ON HARMONIOUS COLORING OF TOTAL GRAPHS OF $C(C_n)$, $C(K_{1,n})$ AND $C(P_n)$

VERNOLD VIVIN J.
SRI SHAKTHI INSTITUTE OF ENGINEERING AND TECH., INDIA

AKBAR ALI M. M.
SRI SHAKTHI INSTITUTE OF ENGINEERING AND TECH., INDIA

and

K. KALIRAJ
R. V. S. COLLEGE OF ENGINEERING AND TECHNOLOGY, INDIA

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Abstract

In this paper, we present the structural properties of total graph of central graph of cycles $C_n$, star graphs $K_{1,n}$ and paths $P_n$ denoted by $T(C(C_n))$, $T(C(K_{1,n}))$ and $T(C(P_n))$ respectively. We mainly focus our discussion on the harmonious chromatic number of $T(C(C_n))$, $T(C(K_{1,n}))$ and $T(C(P_n))$.

Keywords : Central graph, total graph and harmonious coloring.

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1. Introduction

The central graph \([6, 7]\) \(C(G)\) of a graph \(G\) is formed by adding an extra vertex on each edge of \(G\), and then joining each pair of vertices of the original graph which were previously non-adjacent.

A harmonious coloring \([2, 4, 5]\) of a simple graph \(G\) is proper vertex coloring such that each pair of colors appears together on at most one edge. The harmonious chromatic number \(\chi_H(G)\) is the least number of colors in such a coloring.

The total graph of \(G\) has vertex set \(V(G) \cup E(G)\), and edges joining all elements of this vertex set which are adjacent or incident in \(G\). The terminology and notations used in this paper can be found in \([1, 3]\). The purpose of this paper serves to find the harmonious chromatic number for the total graph of central graph of \(C_n, K_1,n\) and \(P_n\).

2. Harmonious Coloring on Total Graph of Central Graph of Cycle \(C_n\)

In \(T(C(C_n))\), there are \(n\) vertices of degree 4, \(2n\) vertices of degree \((n+1)\), \(\binom{n}{2}\) vertices of degree \(2(n-1)\).

Therefore

- The number of vertices, \(p_{T(C(C_n))} = \frac{n^2 + 5n}{2}\).

- The number of edges, \(q_{T(C(C_n))} = \frac{n^3 + 7n}{2}\).

- \(\Delta = 2(n-1)\).

Theorem 2.1. The harmonious chromatic number of total graph of central graph of cycles \(C_n\):

\[
\chi_H(T(C(C_n))) = \begin{cases} 
\frac{(n+1)^2 + 2}{2} & \text{if } n \text{ is odd} \\
\frac{(n+1)^2 + 3}{2} & \text{if } n \text{ is even, } (n > 4).
\end{cases}
\]

Proof. Let \(V(C_n) = \{v_1, v_2, \cdots, v_n\}\). By the definition of central graph \(C(C_n)\) has the vertex set \(V(C_n) \cup \{u_i : 1 \leq i \leq n\}\) where \(u_i\) is a vertex of subdivision of the edge \(v_iv_{i+1}(1 \leq i \leq n-1)\) and \(u_n\) is a vertex of
subdivision of the edge \(v_nv_1\). Also let \(u_iv_i = e_i(1 \leq i \leq n)\) and \(u_nv_{i+1} = e'_i(1 \leq i \leq n-1)\) and \(u_nv_1 = e'_n\). Also let \(e_{ij} = v_iv_j\). Therefore \(E(C(C_n)) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{e_{ij} : 3 \leq i \leq n-1\} \cup \{e_{ij} : 2 \leq i \leq n-2, i+2 \leq j \leq n\}\).

