family $\mathcal{U} \subseteq T$, it follows that

$$\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{\mathcal{U}}} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} \text{cl}(A)(x) \right).$$

For an open $L$-set $A$, by $A \leq \text{int}(\text{cl}(A))$ we can obtain the following theorem.

Theorem 3.3. Fuzzy compactness $\Rightarrow$ almost fuzzy compactness $\Rightarrow$ almost countable fuzzy compactness.

From Definition 3.1 and Definition 3.2 we can obtain the following theorem by using quasi-complement.

Theorem 3.4. Let $(X, T)$ be an $L$-space. $G \in L^X$ is almost (countably) fuzzy compact if and only if for every (countable) family $\mathcal{P} \subseteq T'$, it follows that

$$\bigvee_{x \in X} \left( G(x) \land \bigwedge_{A \in \mathcal{P}} A(x) \right) \geq \bigwedge_{\mathcal{F} \in 2^{\mathcal{P}}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{A \in \mathcal{F}} \text{int}(A)(x) \right).$$

Definition 3.5 ([21]). Let $(X, T)$ be an $L$-space, $a \in L \setminus \{1\}$ and $G \in L^X$. A family $\mathcal{A} \subseteq L^X$ is said to be
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(1) an $a$-shading of $G$ if for any $x \in X$, it follows that
\[ (G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x)) \not\leq a. \]

(2) a strong $a$-shading of $G$ if \( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \not\leq a. \)

(3) an $a$-remote family of $G$ if for any $x \in X$, it follows that
\[ G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \not\geq a. \]

(4) a strong $a$-remote family of $G$ if \( \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \not\geq a. \)

From Definition 3.1, Definition 3.2, Theorem 3.4 and Theorem 3.5 we immediately obtain the following result.

**Theorem 3.6.** Let $(X, T)$ be an $L$-space and $G \in L^X$. Then the following conditions are equivalent:

1. $G$ is almost (countably) fuzzy compact.
2. For any $a \in L\setminus\{1\}$, each (countable) open strong $a$-shading $\mathcal{U}$ of $G$ has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^-$ is a strong $a$-shading of $G$, where $\mathcal{V}^- = \{ \text{cl}(A) \mid A \in \mathcal{V} \}$.
3. For any $a \in L\setminus\{0\}$, each (countable) closed strong $a$-remote family $\mathcal{P}$ of $G$ has a finite subfamily $\mathcal{F}$ such that $\mathcal{F}^\circ$ is a strong $a$-remote family of $G$, where $\mathcal{F}^\circ = \{ \text{int}(A) \mid A \in \mathcal{F} \}$.

Moreover by means of regularly open $L$-sets and regularly closed $L$-sets, we can give the following characterizations of almost (countable) fuzzy compactness.

**Theorem 3.7.** Let $(X, T)$ be an $L$-space and $G \in L^X$. Then the following conditions are equivalent:

1. $G$ is almost (countably) fuzzy compact.
2. For each (countable) family $\mathcal{U}$ of regularly open $L$-sets, it follows that
\[ \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2(\mathcal{U})} \left( G'(x) \lor \bigvee_{A \in \mathcal{V}} \text{cl}(A)(x) \right). \]
3. For each (countable) family $\mathcal{U}$ of regularly closed $L$-sets, it follows that
\[ \bigvee_{x \in X} \left( G(x) \land \bigwedge_{A \in \mathcal{U}} A(x) \right) \geq \bigwedge_{\mathcal{V} \in 2(\mathcal{U})} \left( G(x) \land \bigwedge_{A \in \mathcal{V}} \text{int}(A)(x) \right). \]
Proof. (2) ⇔ (3) is obvious. Because a regularly open \( L \)-set is open, we easily obtain (1) ⇒ (2). Now we prove (2) ⇒ (1). Suppose that \( \mathcal{U} \) is a family of open \( L \)-sets. From Lemma 2.8 we know that \( \text{int}(\text{cl}(A)) \) is a regularly open \( L \)-set for each \( A \in \mathcal{U} \). Hence by (2) we obtain

\[
\bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right)
\]

\[
= \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} \text{int}(A)(x) \right)
\]

\[
\leq \bigvee_{V \in 2(\mathcal{U})} \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in V} \text{int}(\text{cl}(A))(x) \right)
\]

\[
\leq \bigvee_{V \in 2(\mathcal{U})} \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in V} \text{cl}(\text{int}(\text{cl}(A)))(x) \right)
\]

\[
= \bigvee_{V \in 2(\mathcal{U})} \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in V} \text{cl}(A)(x) \right)
\]

This shows that (1) is true.

Analogous to Theorem 3.6 we have the following result.

