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ON HARMONIOUS COLORING OF TOTAL GRAPHS OF $C(C_n)$, $C(K_{1,n})$ AND $C(P_n)$

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Abstract

In this paper, we present the structural properties of total graph of central graph of cycles C_n , star graphs $K_{1,n}$ and paths P_n denoted by $T(C(C_n))$, $T(C(K_{1,n}))$ and $T(C(P_n))$ respectively. We mainly focus our discussion on the harmonious chromatic number of $T(C(C_n))$, $T(C(K_{1,n}))$ and $T(C(P_n))$.

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1. Introduction

The central graph [6, 7] $C(G)$ of a graph G is formed by adding an extra vertex on each edge of G , and then joining each pair of vertices of the original graph which were previously non-adjacent.

A harmonious coloring [2, 4, 5] of a simple graph G is proper vertex coloring such that each pair of colors appears together on at most one edge. The harmonious chromatic number $\chi_H(G)$ is the least number of colors in such a coloring.

The total graph of G has vertex set $V(G) \cup E(G)$, and edges joining all elements of this vertex set which are adjacent or incident in G . The terminology and notations used in this paper can be found in [1, 3]. The purpose of this paper serves to find the harmonious chromatic number for the total graph of central graph of $C_n, K_{1,n}$ and P_n .

2. Harmonious Coloring on Total Graph of Central Graph of Cycle C_n

In $T(C(C_n))$, there are n vertices of degree 4, $2n$ vertices of degree $(n+1)$, $\binom{n}{2}$ vertices of degree $2(n-1)$.

Therefore

- The number of vertices, $p_{T(C(C_n))} = \frac{n^2 + 5n}{2}$.
- The number of edges, $q_{T(C(C_n))} = \frac{n^3 + 7n}{2}$.
- $\Delta = 2(n-1)$.

Theorem 2.1. *The harmonious chromatic number of total graph of central graph of cycles C_n ,*

$$\chi_H(T(C(C_n))) = \begin{cases} \frac{(n+1)^2 + 2}{2} & \text{if } n \text{ is odd} \\ \frac{(n+1)^2 + 3}{2} & \text{if } n \text{ is even, } (n > 4). \end{cases}$$

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$. By the definition of central graph $C(C_n)$ has the vertex set $V(C_n) \cup \{u_i : 1 \leq i \leq n\}$ where u_i is a vertex of subdivision of the edge $v_i v_{i+1}$ ($1 \leq i \leq n-1$) and u_n is a vertex of

subdivision of the edge v_nv_1 . Also let $u_iv_i = e_i (1 \leq i \leq n)$ and $u_iv_{i+1} = e'_i (1 \leq i \leq n-1)$ and $u_nv_1 = e'_n$. Also let $e_{ij} = v_iv_j$. Therefore $E(C(C_n)) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{e_{ij} : 3 \leq i \leq n-1\} \cup \{e_{ij} : 2 \leq i \leq n-2, i+2 \leq j \leq n\}$.

Central Graph of Cycle C_n **Figure 1**

Each v_i is incident with the edges $e_i, e'_{i-1}, e_{ij} (i \neq j)$ and $(2 \leq i \leq n)$. Also v_1 is incident with $e_1, e'_n, e_{13}, e_{1n}, \dots, e_{1(n-1)}$. i.e., Total number of incident edges with v_i is $(n-1) \forall i = 1, 2, \dots, n$. By the definition of total graph the edges incident with v_i together with the vertex v_i induces a clique of n vertices in $T(C(C_n)) (1 \leq i \leq n)$.

Let $K_n^{(i)}$ be the cliques in $T(C(C_n)) (i = 1, 2, \dots, n)$. Since $e_{ij} = e_{ji}$, each $K_n^{(i)}$ shares exactly $(n-3)$ vertices with the remaining cliques. Also the vertices v_i 's are adjacent with u_i 's $(1 \leq i \leq n)$ and u_i 's are adjacent

with v_{i+1} for every $i = 1, 2, \dots, n-1$ in $T(C(C_n))$.