Central Graph of Cycle \(C_n\)

![Central Graph of Cycle \(C_n\)](image)

Each \(v_i\) is incident with the edges \(e_i, e'_{i-1}, e_{ij} (i \neq j)\) and \((2 \leq i \leq n)\). Also \(v_1\) is incident with \(e_1, e'_1, e_{13}, e_{1n}, \ldots, e_{1(n-1)}\). i.e., Total number of incident edges with \(v_i\) is \((n - 1)\forall i = 1, 2, \ldots, n\). By the definition of total graph the edges incident with \(v_i\) together with the vertex \(v_i\) induces a clique of \(n\) vertices in \(T(C(C_n))(1 \leq i \leq n)\).

Let \(K^{(i)}_n\) be the cliques in \(T(C(C_n))(i = 1, 2, \ldots, n)\). Since \(e_{ij} = e_{ji}\), each \(K^{(i)}_n\) shares exactly \((n - 3)\) vertices with the remaining cliques. Also the vertices \(v_i\)'s are adjacent with \(u_i\)'s \((1 \leq i \leq n)\) and \(u_i\)'s are adjacent
with $v_{i+1}$ for every $i = 1, 2, \cdots n - 1$ in $T(C(C_n))$. 

![Graph](image)

**Figure 2**

Therefore in each clique, the harmonious coloring is performed by distinct $n$ colors. $K_n^{(1)}$ is assigned $n$ colors, where as since $K_n^{(2)}$ shares one vertex with $K_n^{(1)}$, it needs distinct $(n - 1)$ colors which are distinct from the set of colors assigned to $K_n^{(1)}$. Now since $K_n^{(3)}$ shares one vertex with $K_n^{(1)}$ and one with $K_n^{(2)}$, it needs only $(n - 2)$ colors and so on. Now we use induction.

**Case (i)**

If $n$ is odd,
If $n = 5, T(C(C_5))$ has the harmonious coloring as follows.

Assign 5 distinct colors to the vertices of $K_5^{(1)}$. One of these 5 colors can be assigned to the vertices of $K_5^{(2)} - \{v_2\}$ and hence 4 new colors are assigned to the vertices of $K_5^{(2)}$. Similarly 4 new colors are assigned to $K_5^{(3)} - \{e_{31}\}$ and 3 new colors are assigned to $K_5^{(4)} - \{e_{14}, e_{24}\}$, 3 new colors are assigned to the vertices of $K_5^{(5)} - \{e_{25}, e_{35}\}$. Hence the total number of minimum colors used is 19. The vertices $v_{12}, v_{23}, v_{34}, v_{45}$ and $v_{51}$ can be colored by
any one of the above said colors. Therefore $\chi_H(T(C_5)) = 19$.

Structure of Total Graph of Central Graph of Cycle $C_5$

**Figure 4**
By the above process the harmonious chromatic number of $T(C(C_5))$ is as follows

\[ \chi_H(T(C(C_5))) = \frac{(n+1)^2 + 2}{2} \]

when $n = 5$.

Assume that the result is valid for $C_n$, when $n$ is odd. i.e., $\chi_H(T(C(C_n))) = \frac{(n+1)^2 + 2}{2}$. Now consider $C_{n+2}$ by introducing two new vertices $v_{n+1}$ and $v_{n+2}$. Consider the incident edges of $v_{n+1}$ and $v_{n+2}$ in $C(C_{n+2})$. These edges together with the vertices $v_{n+1}$ and $v_{n+2}$ induces two more cliques of order $(n+2)$ in $T(C(C_n))$. The vertices $v_{n+1}, v_{n+2}, v_{n+1}', v_{n+2}', u_{n+1}, u_{n+2}, u_{n+1}', u_{n+2}'$ are assigned by 8 colors and the cliques $K_{n+2}^{(n+1)}$ and $K_{n+2}^{(n+2)}$ are assigned by
\((n-2)+(n-2) = (2n-4)\) colors. Therefore \(\chi_H(T(C(C_n))) = \frac{(n+1)^2 + 2}{2} +\)

\[2n-4+8 = \frac{(n+1)^2 + 2}{2} + (2n+4).\] Hence \(\chi_H(T(C(C_{n+2}))) = \frac{(n+3)^2 + 2}{2} .\)

Therefore by induction hypothesis \(\chi_H(T(C(C_n))) = \frac{(n+1)^2 + 2}{2} ,\) if \(n\) is odd.