**Theorem 3.8.** Let \((X, T)\) be an \( L \)-space and \( G \in L^X \). Then the following conditions are equivalent:

1. \( G \) is almost (countably) fuzzy compact.
2. For any \( a \in L \backslash \{1\} \), each (countable) regularly open strong \( a \)-shading \( \mathcal{U} \) of \( G \) has a finite subfamily \( \mathcal{V} \) such that \( \mathcal{V}^- \) is a strong \( a \)-shading of \( G \).
3. For any \( a \in L \backslash \{0\} \), each (countable) regularly closed strong \( a \)-remote family \( \mathcal{P} \) of \( G \) has a finite subfamily \( \mathcal{F} \) such that \( \mathcal{F}^- \) is a strong \( a \)-remote family of \( G \).

**Theorem 3.9.** Let \((X, T)\) be a regular \( L \)-space and \( G \in L^X \). Then \( G \) is fuzzy compact if and only if it is almost fuzzy compact.

Proof. The necessity is obvious. Now we prove the sufficiency. Let \( \{A_i\}_{i \in \Omega} \) be a family of open \( L \)-sets. By regularity of \((X, T)\), we know that for each \( i \in \Omega \), there exists a family \( \{B_{ij} \mid j \in \Delta_i\} \) of open \( L \)-sets such that \( A_i = \bigvee_{j \in \Delta_i} B_{ij} \) and \( \text{cl}(B_{ij}) \leq A_i \). By almost fuzzy compactness of \( G \),
we know
\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{i \in \Omega} A_i(x) \right) = \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{i \in \Omega, j \in \Delta_i} B_{ij}(x) \right) \\
\leq \bigvee_{\Gamma \in 2^{\Omega}} \bigvee_{\Theta_i \in 2^{\Delta_i}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{i \in \Gamma, j \in \Theta_i} \text{cl}(B_{ij})(x) \right) \\
\leq \bigvee_{\Gamma \in 2^{\Omega}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{i \in \Gamma} A_i(x) \right).
\]

Therefore $G$ is fuzzy compact.

4. Some properties of almost fuzzy compactness

**Theorem 4.1.** Let $L$ be a complete Heyting algebra. If both $G$ and $H$ are almost (countably) fuzzy compact, then $G \lor H$ is almost (countably) fuzzy compact.

**Proof.** For any family $\mathcal{P}$ of closed $L$-sets, by Theorem 3.4 we have
\[
\bigvee_{x \in X} \left( (G \lor H)(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \\
= \bigg\{ \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \bigg\} \lor \bigg\{ \bigvee_{x \in X} \left( H(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \bigg\} \\
\geq \bigg\{ \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(x) \right) \bigg\} \lor \\
\bigg\{ \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \bigvee_{x \in X} \left( H(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(x) \right) \bigg\} \\
= \bigwedge_{\mathcal{F} \in 2(\mathcal{P})} \bigvee_{x \in X} \left( (G \lor H)(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(x) \right).
\]

This shows that $G \lor H$ is almost fuzzy compact. \qed

**Theorem 4.2.** If $G$ is almost (countably) fuzzy compact, and $H$ is clopen, then $G \land H$ is almost (countably) fuzzy compact.

**Proof.** Since $G$ is almost fuzzy compact, for any family $\mathcal{P}$ of closed
\[ L\text{-sets, by Theorem 3.4 we have} \]
\[
\bigvee_{x \in X} \left( (G \land H)(x) \land \bigwedge_{B \in \mathcal{P}} B(x) \right) \\
= \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} B(x) \right) \\
\geq \bigwedge_{\mathcal{F} \in 2^{(Y)}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(x) \right) \\
= \left\{ \bigwedge_{\mathcal{F} \in 2^{(Y)}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(x) \right) \right\} \\
\land \left\{ \bigwedge_{\mathcal{F} \in 2^{(Y)}} \bigvee_{x \in X} \left( G(x) \land \text{int}(H)(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(x) \right) \right\} \\
= \left\{ \bigwedge_{\mathcal{F} \in 2^{(Y)}} \bigvee_{x \in X} \left( G(x) \land \text{int}(H)(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(x) \right) \right\} \\
= \bigvee_{\mathcal{F} \in 2^{(Y)}} \left( G \land H \right)(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(x).
\]

This shows that \( G \land H \) is almost fuzzy compact. \( \square \)

**Theorem 4.3.** Let \( L \) be a complete Heyting algebra, and let \( f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2) \) be almost continuous. If \( G \) is almost (countably) fuzzy compact in \( (X, \mathcal{T}_1) \), then so is \( f_L^{-1}(G) \) in \( (Y, \mathcal{T}_2) \).