Total Graph of Central Graph of Cycle C_n

Figure 2

Therefore in each clique, the harmonious coloring is performed by distinct n colors. $K_n^{(1)}$ is assigned n colors, where as since $K_n^{(2)}$ shares one vertex with $K_n^{(1)}$, it needs distinct $(n-1)$ colors which are distinct from the set of colors assigned to $K_n^{(1)}$. Now since $K_n^{(3)}$ shares one vertex with $K_n^{(1)}$ and one with $K_n^{(2)}$, it needs only $(n-2)$ colors and so on. Now we use induction.

Case (i)

If n is odd,

If $n = 5$, $T(C(C_5))$ has the harmonious coloring as follows.

Central Graph of Cycle C_5

Figure 3

Assign 5 distinct colors to the vertices of $K_5^{(1)}$. One of these 5 colors can be assigned to the vertices of $K_5^{(2)} - \{v_2\}$ and hence 4 new colors are assigned to the vertices of $K_5^{(2)}$. Similarly 4 new colors are assigned to $K_5^{(3)} - \{e_{31}\}$ and 3 new colors are assigned to $K_5^{(4)} - \{e_{14}, e_{24}\}$, 3 new colors are assigned to the vertices of $K_5^{(5)} - \{e_{25}, e_{35}\}$. Hence the total number of minimum colors used is 19. The vertices $v_{12}, v_{23}, v_{34}, v_{45}$ and v_{51} can be colored by

Structure of Total Graph of Central Graph of Cycle C_5
Figure 4

By the above process the harmonious chromatic number of $T(C(C_5))$ is as follows

Figure 5

Therefore $\chi_H(T(C(C_n))) = 19 = \frac{(n+1)^2 + 2}{2}$ when $n = 5$.

Assume that the result is valid for C_n , when n is odd. i.e., $\chi_H(T(C(C_n))) = \frac{(n+1)^2 + 2}{2}$. Now consider C_{n+2} by introducing two new vertices v_{n+1} and v_{n+2} . Consider the incident edges of v_{n+1} and v_{n+2} in $C(C_{n+2})$. These edges together with the vertices v_{n+1} and v_{n+2} induces two more cliques of order $(n+2)$ in $T(C(C_n))$. The vertices $v_{n+1}, v_{n+2}, e_{n+1}, e_{n+2}, u_{n+1}, u_{n+2}, e'_{n+1}, e'_{n+2}$ are assigned by 8 colors and the cliques $K_{n+2}^{(n+1)}$ and $K_{n+2}^{(n+2)}$ are assigned by

$(n-2)+(n-2) = (2n-4)$ colors. Therefore $\chi_H(T(C(C_n))) = \frac{(n+1)^2+2}{2} + 2n-4+8 = \frac{(n+1)^2+2}{2} + (2n+4)$. Hence $\chi_H(T(C(C_{n+2}))) = \frac{(n+3)^2+2}{2}$.

Therefore by induction hypothesis $\chi_H(T(C(C_n))) = \frac{(n+1)^2+2}{2}$, if n is odd .

Case (ii)

If n is even, we prove that, $\chi_H(T(C(C_n))) = \frac{(n+1)^2+3}{2}$.

For $n = 6$, the total graph of $C(C_6)$ has harmonious chromatic number 26.

Central Graph of Cycle C_6

Figure 6

The harmonious coloring of $T(C(C_6))$ is as follows. Assign 6 distinct colors to the vertices of $K_6^{(1)}$. One of these 6 colors can be assigned to the vertices of $K_6^{(2)} - \{v_2\}$ and hence 5 new colors are assigned to the vertices of $K_6^{(2)}$. Similarly 5 new colors are assigned to $K_6^{(3)} - \{e_{31}\}$, since e_{31} has already been assigned a color and 4 new colors are assigned to $K_6^{(4)} - \{e_{14}, e_{24}\}$, 3 new colors are assigned to the vertices of $K_6^{(5)} - \{e_{51}, e_{52}, e_{53}\}$, 3 new colors are assigned to the vertices of $K_6^{(6)} - \{e_{62}, e_{63}, e_{64}\}$. Hence the total number of minimum colors used is 26. The vertices $v_{12}, v_{23}, v_{34}, v_{45}, v_{56}$ and v_{61} can be colored by any one of the above said colors. Therefore $\chi_H(T(C(C_6))) = 26$.

