Proyecciones Journal of Mathematics
Vol. 28, N° 3, pp. 271–283, December 2009.
Universidad Católica del Norte
Antofagasta - Chile

M-FUZZIFYING BASES *

XIU XIN

and

FU-GUI SHI

BEIJING INSTITUTE OF TECHNOLOGY, CHINA

Received : October 2009. Accepted : October 2009

Abstract

In this paper, we continue the study of M -fuzzifying matroids. We define the notion of an M -fuzzifying base and discuss some properties of the dual matroids of basic M -fuzzifying matroids.

Keywords : M -fuzzifying bases; M -fuzzifying matroids; Dual matroids

Mathematics Subject Classification (2000) : 05B35, 52B40

*The project is supported by the National Natural Science Foundation of China (10971242)

1. Introduction

In [12, 13], when M is a complete lattice, Shi defined an M -fuzzifying matroid to be the pair $(E, \mathcal{I})$, where $\mathcal{I}$ is a map from 2^E to M satisfying three axioms. Thus each subset of E can be regarded as an independent set to some degree. Moreover, Shi defined the M -fuzzifying rank of a set A as an M -fuzzy natural number $R(A) : \mathbf{N} \rightarrow M$. M -fuzzifying matroids and M -fuzzifying rank functions are one-to-one corresponding. This paper treats the notion of M -fuzzifying bases. In subsequent papers we will deal with M -fuzzifying circuits and other fuzzy concepts related to M -fuzzifying matroids.

2. Preliminaries

We will use the following notation in establishing the results of this paper.

If E is a finite set, $\mathcal{A} \subseteq 2^E$, define

$$Com(\mathcal{A}) = \{A \subseteq E : E - A \in \mathcal{A}\},$$

$$Low(\mathcal{A}) = \{A \subseteq E : \exists B \in \mathcal{A}, A \subseteq B\},$$

$$Max(\mathcal{A}) = \{A \in \mathcal{A} : \forall B \in \mathcal{A}, \text{ if } A \subseteq B, \text{ then } A = B\}.$$

Throughout this paper, M always denotes a completely distributive lattice and M^E is the set of all M -fuzzy sets on E . The smallest element and the largest element in M are denoted by $\perp$ and $\top$, respectively. We often do not distinguish a crisp subset A of E and its characteristic function χ_A .

An element a in M is called a prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. a in M is called co-prime if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$ [2]. The set of non-unit prime elements in M is denoted by $P(M)$. The set of non-zero co-prime elements in M is denoted by $J(M)$.

The binary relation $\prec$ in M is defined as follows: for $a, b \in M$, $a \prec b$ if and only if for every subset $D \subseteq M$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [1]. $\{a \in M : a \prec b\}$ is called the greatest minimal family of b in the sense of [14], denoted by $\beta(b)$, and $\beta^*(b) = \beta(b) \cap J(M)$. Moreover, for $b \in M$, we define $\alpha(b) = \{a \in M : a \prec^{op} b\}$ and $\alpha^*(b) = \alpha(b) \cap P(M)$. In a completely distributive lattice M , α is an $\bigwedge$ - $\bigcup$ map, β is a union-preserving map, and there exist $\alpha(b)$ and $\beta(b)$ for each $b \in M$ such that $b = \bigvee \beta(b) = \bigwedge \alpha(b)$ (see [14]). Note that $\beta(\perp) = \emptyset$ and $\alpha(\top) = \emptyset$.

For any $A \in M^E$ and any $a \in M$, we define

$$\begin{aligned} A_{[a]} &= \{x \in E : A(x) \geq a\}, & A^{(a)} &= \{x \in E : A(x) \not\geq a\}, \\ A_{(a)} &= \{x \in E : a \in \beta(A(x))\}, & A^{[a]} &= \{x \in E : a \notin \alpha(A(x))\}. \end{aligned}$$

Some properties of these cut sets can be found in [4, 6, 8, 9, 10, 11].

In [5, 7, 15], a crisp matroid and a crisp base for a crisp matroid are usually defined as follows:

Definition 2.1. Let E be a finite set. $\mathcal{I} \subseteq 2^E$ is called a system of matroid independent sets on E if it satisfies

- (I1) $\emptyset \in \mathcal{I}$;
 - (I2) $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$;
 - (I3) For any $A, B \in \mathcal{I}$ which satisfy $|A| < |B|$, there exists $e \in B - A$ such that $A \cup \{e\} \in \mathcal{I}$, where $|A|, |B|$ denote the cardinality of A, B .
- The set of all systems of matroid independent sets on E is denoted by $\mathbf{I}(E)$ and $(E, \mathcal{I})$ is called a crisp matroid.

Definition 2.2. Let E be a finite set. A subset $\mathcal{B} \subseteq 2^E$ is called a crisp base on E if it satisfies

- (B1) $\mathcal{B} \neq \emptyset$;
- (B2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$ for some $y \in B_2 - B_1$.

The set of all crisp bases on E is denoted by $\mathbf{B}(E)$.

