Abstract

The idea of difference sequence spaces was introduced by Kizmaz [6], and this concept was generalized by Bektas and Colak [1]. In this paper, we define the sequence spaces $c_0(F, \Delta^m x)$ and $l_\infty(F, \Delta^m x)$, where $F = (f_k)$ is a sequence of modulus functions, and examine some inclusion relations and properties of these spaces.

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1. Definitions and notations

Let $w$ denote the set of all complex sequences $x = (x_k)$, and $l_\infty$, $c$, and $c_0$ be the linear spaces of bounded, convergent, and null sequences with complex terms, respectively, normed by $\| x \| = \sup_k | x_k |$, where $k \in \mathbb{N}$, the set of positive integers.

Kizmaz [6] defined the sequence spaces
\[ l_\infty(\Delta) = \{ x \in w : \Delta x \in l_\infty \}, \]
\[ c(\Delta) = \{ x \in w : \Delta x \in c \}, \] and
\[ c_0(\Delta) = \{ x \in w : \Delta x \in c_0 \}, \] where for any sequence $x = (x_k)$, the difference sequence $\Delta x$ is defined by
\[ \Delta x = (\Delta x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty, \]
\[ \Delta^0 x = (\Delta^0 x_k)_{k=1}^\infty = (x_k)_{k=1}^\infty. \]

Kizmaz [6] proved that these are Banach spaces with the norm $\| x \| = | x_1 | + \| x \|_\infty$. Also, he showed that $E \subset E(\Delta)$, where $E = \{ l_\infty, c, c_0 \}$, since there exists a sequence $x = (x_k)$ such that $x_k = k$, for each $k$, that is, $x = (1, 2, 3, \ldots)$ for which $\Delta x = (-1, -1, -1, \ldots)$, so that although $x$ is not convergent, but it is $\Delta$-convergent.

If $m$ is a nonnegative integer and $x = (x_k)$ is any sequence, then the difference sequences $\Delta x, \Delta^2 x, \ldots, \Delta^m x$ are defined by
\[ \Delta x = (\Delta x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty, \]
\[ \Delta^2 x = \Delta(\Delta x) = \Delta x_k - \Delta x_{k+1}, \]
\[ \vdots \]
\[ \Delta^m x = \Delta(\Delta^{m-1} x) = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}, \]
so that
\[ \Delta^m x_k = \sum_{r=0}^{m} (-1)^r \binom{m}{r} x_{k+r}. \]

Let $U$ be the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ for each $k$. Then Gnanaseelan and Srivastava [4] defined and studied the sequence spaces
\[ E(u, \Delta) = \{ x \in w : u \Delta x \in E \}, \]
where $E = \{ l_\infty, c, c_0 \}$ and $u \Delta x = u_k x_k - u_{k+1} x_{k+1}$.

After then, Et. and Colak [2] defined the sequence spaces
\[ E(\Delta^m) = \{ x \in w : \Delta^m x \in E \}, \]
where $E = \{ l_\infty, c, c_0 \}$.

They proved that these are Banach spaces with the norm $\| x \| = \sum_{i=1}^{m} | x_i | + \| \Delta^m x \|_\infty$.

We recall that a modulus function $f$ is a function from $[0, \infty)$ to $[0, \infty)$ such that
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(i) \( f(x) = 0 \) if and only if \( x = 0 \),
(ii) \( f(x + y) \leq f(x) + f(y) \) for all \( x, y \geq 0 \),
(iii) \( f \) is increasing,
(iv) \( f \) is continuous from the right at 0.

Since \( |f(x) - f(y)| \leq f(|x - y|) \), it follows from conditions (ii) and (iv) that \( f \) is continuous everywhere on \([0, \infty)\).

A modulus function may be bounded or unbounded. For example, \( f(t) = \frac{t}{t+1} \) is bounded but \( f(t) = t^p \) \((0 < p \leq 1)\) is unbounded. Furthermore, we have \( f(nx) \leq nf(x) \), from condition (ii), and so \( f(x) = f(nx\frac{1}{n}) \leq n f(x) \frac{1}{n} \), hence \( \frac{1}{n} f(x) \leq f(x) \), for all \( n \in \mathbb{N} \).

Ruckle [13] used the concept of modulus function to define the sequence space

\[ L(f) = \{ x \in w : \sum_{k=1}^{\infty} f(\{|x_k|\}) < \infty \}. \]

If we take \( f(x) = x^p \), then \( L(f) \) reduces to the familiar space \( l_p \) which is given by

\[ l_p = \{ x \in w : \sum_{k=1}^{\infty} |x_k|^p < \infty \}. \]

In particular, if \( f(x) = x \), then \( L(f) = l_1 \) which is given by

\[ l_1 = \{ x \in w : \sum_{k=1}^{\infty} |x_k| < \infty \}. \]

Several authors including Maddox ([8]-[11]), Ozturk and Bilgin [12], and some others studied some sequence spaces defined by a modulus function. Let \( X \) be a sequence space. Then the sequence space \( X(f) \) is defined by

\[ X(f) = \{ x \in w : f(|x_k|) \in X \}. \]