**Case (ii)**

If \(n\) is even, we prove that, \(\chi_H(T(C(C_n))) = \frac{(n+1)^2 + 3}{2}.\)

For \(n = 6\), the total graph of \(C(C_6)\) has harmonious chromatic number 26.

Central Graph of Cycle \(C_6\)

**Figure 6**
The harmonious coloring of $T(C(C_6))$ is as follows. Assign 6 distinct colors to the vertices of $K_6^{(1)}$. One of these 6 colors can be assigned to the vertices of $K_6^{(2)} - \{v_2\}$ and hence 5 new colors are assigned to the vertices of $K_6^{(2)}$. Similarly 5 new colors are assigned to $K_6^{(3)} - \{e_{31}\}$, since $e_{31}$ has already been assigned a color and 4 new colors are assigned to $K_6^{(4)} - \{e_{14}, e_{24}\}$, 3 new colors are assigned to the vertices of $K_6^{(5)} - \{e_{51}, e_{52}, e_{53}\}$, 3 new colors are assigned to the vertices of $K_6^{(6)} - \{e_{62}, e_{63}, e_{64}\}$. Hence the total number of minimum colors used is 26. The vertices $v_{12}, v_{23}, v_{34}, v_{45}, v_{56}$ and $v_{61}$ can be colored by any one of the above said colors. Therefore $\chi_H(T(C(C_6))) = 26$. 

Structure of Total Graph of Central Graph of Cycle $C_6$

Figure 7
By the above process the harmonious chromatic number of $T(C(C_6))$ is as follows

\[ \chi_H(T(C(C_n))) = \frac{(n+1)^2 + 3}{2} \quad \text{if } n = 6. \]

Now assume that the result is valid for any even $n$. i.e., $\chi_H(T(C(C_n))) = \frac{(n+1)^2 + 3}{2}$ if $n$ is even. Now consider $C_{n+2}$ by introducing two new vertices $v_{n+1}$ and $v_{n+2}$ induce two more cliques of order $n+2$ in $T(C(C_n))$. The vertices $v_{n+1}$, $v_{n+2}$, $\epsilon_{n+1}$, $\epsilon_{n+2}$, $\mu_{n+1}$, $\mu_{n+2}$, $\epsilon'_{n+1}$, $\epsilon'_{n+2}$ are assigned by 8 colors and the cliques $K_{n+2}^{(n+1)}$ and $K_{n+2}^{(n+2)}$ are assigned by $(n-2) + (n-2) = (2n-4)$ colors. Therefore $\chi_H(T(C(C_{n+2}))) = \frac{(n+1)^2 + 3}{2} + 2n - 4 + 8 = \frac{(n+3)^2 + 3}{2}$. Hence $\chi_H(T(C(C_{n+2}))) = \frac{(n+2)(n+1)^2 + 3}{2}$. Therefore by induction hypothesis $\chi_H(T(C(C_n))) = \frac{(n+1)^2 + 3}{2}$, if $n$ is even. □

3. Harmonious Coloring on Total Graph of Central Graph of Star Graph $K_{1,n}$

In $T(C(K_{1,n}))$ there are $n$ vertices of degree 4, $2n$ vertices of degree $\frac{n^2 + n + 2}{2}$, $2n$ vertices of degree $(n + 2)$. 
Therefore

- The number of vertices, \( p_{T(C(K_1,n))} = \frac{n^2 + 7n + 2}{2} \).

- The number of edges, \( q_{T(C(K_1,n))} = \frac{n^3 + 3n^2 + 10n}{2} \).