Theorem 2.3. Let E be a finite set. A subset $\emptyset \neq \mathcal{B} \subseteq 2^E$ is a crisp base on E if and only if it satisfies

- (B2)' If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then $(B_2 - \{y\}) \cup \{x\} \in \mathcal{B}$ for some $y \in B_2 - B_1$.

Theorem 2.4. (1) If $\mathcal{I} \in \mathbf{I}(E)$, then $\text{Max}(\mathcal{I}) \in \mathbf{B}(E)$;

(2) If $\mathcal{B} \in \mathbf{B}(E)$, then $\text{Low}(\mathcal{B}) \in \mathbf{I}(E)$;

(3) $\text{Max} \circ \text{Low}(\mathcal{B}) = \mathcal{B}$ ($\forall \mathcal{B} \in \mathbf{B}(E)$); $\text{Low} \circ \text{Max}(\mathcal{I}) = \mathcal{I}$ ($\forall \mathcal{I} \in \mathbf{I}(E)$).

Definition 2.5. Let $(E, \mathcal{I})$ be a crisp matroid. Define $(\mathcal{B}_{\mathcal{I}})^* = \text{Com}(\text{Max}(\mathcal{I}))$ and $\mathcal{I}^* = \text{Low}((\mathcal{B}_{\mathcal{I}})^*)$, then $(\mathcal{B}_{\mathcal{I}})^*$ is a crisp base on E and $(E, \mathcal{I}^*)$ is a crisp matroid. $(E, \mathcal{I}^*)$ is called the dual matroid of $(E, \mathcal{I})$.

Theorem 2.6. Let $(E, \mathcal{I})$ be a crisp matroid, for every $A \subseteq E$, then

(1) $R_{\mathcal{I}^*}(A) = |A| - R_{\mathcal{I}}(E) + R_{\mathcal{I}}(E - A)$;

(2) $R_{\mathcal{I}}(A) + R_{\mathcal{I}}(E - A) - R_{\mathcal{I}}(E) = R_{\mathcal{I}}(A) + R_{\mathcal{I}^*}(A) - |A| = R_{\mathcal{I}^*}(A) + R_{\mathcal{I}^*}(E - A) - R_{\mathcal{I}^*}(E)$.

Where $R_{\mathcal{I}}$ and $R_{\mathcal{I}^*}$ denote the rank functions for $(E, \mathcal{I})$ and $(E, \mathcal{I}^*)$.

In [12, 13], Shi defined an M -fuzzifying matroid as follows:

Definition 2.7. Let E be a finite set. If a map $\mathcal{I} : 2^E \rightarrow M$ satisfies the following conditions:

- (FI1) $\mathcal{I}(\emptyset) = \top$;
- (FI2) For any $A, B \in 2^E$, $A \subseteq B \Rightarrow \mathcal{I}(A) \geq \mathcal{I}(B)$;
- (FI3) If $A, B \in 2^E$ and $|A| < |B|$, then $\bigvee_{e \in B-A} \mathcal{I}(A \cup \{e\}) \geq \mathcal{I}(A) \wedge \mathcal{I}(B)$.

Then the pair $(E, \mathcal{I})$ is called an M -fuzzifying matroid. $\mathcal{I}$ is called a fuzzy family of independent sets on E . For $A \in 2^E$, $\mathcal{I}(A)$ can be regarded as the degree of the set A to be an independent set. A $[0, 1]$ -fuzzifying matroid is also called a fuzzifying matroid for short.

Theorem 2.8. Let E be a finite set and $\mathcal{I} : 2^E \rightarrow M$ be a map. Then $(E, \mathcal{I})$ is an M -fuzzifying matroid if and only if for each $a \in J(M)$, $(E, \mathcal{I}_{[a]})$ is a crisp matroid.

Definition 2.9. Let $\mathbf{N}$ denote the set of all natural numbers. An M -fuzzy natural number is an antitone map $\lambda : \mathbf{N} \rightarrow M$ satisfying

$$\lambda(0) = \top, \bigwedge_{n \in \mathbf{N}} \lambda(n) = \perp.$$

The set of all M -fuzzy natural numbers is denoted by $\mathbf{N}(M)$.

Definition 2.10. For any $m \in \mathbf{N}$, define $\underline{m} \in \mathbf{N}(M)$ such that

$$\underline{m} = \begin{cases} \top, & \text{if } t \leq m, \\ \perp, & \text{if } t \geq m + 1. \end{cases}$$

Definition 2.11. For any $\lambda, \mu \in \mathbf{N}(M)$, define the addition $\lambda + \mu$ of λ and μ as follows: for any $n \in \mathbf{N}$,

$$(\lambda + \mu)(n) = \bigvee_{k+l=n} (\lambda(k) \wedge \mu(l)).$$

Theorem 2.12. For any $\lambda, \mu \in \mathbf{N}(M)$ and any $a \in J(M)$, it follows that

$$(\lambda + \mu)_{[a]} = \lambda_{[a]} + \mu_{[a]}.$$

Definition 2.13. Let $(E, \mathcal{I})$ be an M -fuzzifying matroid. The map $R_{\mathcal{I}} : 2^E \rightarrow \mathbf{N}(M)$ defined by

$$R_{\mathcal{I}}(A)(n) = \bigvee \{\mathcal{I}(B) : B \subseteq A, |B| \geq n\}$$

is called the M -fuzzifying rank function for $(E, \mathcal{I})$. If $A \in 2^E$, then $R_{\mathcal{I}}(A)$ is called the M -fuzzifying rank of A .