Kolk [7] gave an extension of \( X(f) \) by considering a sequence of modulus functions \( F = (f_k) \) and defined the space

\[ X(F) = \{ x \in w : f_k(|x_k|) \in X \}. \]

Gaur and Mursaleen [3] defined and studied the following sequence spaces

\[ l_\infty(F, \Delta) = \{ x \in w : \Delta x \in l_\infty(F) \}, \]

and

\[ c_0(F, \Delta) = \{ x \in w : \Delta x \in c_0(F) \}. \]

For a nonnegative integer \( m \), Bektas and Colak [1] extended the above mentioned spaces to

\[ l_\infty(F, \Delta^m) = \{ x \in w : \Delta^m x \in l_\infty(F) \}, \]

and

\[ c_0(F, \Delta^m) = \{ x \in w : \Delta^m x \in c_0(F) \}. \]

We further give an extension of the spaces of Bektas and Colak [1] and define the sequence spaces...
\[ l_\infty(F, \Delta^m_u) = \{ x \in w : \Delta^m_u x \in l_\infty(F) \}, \]

and

\[ c_0(F, \Delta^m_u) = \{ x \in w : \Delta^m_u x \in c_0(F) \}, \]

where \( u = (u_k) \) is any sequence such that \( u_k \neq 0 \) for each \( k \), and

\[ \Delta^0_u x = u_k x_k, \]

\[ \Delta^1_u x = u_k x_k - u_{k+1} x_{k+1}, \]

\[ \Delta^2_u x = \Delta(\Delta^1_u x), \]

\[ \vdots \]

\[ \Delta^m_u x = \Delta(\Delta^{m-1}_u x), \]

so that

\[ \Delta^m_u x = \Delta^m_{u_k} x_k = \sum_{r=0}^{m} (-1)^r \binom{m}{r} u_{k+r} x_{k+r}. \]

If \( u = e = (1, 1, 1, \cdots) \), then these spaces will give the spaces of Bektas and Colak [1] as special cases.

2. Main results

For a sequence \( F = (f_k) \) of modulus functions, we will give the necessary and sufficient conditions for the inclusion between \( X(\Delta^m_u) \) and \( Y(F, \Delta^m_u) \), where \( X, Y = l_\infty \) or \( c_0 \).

We need the following Lemmas (see Kolk [7]):

**Lemma 2.1.** The condition \( \sup_k f_k(t) < \infty, t > 0 \) holds if and only if there exists a point \( t_0 > 0 \) such that \( \sup_k f_k(t_0) < \infty \).

**Lemma 2.2.** The condition \( \inf_k f_k(t) > 0, t > 0 \) holds if and only if there exists a point \( t_0 > 0 \) such that \( \inf_k f_k(t_0) > 0 \).

Now, we prove the following theorems

**Theorem 2.3.** For a sequence \( F = (f_k) \) of modulus functions, the following statements are equivalent:

(i) \( l_\infty(\Delta^m_u) \subseteq l_\infty(F, \Delta^m_u) \)

(ii) \( c_0(\Delta^m_u) \subseteq l_\infty(F, \Delta^m_u) \)

(iii) \( \sup_k f_k(t) < \infty, t > 0 \)
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Proof. (i) implies (ii) is obvious

(ii) implies (iii) : Let \( c_0(\Delta_u^m) \subseteq l_\infty(F, \Delta_u^m) \). Suppose that (iii) is not true. Then by Lemma 2.1, we see that \( \sup_k f_k(t) = \infty \), for all \( t > 0 \), and therefore there is a sequence \((k_i)\) of positive integers such that

\[
(f_{k_i}(\frac{1}{i})) > i \quad \text{for} \quad i = 1, 2, 3, \ldots
\]

(2.1)

Define \( x = (x_k) \) as follows

\[
x_k = \begin{cases} 
\frac{1}{i}, & k = k_i \\
0, & \text{otherwise}
\end{cases}
\]

Then \( x \in c_0(\Delta_u^m) \), but by (2.1), \( x \notin l_\infty(F, \Delta_u^m) \) which contradicts (ii). Hence (iii) must hold.

(iii) implies (i) : Let (iii) be satisfied and \( x \in l_\infty(\Delta_u^m) \). Suppose that \( x \notin l_\infty(F, \Delta_u^m) \), then \( \inf_k f_k(t) = 0 \) for \( t > 0 \). Now take \( t = |\Delta_u^m x| \), then \( \sup_k f_k(t) = \infty \) which contradicts (ii). Hence \( l_\infty(\Delta_u^m) \subseteq l_\infty(F, \Delta_u^m) \). This completes the proof of the theorem. □

Theorem 2.4. For a sequence \( F = (f_k) \) of modulus functions, the following statements are equivalent :

(i) \( c_0(F, \Delta_u^m) \subseteq c_0(\Delta_u^m) \)

(ii) \( c_0(F, \Delta_u^m) \subseteq l_\infty(\Delta_u^m) \)