Structure of Total Graph of Central Graph of Cycle C_6

Figure 7

By the above process the harmonious chromatic number of $T(C(C_6))$ is as follows

Total Graph of Central Graph of Cycle C_6

Figure 8

Hence $\chi_H(T(C(C_n))) = \frac{(n+1)^2 + 3}{2}$ if $n = 6$.

Now assume that the result is valid for any even n . i.e., $\chi_H(T(C(C_n))) = \frac{(n+1)^2 + 3}{2}$ if n is even. Now consider C_{n+2} by introducing two new vertices v_{n+1} and v_{n+2} induce two more cliques of order $n+2$ in $T(C(C_n))$. The vertices $v_{n+1}, v_{n+2}, e_{n+1}, e_{n+2}, u_{n+1}, u_{n+2}, e'_{n+1}, e'_{n+2}$ are assigned by 8 colors and the cliques $K_{n+2}^{(n+1)}$ and $K_{n+2}^{(n+2)}$ are assigned by $(n-2) + (n-2) = (2n-4)$ colors. Therefore $\chi_H(T(C(C_{n+2}))) = \frac{(n+1)^2 + 3}{2} + 2n - 4 + 8 = \frac{(n+3)^2 + 3}{2}$. Hence $\chi_H(T(C(C_{n+2}))) = \frac{((n+2)+1)^2 + 3}{2}$. Therefore by induction hypothesis $\chi_H(T(C(C_n))) = \frac{(n+1)^2 + 3}{2}$, if n is even. $\square$

3. Harmonious Coloring on Total Graph of Central Graph of Star Graph $K_{1,n}$

In $T(C(K_{1,n}))$ there are n vertices of degree 4, $2n$ vertices of degree $\frac{n^2 + n + 2}{2}$, $2n$ vertices of degree $(n+2)$.

Therefore

- The number of vertices, $p_{T(C(K_{1,n}))} = \frac{n^2 + 7n + 2}{2}$.
- The number of edges, $q_{T(C(K_{1,n}))} = \frac{n^3 + 3n^2 + 10n}{2}$.
- $\Delta = 2n$.

Theorem 3.1. *The harmonious chromatic number of total graph of central graph of star graph $K_{1,n}$,*

$$\chi_H(T(C(K_{1,n}))) = \begin{cases} \frac{(n^2 + 3n + 8)}{2} & \text{if } n \text{ is even} \\ \frac{(n^2 + 3n + 6)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $V(K_{1,n}) = \{v, v_1, v_2, \dots, v_n\}$ where $\deg v = n$. By the definition of central graph of $K_{1,n}$, we denote the vertices of subdivision by $u_1, u_2, \dots, u_n$. i.e., vv_i is subdivided by $u_i (1 \leq i \leq n)$. Let $e_i = v_i u_i$ and $e'_i = v u_i (1 \leq i \leq n)$. Therefore $V(C(K_{1,n})) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v\}$. By the definition of central graph the subgraph induced by the vertex set $\{v_1, v_2, \dots, v_n\}$ is K_n and let e_{ij} be the edge of $C(K_{1,n})$, connecting the vertex v_i and $v_j (i < j)$.

Central Graph of Star Graph $K_{1,n}$ **Figure 9**

Therefore $E(C(K_{1,n})) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{e_{ij} : 1 \leq i \leq n-1, i+1 \leq j \leq n\}$. By the definition of total graph, $V(T(C(K_{1,n}))) = V(C(K_{1,n})) \cup E(C(K_{1,n}))$. The structure is described below.

Total Graph of Central Graph of Star Graph $K_{1,n}$ **Figure 10**

The vertices $v, e'_1, e'_2, \dots, e'_n$ induce a clique of order $(n + 1)$ in its total graph. The vertices u_i adjacent to e'_1 and $e'_i (1 \leq i \leq n)$. Let $S_i = \{e_{ij} : j = 1, 2, \dots, i-1, i+1, \dots, n\}, (1 \leq i \leq n)$. Clearly $S_i \cap S_j = \{e_{ij}\}$ if $i \neq j$ and let $S^{(n)} = \cup_{i=1}^n S_i$, clearly $|S^{(n)}| = \binom{n}{2}$. Now the vertices v_i and e'_i together with vertices of S_i induce a clique of order $(n + 1), (1 \leq i \leq n)$. Also the vertices $v_1, v_2, \dots, v_n$ induce a clique of order n . Now we prove that harmonious chromatic number of this graph by induction method.