Theorem 2.14. Let $(E, \mathcal{I})$ be an M -fuzzifying matroid and $R_{\mathcal{I}}$ be the M -fuzzifying rank function for $(E, \mathcal{I})$. For each $a \in J(M)$, let $R_{\mathcal{I}_{[a]}}$ denote the rank function for $(E, \mathcal{I}_{[a]})$. Then $R_{\mathcal{I}_{[a]}}(A) = R_{\mathcal{I}}(A)_{[a]}$ for each $A \in 2^E$.

3. M -fuzzifying bases

The obvious M -fuzzifying analog of a crisp base is the following:

Definition 3.1. Let E be a finite set. A map $\mathcal{B} : 2^E \rightarrow M$ is called an M -fuzzifying base on E if it satisfies

$$(\text{FB1}) \quad \bigvee_{B \in 2^E} \mathcal{B}(B) = \top;$$

$$(\text{FB2}) \quad \forall B_1, B_2 \in 2^E, \quad \bigwedge_{x \in B_1 - B_2} \bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2).$$

A $[0, 1]$ -fuzzifying base is also called a fuzzifying base for short.

Example 3.2. Let $E = \{x, y\}$. Define $\mathcal{B} : 2^E \rightarrow [0, 1]$ by

$$\mathcal{B}(A) = \begin{cases} 0, & A \in \{\emptyset, \{x, y\}\}; \\ 1, & A = \{x\}; \\ \frac{1}{2}, & A = \{y\}. \end{cases}$$

Obviously, $\mathcal{B}$ is a fuzzifying base on E .

Theorem 3.3. Let E be a finite set and $\mathcal{B} : 2^E \rightarrow M$ be a map. Then the following conditions are equivalent:

- (1) $\mathcal{B}$ is an M -fuzzifying base on E ;
- (2) For each $a \in J(M)$, $\mathcal{B}_{[a]}$ is a crisp base on E ;
- (3) For each $a \in P(M)$, $\mathcal{B}^{(a)}$ is a crisp base on E .

Proof. (1) $\Rightarrow$ (2). For each $a \in J(M)$. By (FB1), $\bigvee_{B \in 2^E} \mathcal{B}(B) = \top \geq a$, thus $\mathcal{B}(B) \geq a$ for some $B \in 2^E$, hence $\mathcal{B}_{[a]} \neq \emptyset$, i.e. $\mathcal{B}_{[a]}$ satisfies (B1). Let $B_1, B_2 \in \mathcal{B}_{[a]}$ and $x \in B_1 - B_2$, then $\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \geq a$ by (FB2). As $a \in J(M)$, $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}_{[a]}$ for some $y \in B_2 - B_1$. This means that $\mathcal{B}_{[a]}$ satisfies (B2).

(2) $\Rightarrow$ (1). By (2), for each $a \in J(M)$, $\mathcal{B}(B) \geq a$ for some $B \in 2^E$, thus $\bigvee_{B \in 2^E} \mathcal{B}(B) \geq a$, hence $\bigvee_{B \in 2^E} \mathcal{B}(B) = \top$, i.e. (FB1) holds. Let $B_1, B_2 \in 2^E$, $x \in B_1 - B_2$, and $\mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \neq \perp$. For every $a \in J(M)$ and

$a \leq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$, then $B_1, B_2 \in \mathcal{B}_{[a]}$, thus there exists $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}_{[a]}$, hence $\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq a$.

Therefore, $\bigwedge_{x \in B_1 - B_2} \bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$.

(1) $\Rightarrow$ (3). For each $a \in P(M)$. By (FB1), $\bigvee_{B \in 2^E} \mathcal{B}(B) = \top \not\leq a$,

thus $\mathcal{B}(B) \not\leq a$ for some $B \in 2^E$, hence $\mathcal{B}^{(a)} \neq \emptyset$, i.e. $\mathcal{B}^{(a)}$ satisfies (B1).

Let $B_1, B_2 \in \mathcal{B}^{(a)}$ and $x \in B_1 - B_2$, by (FB2) and $a \in P(M)$, we have

$\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \not\leq a$, hence $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}^{(a)}$ for some $y \in B_2 - B_1$. This means that $\mathcal{B}^{(a)}$ satisfies (B2).

(3) $\Rightarrow$ (1). By (3), for each $a \in P(M)$, $B \in \mathcal{B}^{(a)}$ for some $B \in 2^E$, i.e. $\mathcal{B}(B) \not\leq a$ for some $B \in 2^E$, thus $\bigvee_{B \in 2^E} \mathcal{B}(B) \not\leq a$, hence $\bigvee_{B \in 2^E} \mathcal{B}(B) = \top$.

