Abstract

Let $A$ be an algebra. A sequence $\{d_n\}$ of linear mappings on $A$ is called a higher derivation if $d_n(ab) = \sum_{j=0}^{n} d_j(a)d_{n-j}(b)$ for each $a, b \in A$ and each nonnegative integer $n$. Jewell [Pacific J. Math. 68 (1977), 91-98], showed that a higher derivation from a Banach algebra onto a semisimple Banach algebra is continuous provided that $\ker(d_0) \subseteq \ker(d_m)$, for all $m \geq 1$. In this paper, under a different approach using $C^*$-algebraic tools, we prove that each higher derivation $\{d_n\}$ on a $C^*$-algebra $A$ is automatically continuous, provided that it is normal, i.e. $d_0$ is the identity mapping on $A$.

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1. Introduction

Let $\mathcal{A}$ be an algebra. A linear mapping $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is called a derivation if it satisfies the Leibniz rule, i.e. $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. If we define the sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ by $d_0 = I$ and $d_n = \frac{\delta^n}{n!}$, where $I$ is the identity mapping on $\mathcal{A}$, then the Leibniz rule ensures us that $d_n$’s satisfy the condition

$$d_n(ab) = \sum_{j=0}^{n} d_j(a)d_{n-j}(b) \quad (*)$$

for each $a, b \in \mathcal{A}$ and each nonnegative integer $n$. This motivates us to consider the sequences $\{d_n\}$ of linear mappings on an algebra $\mathcal{A}$ satisfying $(*)$. Such a sequence is called a higher derivation. Higher derivations were introduced by Hasse and Schmidt [2], and algebraists sometimes call them Hasse-Schmidt derivations. Though, if $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is a derivation then $d_n = \frac{\delta^n}{n!}$ is a higher derivation, this is not the only example of a higher derivation.

Regarding to a celebrated theorem of Sakai [11, 12], all derivations defined on a $C^*$-algebra are automatically continuous. Some results concerning to the theorem are discussed in [8] and [3]. Regarding to the Sakai’s theorem we can deduce that the higher derivation $d_n = \frac{\delta^n}{n!}$ defined on a $C^*$-algebra is automatically continuous in the sense that each $d_n$ is continuous. This poses the problem of automatic continuity of higher derivations. Many mathematicians could find some affirmative answers to the problem in special cases. Loy [7] proved that if $\mathcal{A}$ is an $(F)$-algebra which is a subalgebra of a Banach algebra $B$ of power series, then every higher derivation $\{d_n\} : \mathcal{A} \rightarrow B$ is automatically continuous. Jewell [5], showed that a higher derivation from a Banach algebra onto a semisimple Banach algebra is continuous provided that $\ker(d_0) \subseteq \ker(d_m)$, for all $m \geq 1$. Villena [14], proved that every higher derivation from a unital Banach algebra $\mathcal{A}$ into $\mathcal{A}/\mathcal{P}$, where $\mathcal{P}$ is a primitive ideal of $\mathcal{A}$ with infinite codimension, is continuous. Hejazian and Shatery [4] prove the automatic continuity of higher derivations in the case of $JB^*$-algebras.

Here, we prove automatic continuity of higher derivations in the domain of $C^*$-algebras. Though, this is a consequence of the Jewell result in [5], our proof just depends on $C^*$-algebraic tools. Prior to that, we need some elementary facts concerning higher derivations. For the definition and elementary properties of $C^*$-algebras we refer the reader to [6, 9] and [10]. One can find a collection of suitable information about automatic
continuity and some applications of higher derivations in [1] and [13].

2. Preliminaries

Let $A$ be an algebra, $\mathbb{Z}_k^+ = \{0,1,\ldots,k\}$ for $k \in \mathbb{N}$ and $\mathbb{Z}^+ = \{0,1,2,\ldots\}$. A higher derivation of order $k$ is a sequence $\{d_n\}_{n \in \mathbb{Z}_k^+}$ of linear mappings from $A$ to $A$ such that

$$d_n(ab) = \sum_{j=0}^{n} d_j(a)d_{n-j}(b)$$

for all $a,b \in A$ and $n \in \mathbb{Z}_k^+$. A sequence $\{d_n\}_{n \in \mathbb{Z}_k^+}$ is a higher derivation of infinite order if $\{d_n\}_{n \in \mathbb{Z}_k^+}$ is a higher derivation of order $k$ for each $k \in \mathbb{N}$.

A higher derivation $\{d_n\}$ is called normal if $d_0 = I$ (the identity mapping on $A$). As a simple example, for a derivation $\delta : A \rightarrow A$ we can assume the sequence $d_0 = I$, $d_n = \frac{\delta^n}{n!}$. The Leibniz rule implies that $\{d_n\}$ is a higher derivation.

A higher derivation $\{d_n\}$ is called continuous if each $d_n$ is continuous. It is said to be onto if $d_0$ is onto.

