COUNTABLE COMPACTNESS AND THE LINDELOF PROPERTY IN L-FUZZY TOPOLOGICAL SPACES *

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Abstract

In this paper, the concepts of L -fuzzy countable compactness and the L -fuzzy Lindelöf property are introduced in L -fuzzy topological spaces, where L is a completely distributive DeMorgan algebra. An L -fuzzy compact L -fuzzy set is L -fuzzy countably compact and has the L -fuzzy Lindelöf property. An L -fuzzy set having the L -fuzzy Lindelöf property is L -fuzzy countably compact if and only if it is L -fuzzy compact. Many characterizations of L -fuzzy countable compactness and the L -fuzzy Lindelöf property are presented.

Keywords : *L -fuzzy topology, L -fuzzy countable compactness, the L -fuzzy Lindelöf property.*

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1. Introduction

In 1976, the concept of fuzzy compactness was introduced in $[0, 1]$ -topological spaces by R. Lowen [5]. Subsequently its characterization was given by G.J. Wang in terms of α -net in [12]. In 1988, it was extended to L -topological spaces [13], where L is a completely distributive DeMorgan algebra. In [9], a new definition of fuzzy compactness was presented by means of open L -sets and their inequality in L -topological spaces. When L is a completely distributive DeMorgan algebra, it is equivalent to the notion of fuzzy compactness in [4, 7, 13]. Recently the concept of L -fuzzy compactness was introduced by Shi and Li [10] in L -fuzzy topological spaces.

In this paper, our aim is to continue the research of L -fuzzy countable compactness and the L -fuzzy Lindelöf property of L -fuzzy sets.

2. Preliminaries

Throughout this paper $(L, \vee, \wedge, ')$ is a completely distributive DeMorgan algebra, X is a nonempty set. L^X is the set of all L -fuzzy sets on X . The smallest element and the largest element in L^X are denoted respectively by $\underline{\perp}$ and $\underline{\top}$. An L -fuzzy set is briefly written as an L -set. We often do not distinguish a crisp subset A from its characteristic function χ_A .

The set of nonunit prime elements in L is denoted by $P(L)$. The set of nonzero co-prime elements in L is denoted by $M(L)$. The set of nonzero co-prime elements in L^X is denoted by $M(L^X)$. The set all L -fuzzy points x_λ (i.e., an L -fuzzy set $A \in L^X$ such that $A(x) = \lambda \neq 0$ and $A(y) = 0$ for $y \neq x$) is denoted by $pt(L^X)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [1]. In a completely distributive DeMorgan algebra L , each member b is a sup of $\{a \in L \mid a \prec b\}$. In the sense of [4, 13], $\{a \in L \mid a \prec b\}$ is the greatest minimal family of b , denoted by $\beta(b)$, and $\beta^*(b) = \beta(b) \cap M(L)$. Moreover for $b \in L$, define $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we define $A_{[a]} = \{x \in X \mid A(x) \geq a\}$.

Definition 2.1 ([2, 3, 6, 11]). An L -fuzzy topology on a set X is a map $\mathcal{T} : L^X \rightarrow L$ such that

- (1) $\mathcal{T}(\underline{\top}) = \mathcal{T}(\underline{\perp}) = \underline{\top}$;
- (2) $\forall U, V \in L^X, \mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$;

$$(3) \quad \forall U_j \in L^X, j \in J, \mathcal{T}(\bigvee_{j \in J} U_j) \geq \bigwedge_{j \in J} \mathcal{T}(U_j).$$

$\mathcal{T}(U)$ can be interpreted as the degree to which U is an open set. $\mathcal{T}^*(U) = \mathcal{T}(U')$ will be called the degree of closedness of U . The pair $(X, \mathcal{T})$ is called an L -fuzzy topological space.

A mapping $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ is said to be L -fuzzy continuous if $\mathcal{T}(f_L^-(B)) \geq \mathcal{U}(B)$ holds for all $B \in L^Y$, where f_L^- is defined by $f_L^-(B)(x) = B(f(x))$ [6].

Theorem 2.2 ([14]). Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be L -fuzzy topological spaces. Then $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be L -fuzzy continuous if and only if $\forall a \in M(L)$, $f : (X, \mathcal{T}_{[a]}) \rightarrow (Y, \mathcal{U}_{[a]})$ be L -continuous.

Definition 2.3 ([8, 9]). Let $a \in L \setminus \{\top\}$ and $G \in L^X$. A subfamily $\mathcal{U}$ in L^X is said to be

- (1) an a -shading of G if for any $x \in X$, it follows that $G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \not\leq a$.
- (2) a strong a -shading of G if $\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \not\leq a$.

Definition 2.4 ([8, 9]). Let $a \in L \setminus \{\perp\}$ and $G \in L^X$. A subfamily $\mathcal{P}$ in L^X is said to be

- (1) an a -remote family of G if for any $x \in X$, it follows that $G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \not\geq a$.
- (2) a strong a -remote family of G if $\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \not\geq a$.

Definition 2.5 ([8, 9]). Let $a \in L \setminus \{\perp\}$ and $G \in L^X$. A subfamily $\mathcal{U}$ in L^X is called

- (1) a β_a -cover of G if for any $x \in X$, it follows that $a \in \beta \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right)$.
- (2) a strong β_a -cover of G if for any $x \in X$, it follows that $a \in \beta \left(\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \right)$.