Let $B_1, B_2 \in 2^E$, $x \in B_1 - B_2$, and $\mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \neq \perp$. For every $a \in P(M)$ and $\mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \not\leq a$, then $\mathcal{B}(B_1) \not\leq a$ and $\mathcal{B}(B_2) \not\leq a$, i.e. $B_1, B_2 \in \mathcal{B}^{(a)}$, thus there exists $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}^{(a)}$, hence

$\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \not\leq a$. Therefore, $\bigwedge_{x \in B_1 - B_2} \bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$. $\square$

By Theorem 2.3 and Theorem 3.3, we can obtain the following theorem.

Theorem 3.4. Let E be a finite set and $\mathcal{B} : 2^E \rightarrow M$ be a map. Then $\mathcal{B}$ is an M -fuzzifying base on E if and only if it satisfies the following conditions:

(FB1) $\bigvee_{B \in 2^E} \mathcal{B}(B) = \top$;

(FB2)' $\forall B_1, B_2 \in 2^E$, $\bigwedge_{x \in B_1 - B_2} \bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$.

Theorem 3.5. Let E be a finite set and $\mathcal{B} : 2^E \rightarrow M$ be a map. If $\alpha(a \vee b) = \alpha(a) \cap \alpha(b)$ for any $a, b \in M$, then the following conditions are equivalent:

- (1) $\mathcal{B}$ is an M -fuzzifying base on E ;
- (2) For each $a \in M \setminus \{\top\}$, $\mathcal{B}^{[a]}$ is a crisp base on E ;
- (3) For each $a \in P(M)$, $\mathcal{B}^{[a]}$ is a crisp base on E .

Proof. (1) $\Rightarrow$ (2). For each $a \in M \setminus \{\top\}$. By (FB1), $\bigvee_{B \in 2^E} \mathcal{B}(B) = \top$,

hence $a \notin \alpha(\top) = \alpha\left(\bigvee_{B \in 2^E} \mathcal{B}(B)\right) = \bigcap_{B \in 2^E} \alpha(\mathcal{B}(B))$, thus $a \notin \alpha(\mathcal{B}(B))$ for

some $B \in 2^E$, i. e. $B \in \mathcal{B}^{[a]}$. This implies $\mathcal{B}^{[a]}$ satisfies (B1). Let

$B_1, B_2 \in \mathcal{B}^{[a]}$ and $x \in B_1 - B_2$, then $a \notin \alpha(\mathcal{B}(B_1)) \cup \alpha(\mathcal{B}(B_2)) = \alpha(\mathcal{B}(B_1) \wedge \mathcal{B}(B_2))$ and $\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$ by (FB2), hence

$$a \notin \alpha \left(\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \right) = \bigcap_{y \in B_2 - B_1} \alpha(\mathcal{B}((B_1 - \{x\}) \cup \{y\})),$$

thus $a \notin \alpha((\mathcal{B}(B_1 - \{x\}) \cup \{y\}))$ for some $y \in B_2 - B_1$, i.e. $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}^{[a]}$ for some $y \in B_2 - B_1$. This means that $\mathcal{B}^{[a]}$ satisfies (B2).

(2) $\Rightarrow$ (1). For each $a \in M \setminus \{\top\}$, by $\mathcal{B}^{[a]}$ satisfies (B1), then $a \notin \alpha(\mathcal{B}(B))$ for some $B \in 2^E$, thus $a \notin \bigcap_{B \in 2^E} \alpha(\mathcal{B}(B)) = \alpha \left(\bigvee_{B \in 2^E} \mathcal{B}(B) \right)$, hence

$$\bigvee_{B \in 2^E} \mathcal{B}(B) = \bigwedge \alpha \left(\bigvee_{B \in 2^E} \mathcal{B}(B) \right) = \top, \text{ i.e. (FB1) holds. Let } B_1, B_2 \in 2^E,$$

$x \in B_1 - B_2$, and $\mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \neq \perp$. For every $a \in M \setminus \{\top\}$ and $a \notin \alpha(\mathcal{B}(B_1) \wedge \mathcal{B}(B_2)) = \alpha(\mathcal{B}(B_1)) \cup \alpha(\mathcal{B}(B_2))$, then $a \notin \alpha(\mathcal{B}(B_1))$ and $a \notin \alpha(\mathcal{B}(B_2))$, i.e. $B_1, B_2 \in \mathcal{B}^{[a]}$, thus there exists $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}^{[a]}$, hence $a \notin \alpha(\mathcal{B}((B_1 - \{x\}) \cup \{y\}))$, $a \notin \alpha \left(\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \right)$. Therefore, $\bigwedge_{x \in B_1 - B_2} \bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$.

Analogously, we can obtain (1) $\Leftrightarrow$ (3). $\square$

Theorem 3.6. *Let E be a finite set and $\mathcal{B} : 2^E \rightarrow M$ be a map. If $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for any $a, b \in M$, then the following conditions are equivalent:*

- (1) $\mathcal{B}$ is an M -fuzzifying base on E ;
- (2) For each $a \in \beta(\top)$, $\mathcal{B}_{(a)}$ is a crisp base on E .