- \( \Delta = 2n \).

**Theorem 3.1.** The harmonious chromatic number of total graph of central graph of star graph \( K_{1,n} \),

\[
\chi_H(T(C(K_1,n))) = \begin{cases} 
\frac{(n^2 + 3n + 8)}{2} & \text{if } n \text{ is even} \\
\frac{(n^2 + 3n + 6)}{2} & \text{if } n \text{ is odd}.
\end{cases}
\]

**Proof.** Let \( V(K_{1,n}) = \{v, v_1, v_2 \ldots, v_n \} \) where \( \text{deg } v = n \). By the definition of central graph of \( K_{1,n} \), we denote the vertices of subdivision by \( u_1, u_2, \ldots, u_n \). i.e., \( vv_i \) is subdivided by \( u_i(1 \leq i \leq n) \). Let \( e_i = v_iu_i \) and \( e'_i = vu_i(1 \leq i \leq n) \). Therefore \( V(C(K_{1,n})) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v\} \). By the definition of central graph the subgraph induced by the vertex set \( \{v_1, v_2 \ldots, v_n\} \) is \( K_n \) and let \( e_{ij} \) be the edge of \( C(K_{1,n}) \), connecting the vertex \( v_i \) and \( v_j(i < j) \).
Therefore $E(C(K_1,n)) = \{ e_i : 1 \leq i \leq n \} \cup \{ e'_i : 1 \leq i \leq n \} \cup \{ e_{ij} : 1 \leq i \leq n-1, i+1 \leq j \leq n \}$. By the definition of total graph, $V(T(C(K_1,n))) = V(C(K_1,n)) \cup E(C(K_1,n))$. The structure is described below.
The vertices $v, e'_1, e'_2, \cdots, e'_n$ induce a clique of order $(n+1)$ in its total graph. The vertices $u_i$ adjacent to $e'_1$ and $e_j (1 \leq i \leq n)$. Let $S_i = \{e_{ij} : j = 1, 2, \cdots i-1, i+1, \cdots, n\}, (1 \leq i \leq n)$. Clearly $S_i \cap S_j = \{e_{ij}\}$ if $i \neq j$ and let $S^{(n)} = \cup_{i=1}^n S_i$, clearly $|S^{(n)}| = \binom{n}{2}$. Now the vertices $v_i$ and $e'_i$ together with vertices of $S_i$ induce a clique of order $(n+1), (1 \leq i \leq n)$. Also the vertices $v_1, v_2, \cdots v_n$ induce a clique of order $n$. Now we prove that harmonious chromatic number of this graph by induction method.

**Case (i)**

If $n$ is even

Then we prove $\chi_H(T(C(K_{1,n}))) = \frac{n^2 + 3n + 8}{2}$.

If $n = 2$, then $C(K_{1,2})$ has 5 vertices. The harmonious coloring for the total graph of $C(K_{1,2})$, is shown below.
From the Fig.11, 9 is the minimum number of colors as the 5 triangles shares atleast one vertex in common. Therefore $\chi_H(T(C(K_{1,2}))) = 9 = \frac{n^2 + 3n + 8}{2}$, when $n = 2$. Therefore the result is true if $n = 2$.