**Proof.** Suppose that \( \mathcal{P} \) be a family of regularly closed \( L \)-sets, by Lemma 2.5 and almost fuzzy compactness of \( G \), we have
\[
\bigvee_{y \in Y} \left( f_L^{-1}(G)(y) \land \bigwedge_{B \in \mathcal{P}} B(y) \right) \\
= \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} f_L^{-1}(B)(x) \right) \\
\geq \bigwedge_{\mathcal{F} \in 2^{(Y)}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} \text{int}(f_L^{-1}(B))(x) \right) \\
\geq \bigwedge_{\mathcal{F} \in 2^{(Y)}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} f_L^{-1}(\text{int}(B))(x) \right) \\
= \bigwedge_{\mathcal{F} \in 2^{(Y)}} \bigvee_{y \in Y} \left( f_L^{-1}(G)(y) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(y) \right).
\]

Therefore \( f_L^{-1}(G) \) is almost fuzzy compact.
Theorem 4.4. Let $L$ be a complete Heyting algebra, and let $f : (X, T_1) \to (Y, T_2)$ be weakly continuous. If $G$ is (countably) fuzzy compact in $(X, T_1)$, then $f_L^\ast(G)$ is almost (countably) fuzzy compact in $(Y, T_2)$.

**Proof.** Let $\mathcal{P}$ be a family of regularly closed $L$-sets, by Lemma 2.5 and fuzzy compactness of $G$, we have

\[
\bigvee_{y \in Y} \left( f_L^\ast(G)(y) \land \bigwedge_{B \in \mathcal{P}} B(y) \right) = \bigwedge_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{P}} f_L^\ast(B)(x) \right) \geq \bigwedge_{F \in \mathcal{P}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} \text{cl}(f_L^\ast(\text{int}(B)))(x) \right) \geq \bigwedge_{F \in \mathcal{P}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{B \in \mathcal{F}} f_L^\ast(\text{int}(B))(x) \right) = \bigwedge_{F \in \mathcal{P}} \bigvee_{y \in Y} \left( f_L^\ast(G)(y) \land \bigwedge_{B \in \mathcal{F}} \text{int}(B)(y) \right).
\]

Therefore $f_L^\ast(G)$ is almost fuzzy compact.

5. Further characterizations of almost fuzzy compactness

In this section, we assume that $L$ is a completely distributive DeMorgan algebra.

**Definition 5.1 ([21]).** Let $(X, T)$ be an $L$-space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a $\beta_a$-cover of $G$ if for any $x \in X$, it follows that $a \in \beta \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right)$. $\mathcal{U}$ is called a strong $\beta_a$-cover of $G$ if $a \in \beta \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \right)$.

**Definition 5.2 ([21]).** Let $(X, T)$ be an $L$-space, $a \in L \setminus \{0\}$ and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a $Q_a$-cover of $G$ if for any $x \in X$, it follows that $G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \geq a$.

Analogous to [21] we can obtain the following theorem.
**Theorem 5.3.** Let $(X, T)$ be an $L$-space and $G \in L^X$. Then the following conditions are equivalent.

1. $G$ is almost (countably) fuzzy compact.
2. For any $a \in L\{0\}$ (or $a \in M(L)$), each (countable) closed strong $a$-remote family $\mathcal{P}$ of $G$ has a finite subfamily $\mathcal{F}$ such that $\mathcal{F}^\circ$ is an (a strong) $a$-remote family of $G$.
3. For any $a \in L\{0\}$ (or $a \in M(L)$) and any (countable) closed strong $a$-remote family $\mathcal{P}$ of $G$, there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta(a)$ (or $b \in \beta^*(a)$) such that $\mathcal{F}^\circ$ is a (strong) $b$-remote family of $G$.
4. For any $a \in L\{1\}$ (or $a \in P(L)$), each (countable) open strong $a$-shading $\mathcal{U}$ of $G$ has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^\circ$ is an (a strong) $a$-shading of $G$.
5. For any $a \in L\{1\}$ (or $a \in P(L)$) and any (countable) open strong $a$-shading $\mathcal{U}$ of $G$, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha(a)$ (or $b \in \alpha^*(a)$) such that $\mathcal{V}^\circ$ is a (strong) $b$-shading of $G$.
6. For any $a \in L\{0\}$ (or $a \in M(L)$), each (countable) open strong $\beta_a$-cover $\mathcal{U}$ of $G$ has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^\circ$ is a (strong) $\beta_a$-cover of $G$.
7. For any $a \in L\{0\}$ (or $a \in M(L)$) and any (countable) open strong $\beta_a$-cover $\mathcal{U}$ of $G$, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in L$ (or $b \in M(L)$) with $a \in \beta(b)$ such that $\mathcal{V}^\circ$ is a (strong) $\beta_b$-cover of $G$.
8. For any $a \in L\{0\}$ (or $a \in M(L)$) and any $b \in \beta(a)\{0\}$, each (countable) open $Q_a$-cover of $G$ has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^\circ$ is a $Q_b$-cover of $G$.
9. For any $a \in L\{0\}$ (or $a \in M(L)$) and any $b \in \beta(a)\{0\}$ (or $b \in \beta^*(a)$), each (countable) open $Q_a$-cover of $G$ has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^\circ$ is a (strong) $\beta_b$-cover of $G$.