Proof. (1) $\Rightarrow$ (2). Suppose that $\mathcal{B}$ is an M -fuzzifying base on E . Then for any $a \in \beta(\top)$, $a \in \beta \left(\bigvee_{B \in 2^E} \mathcal{B}(B) \right) = \bigcup_{B \in 2^E} \beta(\mathcal{B}(B))$ by (FB1), thus $a \in \beta(\mathcal{B}(B))$ for some $B \in 2^E$, hence $B \in \mathcal{B}_{(a)}$, which means that $\mathcal{B}_{(a)}$ satisfies (B1). Let $B_1, B_2 \in \mathcal{B}_{(a)}$ and $x \in B_1 - B_2$, then $a \in \beta(\mathcal{B}(B_1)) \cap \beta(\mathcal{B}(B_1)) = \beta(\mathcal{B}(B_1) \wedge \mathcal{B}(B_2))$, and $\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$ by (FB2), hence $a \in \beta(\mathcal{B}((B_1 - \{x\}) \cup \{y\}))$ for some $y \in B_2 - B_1$, i.e. $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}_{(a)}$ for some $y \in B_2 - B_1$. This means that $\mathcal{B}_{(a)}$ satisfies (B2).

(2) $\Rightarrow$ (1). By (B1), for each $a \in \beta(\top)$, $B \in \mathcal{B}_{(a)}$ for some $B \in 2^E$, thus $a \in \beta(\mathcal{B}(B)) \subseteq \beta\left(\bigvee_{B \in 2^E} \mathcal{B}(B)\right)$, hence $\bigvee_{B \in 2^E} \mathcal{B}(B) = \bigvee \beta\left(\bigvee_{B \in 2^E} \mathcal{B}(B)\right) \geq \bigvee_{a \in \beta(\top)} a = \top$, i.e. (FB1) holds. Let $B_1, B_2 \in 2^E$, $\mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \neq \perp$, $x \in B_1 - B_2$, and $a \in \beta(\mathcal{B}(B_1) \wedge \mathcal{B}(B_2))$. Then $a \in \beta(\mathcal{B}(B_1))$ and $a \in \beta(\mathcal{B}(B_2))$, i.e. $B_1, B_2 \in \mathcal{B}_{(a)}$. Hence there exists $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}_{(a)}$ by $\mathcal{B}_{(a)}$ is a crisp base on E . This shows that $a \in \beta(\mathcal{B}((B_1 - \{x\}) \cup \{y\})) \subseteq \beta\left(\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\})\right)$. Therefore, $\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$. This means that (FB2) holds. $\square$

Theorem 3.7. *Let E be a finite set, Then the following conditions are equivalent:*

- (1) $\mathcal{B}$ is a fuzzifying base on E ;
- (2) For each $a \in (0, 1]$, $\mathcal{B}_{[a]}$ is a crisp base on E ;
- (3) For each $a \in [0, 1)$, $\mathcal{B}_{(a)}$ is a crisp base on E .

4. The relation between M -fuzzifying bases and M -fuzzifying matroids

Definition 4.1. *Let $(E, \mathcal{I})$ be an M -fuzzifying matroid on E , if it satisfies $\text{Max}(\mathcal{I}_{[b]}) \subseteq \text{Max}(\mathcal{I}_{[a]})$ for every $a, b \in J(M)$ and $a \leq b$, then it is called a basic M -fuzzifying matroid.*

Remark 4.2. *Some M -fuzzifying matroids are not basic M -fuzzifying matroids. For example, let $E = \{x, y, z\}$ and $M = [0, 1]$. Define $\mathcal{I} : 2^E \rightarrow M$ by*

$$\mathcal{I}(A) = \begin{cases} 1, & A \in \{\emptyset, \{x\}, \{y\}\}; \\ \frac{1}{2}, & A \in \{\{x, y\}\}; \\ \frac{1}{3}, & A \in \{\{z\}, \{x, z\}\}; \\ 0, & A \in \{\{y, z\}, E\}. \end{cases}$$

Then

$$\mathcal{I}_{[r]} = \begin{cases} \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}\}, & r \in (0, \frac{1}{3}]; \\ \{\emptyset, \{x\}, \{y\}, \{x, y\}\}, & r \in (\frac{1}{3}, \frac{1}{2}]; \\ \{\emptyset, \{x\}, \{y\}\}, & r \in (\frac{1}{2}, 1]. \end{cases}$$

Hence $(E, \mathcal{I})$ is an *M-fuzzifying matroid*, but it is not a basic *M-fuzzifying matroid* since $\text{Max}(\mathcal{I}_{[1]}) \not\subseteq \text{Max}(\mathcal{I}_{[\frac{1}{2}]})$.