Assume that the result is true for any even integer $n$ and we prove the same for $n+2$. i.e., $\chi_H(T(C(K_{1,n}))) = \frac{n^2 + 3n + 8}{2}$. Let $v_{n+1}, v_{n+2}$ be two non adjacent vertices introduced in $K_{1,n}$ which are adjacent to $v$. Let $u_{n+1}$ and $u_{n+2}$ be the vertices of subdivision in its centralization. Clearly by the structure given in Fig.10, the total graph $C(K_{1,n+2})$ has the following structural property. (i) There are $(n+3)$ cliques $K^{(1)}_{n+3}, K^{(2)}_{n+3}, \cdots, K^{(n+3)}_{n+3}$ of order $(n+3)$. (ii) There is a clique of order $(n+2)$. (iii) Each $K^{(i)}_{n+3}$ has exactly one vertex common with $K^{(j)}_{n+3}$ where $(2 \leq i \leq n+3), (2 \leq j \leq n+3)$ and $i \neq j$. By induction hypothesis the minimum number of colors for harmonious coloring in $T(C(K_{1,n})) = \frac{n^2 + 3n + 8}{2}$. By the above said structure of $T(C(K_{1,n})), |S^{(n)}| = |S_1 \cup S_2 \cup \cdots S_n| = \binom{n}{2} = \frac{n(n-1)}{2}$. Also the new
vertices in $T(C(K_{1,n+2}))$ are $e_{n+1}, e_{n+2}, u_{n+1}, u_{n+2}, e'_{n+1}, e'_{n+2}, v_{n+1}, v_{n+2}$. Therefore the total number of vertices in $T(C(K_{1,n+2})) = \frac{(n+2)(n+1) - n(n-1)}{2} + 8 = 2n + 1 + 8 = 2n + 9$. Now we find the minimal harmonious coloring in $T(C(K_{1,n+2}))$ as below. By induction hypothesis $T(C(K_{1,n}))$ has an harmonious coloring with the minimum number of $\frac{n^2 + 3n + 8}{2}$ colors. With this same color assigned to the vertices of $T(C(K_{1,n+2}))$, we assign some new colors to the remaining vertices as below. The vertices $u_{n+1}$ and $u_{n+2}$ are assigned the same color as in $u_i (1 \leq i \leq n)$. Then all the remaining vertices are assigned $2n + 5$ colors. Therefore $\chi_H(T(C(K_{1,n+2}))) = \frac{n^2 + 3n + 8}{2} + 2n + 5 = (n + 2)^2 + 3(n + 2) + 8$. By induction hypothesis $\chi_H(T(C(K_{1,n}))) = \frac{n^2 + 3n + 8}{2}$ if $n$ is even.

**Case (ii)**

If $n$ is odd, we prove $\chi_H(T(C(K_{1,n}))) = \frac{n^2 + 3n + 6}{2}$ by induction following the same procedure as above. □

4. Harmonious Coloring on Total Graph of Central Graph of Path $P_n$

In $T(C(P_n))$ there are $n - 1$ vertices of degree 4, $2n - 2$ vertices of degree $n + 1$, $\frac{n^2 - n + 2}{2}$ vertices of degree $2n - 2$.

Therefore

- The number of vertices, $p_{T(C(P_n))} = \frac{n^2 + 5n - 4}{2}$.
- The number of edges, $q_{T(C(P_n))} = \frac{n^3 + 7n^2 - 8}{2}$.
- $\Delta = 2n - 2$ for $n \geq 3$.

**Theorem 4.1.** The harmonious chromatic number of total graph of central graph of path $P_n$, $\chi_H(T(C(P_n))) = 4n-3$, where $n$ denotes the number of vertices in $P_n$. 
**Proof.** Let \( V(P_n) = \{v_1, v_2, \ldots, v_n\} \). On the process of centralization of \( P_n \), let \( u_i \) be the vertex of subdivision of the edges \( v_i v_{i-1} (1 \leq i \leq n) \). Also let \( v_i u_i = e_i \) and \( u_i v_{i+1} = e'_i(1 \leq i \leq n - 1) \). By the definition of central graph the non adjacent vertices \( v_i \) and \( v_j \) of \( P_n \), are adjacent in \( C(P_n) \) by the edge \( e_{ij} \).