Lemma 2.1. If $\{d_n\}$ is a normal higher derivation on a unital $C^*$-algebra with unit $\iota$, then $d_n(\iota) = 0$ for $n \geq 1$.

Proof. Since $\{d_n\}$ is normal, $d_1$ is a derivation and so $d_1(\iota) = 0$. Let $d_j(\iota) = 0$ for $1 \leq j \leq n - 1$. Then we have

$$d_n(\iota) = d_n(\iota.\iota) = \iota.d_n(\iota) + \sum_{j=1}^{n-1} d_j(\iota)d_{n-j}(\iota) + d_n(\iota)\iota = d_n(\iota) + d_n(\iota)$$

Hence $d_n(\iota) = 0$. □

From now on, we assume that $A$ is a unital $C^*$-algebra. In fact, if $A$ has no identity, we shall consider the $C^*$-unitization $A_1$ of $A$, and define $d_n(\iota) = 0$ for each $n$.

Recall that if $T$ is a linear mapping and we define $T^*$ by $T^*(a) = T(a^*)^*$ for all $a \in A$, then $T^*$ is a linear mapping on $A$.

Lemma 2.2. Let $\{d_n\}$ be a higher derivation on a $C^*$-algebra $A$. Then $\{d_n^*\}$ is also a higher derivation on $A$. 


Proof. For each \(a, b \in \mathcal{A}\) and \(n \in \mathbb{Z}^+\) we have

\[
d_n(ab) = (d_n(b^*a^*))^* = \left( \sum_{j=0}^{n} d_j(b^*)d_{n-j}(a^*) \right)^* = \sum_{j=0}^{n} d_{n-j}(a)d_j(b)
\]

\[= \sum_{k=0}^{n} d_k^*(a)d_{n-k}^*(b).
\]

Thus \(\{d_n^*\}\) is a higher derivation. \(\Box\)

It is known that the derivation \(d : C^1([0,1]) \to C([0,1])\) defined by \(d(f) = f'\) on the dense subalgebra \(C^1([0,1])\) of \(C([0,1])\) is not continuous. So the higher derivation \(\{d_n^*\}\) is an example of a discontinuous densely defined normal higher derivation in the \(C^*-\)algebra \(C([0,1])\). In the next section, we will show that this is not the case for everywhere defined higher derivations on \(C^*-\)algebras.

3. The Result

Theorem 3.1. Let \(\mathcal{A}\) be a unital \(C^*-\)algebra. Then every normal higher derivation \(\{d_n\}\) on \(\mathcal{A}\) is continuous.

Proof. For each \(n \in \mathbb{Z}^+\) we can write

\[
d_n(ab) = \frac{d_n^* + d_n}{2} + i\frac{d_n^* - id_n}{2}.
\]

Put \(d_n^1 = \frac{d_n^* + d_n}{2}\) and \(d_n^2 = \frac{id_n^* - id_n}{2}\). Then \(d_n^1\)’s and \(d_n^2\)’s are \(*\)-mappings and \(d_n^1(\iota) = d_n^2(\iota) = 0\) for all \(n \in \mathbb{N}\). We also have

\[
d_n^1(ab) = ad_n^1(b) + d_n^1(a)b + \frac{1}{2} \sum_{j=1}^{n-1} d_j(a)d_{n-j}(b) + \frac{1}{2} \sum_{j=1}^{n-1} d_j^*(a)d_{n-j}^*(b),
\]

\[
d_n^2(ab) = ad_n^2(b) + d_n^2(a)b - \frac{i}{2} \sum_{j=1}^{n-1} d_j(a)d_{n-j}(b) + \frac{i}{2} \sum_{j=1}^{n-1} d_j^*(a)d_{n-j}^*(b).
\]

It suffices to show that \(d_n^1\) and \(d_n^2\) are continuous for all \(n \in \mathbb{Z}^+\). At first we prove continuity of \(d_n^1\)’s by induction:

Since \(d_0^1 = I\), \(d_0^1\) is continuous. Suppose that \(d_j^1\) is continuous for \(j \leq n-1\). Let \(a\) be a self-adjoint element of \(\mathcal{A}\) and \(\varphi\) be a state on \(\mathcal{A}\) such that \(|\varphi(a)| = \|a\|\). We may assume that \(\varphi(a) = \|a\|\) (If \(-\varphi(a) = \|a\|\) then we
can write $\varphi(-a) = \| -a \|$ and choose the self-adjoint element $-a$ instead of $a$). Put $\|a\| \leq a - h^2$ ($h \geq 0$, $h \in A$. Then $\varphi(h^2) = 0$ and

$$
| - \varphi(d_n^1(a)) - \varphi\left(\frac{1}{2} \sum_{j=1}^{n-1} d_j(h) d_{n-j}(h) + \frac{1}{2} \sum_{j=1}^{n-1} d_j^*(h) d_{n-j}^*(h)\right) | 
$$