Theorem 4.3. *Let $\mathcal{B}$ be an *M-fuzzifying base* on E . Define a map $\mathcal{I}_{\mathcal{B}} : 2^E \rightarrow M$ by*

$$\mathcal{I}_{\mathcal{B}}(A) = \bigvee_{A \subseteq B} \mathcal{B}(B).$$

*Then $\mathcal{I}_{\mathcal{B}}$ is a basic *M-fuzzifying matroid* on E .*

Proof. For every $a \in J(M)$, $A \in (\mathcal{I}_{\mathcal{B}})_{[a]} \Leftrightarrow \bigvee_{A \subseteq B} \mathcal{B}(B) \geq a \Leftrightarrow A \subseteq B$ and $\mathcal{B}(B) \geq a$ for some $B \in 2^E \Leftrightarrow A \subseteq B$ and $B \in \mathcal{B}_{[a]}$ for some $B \in 2^E \Leftrightarrow A \in \text{Low}(\mathcal{B}_{[a]})$. Therefore, $(\mathcal{I}_{\mathcal{B}})_{[a]} = \text{Low}(\mathcal{B}_{[a]})$ and $\text{Max}((\mathcal{I}_{\mathcal{B}})_{[a]}) = \mathcal{B}_{[a]}$. By Theorem 3.3, $(E, (\mathcal{I}_{\mathcal{B}})_{[a]})$ is a crisp matroid for every $a \in J(M)$. Therefore, $\mathcal{I}_{\mathcal{B}}$ is a basic *M-fuzzifying matroid* on E . $\square$

When M is a boolean algebra, we have Theorems 4.4-4.6.

Theorem 4.4. *Let $(E, \mathcal{I})$ be an *M-fuzzifying matroid*. Define a map $\mathcal{B}_{\mathcal{I}} : 2^E \rightarrow M$ by*

$$\mathcal{B}_{\mathcal{I}}(B) = \bigwedge_{B \subset A} (\mathcal{I}(A))' \wedge \mathcal{I}(B).$$

*Then $\mathcal{B}_{\mathcal{I}}$ is an *M-fuzzifying base*.*

Proof. $\forall a \in J(M)$, $B \in (\mathcal{B}_{\mathcal{I}})_{[a]} \Leftrightarrow (\mathcal{B}_{\mathcal{I}})(B) \geq a \Leftrightarrow (\mathcal{I}(A))' \wedge \mathcal{I}(B) \geq a$ for every $A \supset B \Leftrightarrow \mathcal{I}(B) \geq a$ and $(\mathcal{I}(A))' \geq a$ for every $A \supset B \Leftrightarrow \mathcal{I}(B) \geq a$ and $\mathcal{I}(A) \not\geq a$ for every $A \supset B \Leftrightarrow B \in \mathcal{I}_{[a]}$ and $A \notin \mathcal{I}_{[a]}$ for every $A \supset B \Leftrightarrow B \in \text{Max}(\mathcal{I}_{[a]})$. Hence $(\mathcal{B}_{\mathcal{I}})_{[a]} = \text{Max}(\mathcal{I}_{[a]})$ ($\forall a \in J(M)$), and $\mathcal{B}_{\mathcal{I}}(B) = \bigvee \{a \in J(M) : B \in \text{Max}(\mathcal{I}_{[a]})\}$. By Theorem 3.3, $\mathcal{B}_{\mathcal{I}}$ is an *M-fuzzifying base*. $\square$

Theorem 4.5. (1) *For an *M-fuzzifying matroid* $(E, \mathcal{I})$, it follows that $\mathcal{I}_{\mathcal{B}_{\mathcal{I}}} = \mathcal{I}$;*

(2) *For an *M-fuzzifying base* $\mathcal{B}$, it follows that $\mathcal{B}_{\mathcal{I}_{\mathcal{B}}} = \mathcal{B}$.*

Proof. (1) For every $a \in J(M)$, by Theorem 4.3 and Theorem 4.4, $(\mathcal{I}_{\mathcal{B}_{\mathcal{I}}})_{[a]} = \text{Low}((\mathcal{B}_{\mathcal{I}})_{[a]}) = \text{Low}(\text{Max}(\mathcal{I}_{[a]})) = \mathcal{I}_{[a]}$.

(2) For every $a \in J(M)$, by Theorem 4.3 and Theorem 4.4, $(\mathcal{B}_{\mathcal{I}_{\mathcal{B}}})_{[a]} = \text{Max}((\mathcal{I}_{\mathcal{B}})_{[a]}) = \text{Max}(\text{Low}(\mathcal{B}_{[a]})) = \mathcal{B}_{[a]}$. $\square$

By Theorems 4.3-4.5, we can obtain the following two results:

Theorem 4.6. *There is a one-to-one correspondence between M -fuzzifying matroids and M -fuzzifying bases. That is, an M -fuzzifying matroid can be completely characterized by an M -fuzzifying base.*

Remark 4.7. *When M is a boolean algebra, an M -fuzzifying matroid is equivalent to a basic M -fuzzifying matroid.*

When $M = [0, 1]$, we have Theorems 4.8-4.10.

Theorem 4.8. *Let $(E, \mathcal{I})$ be a basic fuzzifying matroid. Define a map $\mathcal{B}_{\mathcal{I}} : 2^E \rightarrow [0, 1]$ by*

$$\mathcal{B}_{\mathcal{I}}(B) = \bigvee \{a \in (0, 1] : B \in \text{Max}(\mathcal{I}_{[a]})\}.$$

Then $\mathcal{B}_{\mathcal{I}}$ is a fuzzifying base on E .