Therefore \( V(C(P_n)) = \{v_i / 1 \leq i \leq n\} \cup \{u_i / 1 \leq i \leq n - 1\} \) and \( E(C(P_n)) = \{e_i : 1 \leq i \leq n - 1\} \cup \{e'_i : 1 \leq i \leq n - 1\} \cup \{e_{ij} : 1 \leq i \leq n - 2, i + 2 \leq j \leq n\} \). By the definition of total graph, \( V(T(C(P_n))) = V(C(P_n)) \cup E(C(P_n)) \). In \( C(P_n) \), each \( v_i \) is incident with the \((n - 1)\) edges \( e_i, e'_i, e_{ij} \) for \((2 \leq i \leq n - 1), (i \neq j)\), \( v_j \) is incident with the \((n - 1)\) edges \( e_1, e_{13}, e_{14}, \ldots e_{1n} \) and \( v_n \) is incident with \((n - 1)\) edges \( e_{n-1}, e_{2n}, e_{3n}, \ldots e_{(n-2)n}, e_{1n} \). The edges incident with \( v_i \) together with vertex \( v_i \) induces a clique of \( n \) vertices in \( T(C(P_n)) \) for \( i = 1, 2, \ldots n \).

Let \( K^{(i)}_n \) be the cliques in \( T(C(C_n))(i = 1, 2, \ldots n) \). Since \( e_{ij} = e_{ji} \), each \( K^{(i)}_n \) shares exactly \((n - 3)\) vertices with the remaining cliques. Also the vertices \( v_i \)'s are adjacent with \( u_i \)'s for \( i = 1, 2, \ldots, n - 1 \) and \( u_i \)'s are adjacent with \( v_{i+1} \) for \( i = 1, 2, \ldots, n - 1 \) in \( T(C(P_n)) \). Also \( v_i \) is adjacent with \( v_{i+2}, v_{i+3}, \ldots, v_n \) and \( v_{i-2}, v_{i-3}, \ldots, v_1 \).
In each of the above said clique, harmonious coloring is performed by distinct $n$ colors. $K_n^{(1)}$ is assigned $n$ colors, where as since $K_n^{(2)}$ shares one vertex with $K_n^{(1)}$, it needs distinct $(n - 1)$ colors which are distinct from the set of colors assigned to $K_n^{(1)}$. Now since $K_n^{(3)}$ shares one vertex with $K_n^{(1)}$ and one with $K_n^{(2)}$, it needs only $(n - 2)$ colors and so on. Now we use induction on $n$.

For $n = 1$, the results follows obviously. For $n = 2, C(P_2)$ is $P_3$ and $T(C(P_2))$ and its harmonious coloring is as follows.

Total Graph of Central Graph of Path $P_n$

Figure 14

Total Graph of Central Graph of Path $P_2$

Figure 15
Assume that the result is valid for any $n$, $\chi_H(T(C(P_n))) = 4n - 3$. Now consider $P_{n+1}$ by introducing a vertex $v_{n+1}$ and consider the incident edges of $v_{n+1}$ in $C(P_{n+2})$. These edges together with the vertex $v_{n+1}$ induces one more clique of order $n$ in $T(C(P_{n+1}))$ and it shares $n - 4$ vertices with other cliques. Therefore we need 4 colors for the harmonious coloring of $T(C(P_n))$. Therefore $\chi_H(T(C(P_{n+1}))) = 4n - 3 + 4 = 4n + 1 = 4(n + 1) - 3$. Hence by induction hypothesis $\chi_H(T(C(P_{n+1}))) = 4n - 3$. □

References


Vernold Vivin. J
Department of Mathematics
Sri Shakthi Institute of Engineering and Technology
Coimbatore- 641 062
e-mail: vernold_vivin@yahoo.com ; vernoldvivin@yahoo.in
Akbar Ali.M.M
Department of Mathematics
Sri Shakthi Institute of Engineering and Technology
Coimbatore- 641 062
e-mail : um_akbar@yahoo.co.in

and

K.Kaliraj
Department of Mathematics
R. V. S. College of Engineering and Technology
Coimbatore-641 402
e-mail : k_kaliraj@yahoo.com