Case (i)

If n is even

Then we prove $\chi_H(T(C(K_{1,n}))) = \frac{n^2 + 3n + 8}{2}$.

If $n = 2$, then $C(K_{1,2})$ has 5 vertices. The harmonious coloring for the total graph of $C(K_{1,2})$, is shown below.

Total Graph of Central Graph of Star Graph $K_{1,2}$ **Figure 11**

From the Fig.11, 9 is the minimum number of colors as the 5 triangles shares atleast one vertex in common. Therefore $\chi_H(T(C(K_{1,2}))) = 9 = \frac{n^2 + 3n + 8}{2}$, when $n = 2$. Therefore the result is true if $n = 2$.

Assume that the result is true for any even integer n and we prove the same for $n + 2$. i.e., $\chi_H(T(C(K_{1,n}))) = \frac{n^2 + 3n + 8}{2}$. Let v_{n+1}, v_{n+2} be two non adjacent vertices introduced in $K_{1,n}$ which are adjacent to v . Let u_{n+1} and u_{n+2} be the vertices of subdivision in its centralization. Clearly by the structure given in Fig.10, the total graph $C(K_{1,n+2})$ has the following structural property. (i) There are $(n+3)$ cliques $K_{n+3}^{(1)}, K_{n+3}^{(2)}, \dots, K_{n+3}^{(n+3)}$ of order $(n+3)$. (ii) There is a clique of order $(n+2)$. (iii) Each $K_{n+3}^{(i)}$ has exactly one vertex common with $K_{n+3}^{(j)}$ where $(2 \leq i \leq n+3), (2 \leq j \leq n+3)$ and $i \neq j$. By induction hypothesis the minimum number of colors for harmonious coloring in $T(C(K_{1,n})) = \frac{n^2 + 3n + 8}{2}$. By the above said structure of $T(C(K_{1,n}))$, $|S^{(n)}| = |S_1 \cup S_2 \cup \dots \cup S_n| = \binom{n}{2} = \frac{n(n-1)}{2}$. $|S^{(n+2)}| = |S_1 \cup S_2 \cup \dots \cup S_{n+2}| = \binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$. Also the new

vertices in $T(C(K_{1,n+2}))$ are $e_{n+1}, e_{n+2}, u_{n+1}, u_{n+2}, e'_{n+1}, e'_{n+2}, v_{n+1}, v_{n+2}$. Therefore the total number of vertices in $T(C(K_{1,n+2})) = \frac{(n+2)(n+1)}{2} - \frac{n(n-1)}{2} + 8 = 2n + 1 + 8 = 2n + 9$. Now we find the minimal harmonious coloring in $T(C(K_{1,n+2}))$ as below. By induction hypothesis $T(C(K_{1,n}))$ has an harmonious coloring with the minimum number of $\frac{n^2 + 3n + 8}{2}$ colors. With this same color assigned to the vertices of $T(C(K_{1,n+2}))$, we assign some new colors to the remaining vertices as below. The vertices u_{n+1} and u_{n+2} are assign the same color as in $u_i (1 \leq i \leq n)$. Then all the remaining vertices are assigned $2n + 5$ colors. Therefore $\chi_H(T(C(K_{1,n+2}))) = \frac{n^2 + 3n + 8}{2} + 2n + 5 = \frac{(n+2)^2 + 3(n+2) + 8}{2}$. By induction hypothesis $\chi_H(T(C(K_{1,n}))) = \frac{n^2 + 3n + 8}{2}$ if n is even.

Case (ii)

If n is odd, we prove $\chi_H(T(C(K_{1,n}))) = \frac{n^2 + 3n + 6}{2}$ by induction following the same procedure as above. $\square$

4. Harmonious Coloring on Total Graph of Central Graph of Path P_n

In $T(C(P_n))$ there are $n - 1$ vertices of degree 4, $2n - 2$ vertices of degree $n + 1$, $\frac{n^2 - n + 2}{2}$ vertices of degree $2n - 2$. Therefore

- The number of vertices, $p_{T(C(P_n))} = \frac{n^2 + 5n - 4}{2}$.
- The number of edges, $q_{T(C(P_n))} = \frac{n^3 + 7n^2 - 8}{2}$.
- $\Delta = 2n - 2$ for $n \geq 3$.