**Remark 5.4.** In Theorem 5.3, ‘open’ can be replaced by ‘regularly open’, and ‘closed’ can be replaced by ‘regularly closed’.

**Remark 5.5.** From (2) of Theorem 5.3 we know that our notion of almost fuzzy compactness is a generalization of almost $F$-compactness in [3, 9].

The following theorem shows that almost (countable) fuzzy compactness is a good extension.

**Theorem 5.6.** Let $(X, \tau)$ be a topological space and $(X, \omega(\tau))$ be generated topologically by $(X, \tau)$. Then $(X, \omega(\tau))$ is almost (countably) fuzzy compact if and only if $(X, \tau)$ is almost (countably) compact.
Proof. (Necessity) Let $\mathcal{A}$ be an open cover of $(X, \tau)$. Then \( \{ \chi_A \mid A \in \mathcal{A} \} \) is a family of open L-sets in $(X, \omega(\tau))$ with \( \bigwedge_{x \in X} \left( \bigvee_{A \in \mathcal{A}} \chi_A(x) \right) = 1 \).

From almost fuzzy compactness of $(X, \omega(\tau))$ we know that

\[
\bigwedge_{\mathcal{V} \in 2^{(U)}} \left( \bigvee_{x \in X} \chi_{\text{cl}(A)}(x) \right) = \bigwedge_{\mathcal{V} \in 2^{(U)}} \left( \bigvee_{A \in \mathcal{V}} \chi_{\text{cl}(A)}(x) \right) = 1.
\]

This implies that there exists $\mathcal{V} \in 2^{(U)}$ such that \( \bigwedge_{x \in X} \left( \bigvee_{A \in \mathcal{V}} \chi_{\text{cl}(A)}(x) \right) = 1 \).

Hence $\{ \chi_{\text{cl}(A)} \mid A \in \mathcal{V} \}$ is a cover of $(X, \tau)$. Therefore $(X, \tau)$ is almost compact.

(Sufficiency) Let $\mathcal{U}$ be a family of open L-sets in $(X, \omega(\tau))$ and let \( \bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{U}} B(x) \right) = a \). If $a = 0$, then obviously we have

\[
\bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{U}} B(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(U)}} \left( \bigvee_{x \in X} \bigwedge_{A \in \mathcal{V}} \chi_{\text{cl}(A)}(x) \right).
\]

Now we suppose that $a \neq 0$. In this case, for any $b \in \beta(a) \setminus \{0\}$ we have

\[
b \in \beta \left( \bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{U}} B(x) \right) \right) \subseteq \bigcap_{x \in X} \beta \left( \bigvee_{B \in \mathcal{U}} B(x) \right) = \bigcap_{x \in X} \bigcup_{B \in \mathcal{U}} \beta(B(x)).
\]

From Lemma 2.3 this implies that $\{ B(b) \mid B \in \mathcal{U} \}$ is an open cover of $(X, \tau)$. From almost fuzzy compactness of $(X, \tau)$ we know that there exists $\mathcal{V} \in 2^{(U)}$ such that $\{ \chi_{\text{cl}(B(b))} \mid B \in \mathcal{V} \}$ is a cover of $(X, \tau)$. From [17] we can obtain that $\chi_{\text{cl}(B(b))} \subseteq \chi_{\text{cl}(B)[b]}$. This shows that $\{ \chi_{\text{cl}(B)[b]} \mid B \in \mathcal{V} \}$ is a cover of $(X, \tau)$. Hence $b \leq \bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{V}} \chi_{\text{cl}(B)[b]}(x) \right)$. Further we have

\[
b \leq \bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{V}} \chi_{\text{cl}(B)(x)} \right) \leq \bigvee_{\mathcal{V} \in 2^{(U)}} \left( \bigwedge_{x \in X} \bigvee_{B \in \mathcal{V}} \chi_{\text{cl}(B)(x)} \right).
\]

This implies

\[
\bigwedge_{x \in X} \left( \bigvee_{B \in \mathcal{U}} B(x) \right) = a = \bigvee \{ b \mid b \in \beta(a) \} \leq \bigvee_{\mathcal{V} \in 2^{(U)}} \left( \bigvee_{B \in \mathcal{V}} \chi_{\text{cl}(B)(x)} \right).
\]

Therefore $(X, \omega(\tau))$ is almost fuzzy compact.
References


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