$$
= | \varphi(d_n^1(\|a\| - a)) - \varphi\left(\frac{1}{2} \sum_{j=1}^{n-1} d_j(h) d_{n-j}(h) + \frac{1}{2} \sum_{j=1}^{n-1} d_j^*(h) d_{n-j}^*(h)\right) | 
$$

$$
= | \varphi(h d_n^1(h^2)) + \varphi(d_n^1(h) h) | 
$$

$$
\leq \varphi(h^2)^{1/2} \varphi(d_n^1(h^2)^{1/2}) + \varphi(d_n^1(h^2)^{1/2}) \varphi(h^2)^{1/2} 
$$

$$
= 0.
$$

Hence $\varphi(d_n^1(a)) = -\varphi\left(\frac{1}{2} \sum_{j=1}^{n-1} d_j(h) d_{n-j}(h) + \frac{1}{2} \sum_{j=1}^{n-1} d_j^*(h) d_{n-j}^*(h)\right)$.

Suppose that $\{a_m\}$ is a sequence of self-adjoint elements in $A$ such that $a_m \to 0$ and $d_n^1(a_m) \to b(\neq 0)$. Let $\varphi_m$ be a state on $A$ such that $|\varphi_m(b + a_m)| = \| b + a_m \|$, and let $\varphi_0$ be an accumulation point of $\{\varphi_m\}$ in the state space of $A$. Then we have

$$
| \varphi_m(b + a_m) - \varphi(b) | = | \varphi_m(b + a_m) - \varphi_m(b) + \varphi_m(b) - \varphi(b) | 
$$

$$
\leq | \varphi_m(b + a_m) - \varphi_m(b) | + | \varphi_m(b) - \varphi(b) | 
$$

$$
\leq \| b + a_m - b \| + | \varphi_m(b) - \varphi(b) | \to 0
$$

for some subsequence $\{m_k\}$ of $\{m\}$. Hence $|\varphi_0(b)| = \| b \|$ and so

$$
\varphi_0(d_n^1(b)) = -\varphi_0\left(\frac{1}{2} \sum_{j=1}^{n-1} d_j(h_b) d_{n-j}(h_b) + \frac{1}{2} \sum_{j=1}^{n-1} d_j^*(h_b) d_{n-j}^*(h_b)\right),
$$

where $h_b = (\| b \| \leq b - h^1/2$. Similarly one can show that

$$
| \varphi_m(d_n^1(a_m)) - \varphi_0(b) | \to 0.
$$

Also if $(h_b + a_m)^2 = \| b + a_m \| \leq b + a_m$ then $h_b^2 + a_m \to h_b^2$ and since $h_b + a_m$'s and $h_b$ are positive, $h_b + a_m \to h_b$. So continuity of $d_0^1, d_1^1, \ldots, d_{n-1}^1$ implies that

$$
-\varphi_0\left(\frac{1}{2} \sum_{j=1}^{n-1} d_j(h_b) d_{n-j}(h_b) + \frac{1}{2} \sum_{j=1}^{n-1} d_j^*(h_b) d_{n-j}^*(h_b)\right)
$$
\[
\begin{align*}
&= \lim_{m_k \to \infty} -\varphi_{m_k} \left( \frac{1}{2} \sum_{j=1}^{n-1} d_j \left( h_{b+a_{m_k}} \right) d_{n-j} \left( h_{b+a_{m_k}} \right) \right) \\
&\quad + \frac{1}{2} \sum_{j=1}^{n-1} d_j^* \left( h_{b+a_{m_k}} \right) d^*_{n-j} \left( h_{b+a_{m_k}} \right) \\
&= \lim_{m_k \to \infty} \varphi_{m_k} \left( d_n^1 \left( b + a_{m_k} \right) \right) \\
&= \varphi_0 \left( d_n^1 \left( b \right) \right) \\
&= \varphi_0 \left( \frac{1}{2} \sum_{j=1}^{n-1} d_j \left( h_b \right) d_{n-j} \left( h_b \right) + \frac{1}{2} \sum_{j=1}^{n-1} d_j^* \left( h_b \right) d^*_{n-j} \left( h_b \right) \right) + \varphi_0 \left( b \right).
\end{align*}
\]

Hence \( \varphi_0 \left( b \right) = 0 \), which is a contradiction. So the closed graph theorem guarantees that \( d_n^1 \) is continuous.

Similarly we can show that \( d_n^2 \)'s are continuous. Whence the continuity of the higher derivation \( \{ d_n \} \) is deduced. \( \square \)

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**References**


Shirin Hejazian
Department of Mathematics
Ferdowsi University
P. O. Box 1159
Mashhad 91775
Iran
e-mail : hejazian@math.um.ac.ir
Majid Mirzavaziri  
Department of Mathematics  
Ferdowsi University  
P. O. Box 1159  
Mashhad 91775  
Iran  
e-mail: mirzavaziri@gmail.com  

and  

Elahe Omidvar Tehrani  
Department of Mathematics  
Ferdowsi University  
P. O. Box 1159  
Mashhad 91775  
Iran  
e-mail: el_om3@stu-mail.um.ac.ir