Theorem 4.1. *The harmonious chromatic number of total graph of central graph of path P_n , $\chi_H(T(C(P_n))) = 4n - 3$, where n denotes the number of vertices in P_n .*

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$. On the process of centralization of P_n , let u_i be the vertex of subdivision of the edges $v_i v_{i-1}$ ($1 \leq i \leq n$). Also let $v_i u_i = e_i$ and $u_i v_{i+1} = e'_i$ ($1 \leq i \leq n-1$). By the definition of central graph the non adjacent vertices v_i and v_j of P_n , are adjacent in $C(P_n)$ by the edge e_{ij} .

Path P_n
Figure 12

Central Graph of Path P_n
Figure 13

Therefore $V(C(P_n)) = \{v_i / 1 \leq i \leq n\} \cup \{u_i / 1 \leq i \leq n-1\}$ and $E(C(P_n)) = \{e_i : 1 \leq i \leq n-1\} \cup \{e'_i : 1 \leq i \leq n-1\} \cup \{e_{ij} : 1 \leq i \leq n-2, i+2 \leq j \leq n\}$. By the definition of total graph, $V(T(C(P_n))) = V(C(P_n)) \cup E(C(P_n))$. In $C(P_n)$, each v_i is incident with the $(n-1)$ edges e_i, e'_{i-1}, e_{ij} for $(2 \leq i \leq n-1), (i \neq j)$, v_1 is incident with the $(n-1)$ edges $e_1, e_{13}, e_{14}, \dots, e_{1n}$ and v_n is incident with $(n-1)$ edges $e'_{n-1}, e_{2n}, e_{3n}, \dots, e_{(n-2)n}, e_{1n}$. The edges incident with v_i together with vertex v_i induces a clique of n vertices in $T(C(P_n))$ for $i = 1, 2, \dots, n$.

Let $K_n^{(i)}$ be the cliques in $T(C(P_n))$ ($i = 1, 2, \dots, n$). Since $e_{ij} = e_{ji}$, each $K_n^{(i)}$ shares exactly $(n-3)$ vertices with the remaining cliques. Also the vertices v_i 's are adjacent with u_i 's for $i = 1, 2, \dots, n-1$ and u_i 's are adjacent with v_{i+1} for $i = 1, 2, \dots, n-1$ in $T(C(P_n))$. Also v_i is adjacent with $v_{i+2}, v_{i+3}, \dots, v_n$ and $v_{i-2}, v_{i-3}, \dots, v_1$.

Total Graph of Central Graph of Path P_n **Figure 14**

In each of the above said clique, harmonious coloring is performed by distinct n colors. $K_n^{(1)}$ is assigned n colors, where as since $K_n^{(2)}$ shares one vertex with $K_n^{(1)}$, it needs distinct $(n - 1)$ colors which are distinct from the set of colors assigned to $K_n^{(1)}$. Now since $K_n^{(3)}$ shares one vertex with $K_n^{(1)}$ and one with $K_n^{(2)}$, it needs only $(n - 2)$ colors and so on. Now we use induction on n .

For $n = 1$, the results follows obviously. For $n = 2$, $C(P_2)$ is P_3 and $T(C(P_2))$ and its harmonious coloring is as follows.

Total Graph of Central Graph of Path P_2 **Figure 15**

Assume that the result is valid for any n , $\chi_H(T(C(P_n))) = 4n - 3$. Now consider P_{n+1} by introducing a vertex v_{n+1} and consider the incident edges of v_{n+1} in $C(P_{n+2})$. These edges together with the vertex v_{n+1} induces one more clique of order n in $T(C(P_{n+1}))$ and it shares $n - 4$ vertices with other cliques. Therefore we need 4 colors for the harmonious coloring of $T(C(P_n))$. Therefore $\chi_H(T(C(P_{n+1}))) = 4n - 3 + 4 = 4n + 1 = 4(n + 1) - 3$. Hence by induction hypothesis $\chi_H(T(C(P_{n+1}))) = 4n - 3$. $\square$

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