(iii) \( \inf_k f_k(t) > 0 \), \( (t > 0) \)

Proof. (i) implies (ii) is obvious

(ii) implies (iii) : Let \( c_0(F, \Delta_u^m) \subseteq l_\infty(\Delta_u^m) \). Suppose that (iii) does not hold. Then by Lemma 2.2, we see that

\[
(2.2) \quad \inf_k f_k(t) = \infty, \quad (t > 0)
\]

and therefore there is a sequence \((k_i)\) of positive integers such that \( f_{k_i}(i^2) < \frac{1}{i} \) for \( i = 1, 2, 3, \ldots \). Define \( x = (x_k) \) as follows

\[
x_k = \begin{cases} 
i^2, & k = k_i \quad \text{for} \quad i = 1, 2, 3, \ldots \\
0, & \text{otherwise}
\end{cases}
\]

Then by (2.2), \( x \in c_0(F, \Delta_u^m) \) but \( x \notin l_\infty(\Delta_u^m) \) which contradicts (ii). Hence (iii) must hold.

(iii) implies (i) : Let (iii) be satisfied and \( x \in c_0(F, \Delta_u^m) \) that is \( \lim_k f_k(|\Delta_u^m x_k|) = 0 \). Suppose that \( x \notin c_0(\Delta_u^m) \), then for some number \( \varepsilon_0 > 0 \) and positive integer \( k_0 \), we have \( |\Delta_u^m x_k| \leq \varepsilon_0 > 0 \) for \( k \geq k_0 \). Therefore \( f_k(\varepsilon_0) \leq \)
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\[ f_k(\Delta^m u x_k) \] for \( k \geq k_0 \) and hence \( \lim_k f_k(\varepsilon_0) = 0 \) which contradicts (iii). Thus \( c_0(F, \Delta^m u) \subseteq c_0(\Delta^m u) \) and this completes the proof of the theorem.

**Theorem 2.5.** The inclusion \( l_\infty(F, \Delta^m u) \subseteq c_0(\Delta^m u) \) holds if and only if

\[
\lim_k f_k(t) = \infty \text{ for } t > 0
\]  

**Proof.** Let \( l_\infty(F, \Delta^m u) \subseteq c_0(\Delta^m u) \) and suppose that (2.3) does not hold. Then there is a number \( t_0 > 0 \) and a sequence \( (k_i) \) of positive integers such that

\[
f_{k_i}(t_0) \leq L < \infty
\]  

Define the sequence \( x = (x_k) \) by

\[
x_k = \begin{cases} 
  t_0, & k = k_i \text{ for } i = 1, 2, 3, \ldots \\
  0, & \text{otherwise}
\end{cases}
\]

Then \( x \in l_\infty(F, \Delta^m u) \) by (2.4) but \( x \notin c_0(\Delta^m u) \) so that (2.4) must hold if \( l_\infty(F, \Delta^m u) \subseteq c_0(\Delta^m u) \).

Conversely, Suppose that (2.3) is satisfied. If \( x \in l_\infty(F, \Delta^m u) \), then \( f_k(| \Delta^m u x_k |) \leq L < \infty \) for \( k = 1, 2, 3, \ldots \)

Now suppose that \( x \notin c_0(\Delta^m u) \). Then for some \( \varepsilon_0 > 0 \) and positive integer \( k_0 \), we have \( | \Delta^m u x_k | > \varepsilon_0 \) for \( k \geq k_0 \). Therefore \( f_k(\varepsilon_0) \leq f_k(| \Delta^m u x_k |) \leq L \) for \( k \geq k_0 \) which contradicts (2.3). Hence \( x \in c_0(\Delta^m u) \).

**Theorem 2.6.** The inclusion \( l_\infty(\Delta^m u) \subseteq c_0(F, \Delta^m u) \) holds if and only if

\[
\lim_k f_k(t) = 0 \text{ for } t > 0
\]  

**Proof.** Let \( l_\infty(\Delta^m u) \subseteq c_0(F, \Delta^m u) \) and suppose that (2.5) does not hold.

Then

\[
\lim_k f_k(t) = l \neq 0
\]  

for some \( t_0 > 0 \).

Define the sequence \( x = (x_k) \) by
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$x_k = t_0 \sum_{r=0}^{m} (-1)^m \binom{m+k-u-1}{k-u}$ for $k = 1, 2, 3, \cdots$

Then $x \not\in c_0(F, \Delta^m_u)$ by (2.6). Hence (2.5) must hold.

Conversely, Suppose that (2.5) is satisfied. If $x \in l_{\infty}(\Delta^m_u)$, then $|\Delta^m_u x_k| \leq L < \infty$ for $k = 1, 2, 3, \cdots$.

Therefore $f_k (|\Delta^m_u x_k|) \leq f_k (L)$ for $k = 1, 2, 3, \cdots$ and $\lim_k f_k (|\Delta^m_u x_k|) \leq \lim_k f_k (L) = 0$, by (2.5). Hence $x \in c_0(F, \Delta^m_u)$. □

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