Proof. For each $a \in (0, 1]$, obviously, $\text{Max}(\mathcal{I}_{[a]}) \subseteq (\mathcal{B}_{\mathcal{I}})_{[a]}$. $B \in (\mathcal{B}_{\mathcal{I}})_{[a]} \Leftrightarrow \bigvee \{a \in (0, 1] : B \in \text{Max}(\mathcal{I}_{[a]})\} \geq a \Leftrightarrow a \leq b$ for some $b \in (0, 1]$ and $B \in \text{Max}(\mathcal{I}_{[b]})$. As $(E, \mathcal{I})$ is a basic fuzzifying matroid, $B \in \text{Max}(\mathcal{I}_{[b]}) \subseteq \text{Max}(\mathcal{I}_{[a]})$. This means that $(\mathcal{B}_{\mathcal{I}})_{[a]} \subseteq \text{Max}(\mathcal{I}_{[a]})$. Therefore, $(\mathcal{B}_{\mathcal{I}})_{[a]} = \text{Max}(\mathcal{I}_{[a]})$. By Theorem 3.3, $\mathcal{B}_{\mathcal{I}}$ is a fuzzifying base on E . $\square$

Theorem 4.9. (1) *For a basic fuzzifying matroid $(E, \mathcal{I})$, it follows that $\mathcal{I}_{\mathcal{B}_{\mathcal{I}}} = \mathcal{I}$;*

(2) *For a fuzzifying base $\mathcal{B}$, it follows that $\mathcal{B}_{\mathcal{I}_{\mathcal{B}}} = \mathcal{B}$.*

Proof. (1) For every $a \in (0, 1]$, by Theorem 4.3 and Theorem 4.8, $(\mathcal{I}_{\mathcal{B}_{\mathcal{I}}})_{[a]} = \text{Low}((\mathcal{B}_{\mathcal{I}})_{[a]}) = \text{Low}(\text{Max}(\mathcal{I}_{[a]})) = \mathcal{I}_{[a]}$.

(2) For every $a \in (0, 1]$, by Theorem 4.3 and Theorem 4.8, $(\mathcal{B}_{\mathcal{I}_{\mathcal{B}}})_{[a]} = \text{Max}((\mathcal{I}_{\mathcal{B}})_{[a]}) = \text{Max}(\text{Low}(\mathcal{B}_{[a]})) = \mathcal{I}_{[a]}$. $\square$

By Theorem 4.3, Theorem 4.8 and Theorem 4.9, we can obtain the following result:

Theorem 4.10. *There is a one-to-one correspondence between basic fuzzifying matroids and fuzzifying bases. That is, a basic fuzzifying matroid can be completely characterized by a fuzzifying base.*

5. The dual matroids of basic M -fuzzifying matroids

In this section, M is a boolean algebra or $M = [0, 1]$.

Definition 5.1. Let $(E, \mathcal{I})$ be a basic M -fuzzifying matroid. Define a map $(\mathcal{B}_{\mathcal{I}})^* : 2^E \rightarrow M$ by $(\mathcal{B}_{\mathcal{I}})^*(B) = \mathcal{B}_{\mathcal{I}}(E - B)$ and let $\mathcal{I}^* = \mathcal{I}_{(\mathcal{B}_{\mathcal{I}})^*}$, then $(\mathcal{B}_{\mathcal{I}})^*$ is an M -fuzzifying base on E and $(E, \mathcal{I}^*)$ is a basic M -fuzzifying matroid. $(E, \mathcal{I}^*)$ is called the dual matroid of $(E, \mathcal{I})$.

Proof. By the definition of $(\mathcal{B}_{\mathcal{I}})^*$, $(\mathcal{B}_{\mathcal{I}})^*_{[a]} = \text{Com}((\mathcal{B}_{\mathcal{I}})_{[a]})$ for every $a \in J(M)$. Therefore, for every $a \in J(M)$, $(\mathcal{B}_{\mathcal{I}})^*_{[a]}$ is a crisp base on E . By Theorem 3.3, $(\mathcal{B}_{\mathcal{I}})^*$ is an M -fuzzifying base on E . By Theorem 4.3, $(E, \mathcal{I}^*)$ is a basic M -fuzzifying matroid. $\square$

Theorem 5.2. Let $(E, \mathcal{I})$ be a basic M -fuzzifying matroid, then

- (1) $(\mathcal{I}^*)^* = \mathcal{I}$;
- (2) $\text{Com}((\mathcal{B}_{\mathcal{I}})_{[a]}) = (\mathcal{B}_{\mathcal{I}^*})_{[a]}$ for every $a \in J(M)$;
- (3) $\mathcal{I}^*_{[a]} = (\mathcal{I}_{[a]})^*$ for every $a \in J(M)$.

Proof. (1) By the definition of $\mathcal{I}^*$, Theorem 4.5 and Theorem 4.9, $(\mathcal{I}^*)^* = (\mathcal{I}_{(\mathcal{B}_{\mathcal{I}})^*})^* = \mathcal{I}_{(\mathcal{B}_{\mathcal{I}})^*} = \mathcal{I}_{(\mathcal{B}_{\mathcal{I}})^*} = \mathcal{I}_{\mathcal{B}_{\mathcal{I}}} = \mathcal{I}$.

(2) By Theorem 4.5, Theorem 4.9 and the definition of $(\mathcal{B}_{\mathcal{I}})^*$, for every $a \in J(M)$, $(\mathcal{B}_{\mathcal{I}^*})_{[a]} = (\mathcal{B}_{\mathcal{I}})^*_{[a]} = \text{Com}((\mathcal{B}_{\mathcal{I}})_{[a]})$.

(3) For every $a \in J(M)$, $A \in \mathcal{I}^*_{[a]} \Leftrightarrow A \in \text{Low}((\mathcal{B}_{\mathcal{I}})^*_{[a]}) \Leftrightarrow A \in \text{Low}(\text{Com}((\mathcal{B}_{\mathcal{I}})_{[a]})) \Leftrightarrow A \in \text{Low}(\text{Com}(\text{Max}(\mathcal{I}_{[a]}))) \Leftrightarrow A \in (\mathcal{I}_{[a]})^*$. $\square$

Theorem 5.3. Let $(E, \mathcal{I})$ be a basic M -fuzzifying matroid, for every $A \subseteq E$, then

- (1) $R_{\mathcal{I}^*}(A) + R_{\mathcal{I}}(E) = \underline{|A|} + R_{\mathcal{I}}(E - A)$;
- (2) $R_{\mathcal{I}}(A) + R_{\mathcal{I}^*}(E) = \underline{|A|} + R_{\mathcal{I}^*}(E - A)$.

Proof. (1) $\forall a \in J(M)$, $(R_{\mathcal{I}^*}(A) + R_{\mathcal{I}}(E))_{[a]} = (R_{\mathcal{I}^*}(A))_{[a]} + (R_{\mathcal{I}}(E))_{[a]} = R_{\mathcal{I}^*_{[a]}}(A) + R_{\mathcal{I}_{[a]}}(E) = R_{(\mathcal{I}_{[a]})^*_{[a]}}(A) + R_{\mathcal{I}_{[a]}}(E) = |A| + R_{\mathcal{I}_{[a]}}(E - A) = (\underline{|A|})_{[a]} + R_{\mathcal{I}}(E - A)_{[a]} = (\underline{|A|} + R_{\mathcal{I}}(E - A))_{[a]}$.

(2) By (1) and Theorem 5.2(1), $R_{\mathcal{I}}(A) + R_{\mathcal{I}^*}(E) = R_{(\mathcal{I}^*)^*}(A) + R_{\mathcal{I}^*}(E) = \underline{|A|} + R_{\mathcal{I}^*}(E - A)$. $\square$

References

- [1] P. Dwinger, *Characterizations of the complete homomorphic images of a completely distributive complete lattice I*, Indagationes Mathematicae (Proceedings) 85, pp. 403-414, (1982).
- [2] G. Gierz, et al., *Continuous Lattices and Domains*, Cambridge University Press, Cambridge, (2003).
- [3] R. Goetschel, W. Voxman, *Bases of fuzzy matroids*, Fuzzy Sets and Systems 31, pp. 253-261, (1989).
- [4] H.-L. Huang, F.-G. Shi, *M-fuzzy numbers and their properties*, Information Sciences 178, pp. 1141-1151, (2008).
- [5] H.-J. Lai, *Matroid Theory*, Higher Education Press, Beijing (in Chinese), (2002).
- [6] C. V. Negoita, D. A. Ralescu, *Applications of Fuzzy Sets to Systems Analysis*, Interdisciplinary Systems Research Series, vol. 11, Birkhaeuser, Basel, Stuttgart and Halsted Press, New York, (1975).
- [7] J. G. Oxley, *Matroid Theory*, Oxford University Press, New York, (1992).
- [8] F.-G. Shi, *Theory of L_β -nested sets and L_α -nested sets and its applications*, Fuzzy Systems and Mathematics 4, pp. 65-72 (in Chinese), (1995).
- [9] F.-G. Shi, *M-fuzzy sets and prime element nested sets*, J. Mathematical Research and Exposition 16, pp. 398-402 (in Chinese), (1996).
- [10] F.-G. Shi, *Theory of molecular nested sets and its applications*, J. Yantai Teachers University (Natural Science) 1, pp. 33-36 (in Chinese), (1996).
- [11] F.-G. Shi, *M-fuzzy relation and M-fuzzy subgroup*, J. Fuzzy Mathematics 8, pp. 491-499, (2000).
- [12] F.-G. Shi, *A new approach to the fuzzification of matroids*, Fuzzy Sets and Systems 160, pp. 696-705, (2009).

- [13] F.-G. Shi, (L, M) -fuzzy matroids, Fuzzy Sets and Systems 160, pp. 2387-2400, (2009).
- [14] G.-J. Wang, *Theory of topological molecular lattices*, Fuzzy Sets and Systems 47, pp. 351-376, (1992).
- [15] D. J. A. Welsh, *Matroid Theory*, Oxford University Press, New York, (1976).

Xiu Xin

Department of Mathematics
Beijing Institute of Technology
Beijing, 100081,
P. R. China
e-mail : xinxiu518@163.com;

and

Fu-Gui Shi

Department of Mathematics
Beijing Institute of Technology
Beijing, 100081,
P. R. China
e-mail : fuguishi@bit.edu.cn