Definition 2.6 ([8, 9]). Let $a \in L \setminus \{\perp\}$ and $G \in L^X$. A subfamily $\mathcal{U}$ in L^X is called a Q_a -cover of G if $a \leq \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right)$.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ . $2^{[\Phi]}$ denotes the set of countable subfamilies of Φ .

Definition 2.7 ([8]). Let $(X, \mathcal{T})$ be an L -topological space. $G \in L^X$ is said to be countably compact if for every countable family $\mathcal{U} \subseteq L^X$, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{U}} F(x) \right) \leq \bigvee_{V \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in V} F(x) \right).$$

Definition 2.8 ([8]). Let $(X, \mathcal{T})$ be an L -topological space. $G \in L^X$ is said to have the Lindelöf property if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{U}} F(x) \right) \leq \bigvee_{V \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in V} F(x) \right).$$

Definition 2.9 ([10]). Let $(X, \mathcal{T})$ be an L -fuzzy topological space. $G \in L^X$ is said to be L -fuzzy compact if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$\bigwedge_{F \in \mathcal{U}} \mathcal{T}(F) \wedge \left(\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{U}} F(x) \right) \right) \leq \bigvee_{V \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in V} F(x) \right).$$

3. L -fuzzy countable compactness

Definition 3.1. Let $(X, \mathcal{T})$ be an L -fuzzy topological space. $G \in L^X$ is said to be L -fuzzy countably compact if for every countable family $\mathcal{U} \subseteq L^X$, it follows that

$$\bigwedge_{F \in \mathcal{U}} \mathcal{T}(F) \wedge \left(\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{U}} F(x) \right) \right) \leq \bigvee_{V \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in V} F(x) \right).$$

Obviously L -fuzzy compactness implies L -fuzzy countable compactness.

Let $(X, \mathcal{T})$ be an L -topological space. Let $\chi_{\mathcal{T}} : L^X \rightarrow L$

$$\chi_{\mathcal{T}} = \begin{cases} 1, & A \in \mathcal{T}, \\ 0, & A \notin \mathcal{T}. \end{cases}$$

Obviously, (X, χ_T) is a special L -fuzzy topological spaces. So we can easily prove the following theorem.

Theorem 3.2. *Let (X, T) be an L -topological space and $G \in L^X$. G is L -fuzzy countably compact in (X, χ_T) if and only if G is countably compact [8] in (X, T) .*

From Definition 2.1 we easily obtain the following theorem by simply using quasi-complement.

Theorem 3.3. *Let (X, T) be an L -fuzzy topological space. $G \in L^X$ is L -fuzzy countably compact if and only if for every countably family $\mathcal{P} \subseteq L^X$ it follows that*

$$\bigvee_{F \in \mathcal{P}} T'(F) \vee \left(\bigvee_{x \in X} (G(x) \wedge \bigwedge_{F \in \mathcal{P}} F(x)) \right) \geq \bigwedge_{\mathcal{H} \in 2^{(P)}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{F \in \mathcal{H}} F(x) \right).$$

By Definition 2.1 and Theorem 2.2 and analogous to [8] we immediately obtain the following result.

Theorem 3.4. *Let (X, T) be an L -fuzzy topological space and $G \in L^X$. Then the following conditions are equivalent to each other.*

- (1) G is L -fuzzy countably compact.
- (2) For any $a \in M(L)$, each countable strong a -remote family $\mathcal{P}$ of G with $\bigwedge_{F \in \mathcal{P}} T^*(F) \not\leq a'$ has a finite subfamily $\mathcal{H}$ which is a (strong) a -remote family of G .
- (3) For any $a \in M(L)$, and any countable strong a -remote family $\mathcal{P}$ of G with $\bigwedge_{F \in \mathcal{P}} T^*(F) \not\leq a'$, there exists a finite subfamily $\mathcal{H}$ of $\mathcal{P}$ and $b \in \beta^*(a)$ such that $\mathcal{H}$ is a (strong) b -remote family of G .
- (4) For any $a \in P(L)$, each countable strong a -shading $\mathcal{U}$ of G with $\bigwedge_{F \in \mathcal{U}} T(F) \not\leq a$ has a finite subfamily $\mathcal{V}$ which is a (strong) a -shading of G .
- (5) For any $a \in P(L)$ and any countable strong a -shading $\mathcal{U}$ of G with $\bigwedge_{F \in \mathcal{U}} T(F) \not\leq a$, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha^*(a)$ such that $\mathcal{V}$ is a (strong) b -shading of G .

- (6) For any $a \in M(L)$ and any $b \in \beta^*(a)$, each countable Q_a -cover $\mathcal{U}$ of G with $\mathcal{T}(F) \geq a$ ($\forall F \in \mathcal{U}$) has a finite subfamily $\mathcal{V}$ which is a Q_b -cover of G .
- (7) For any $a \in M(L)$ and any $b \in \beta^*(a)$, each countable Q_a -cover $\mathcal{U}$ of G with $\mathcal{T}(F) \geq a$ ($\forall F \in \mathcal{U}$) has a finite subfamily $\mathcal{V}$ which is a (strong) β_b -cover of G .

Theorem 3.5. Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $G \in L^X$. If $\beta(c \wedge d) = \beta(c) \cap \beta(d)$ ($\forall c, d \in L$), then the following conditions are equivalent to each other.

- (1) G is L -fuzzy countably compact.
- (2) For any $a \in M(L)$, each countable strong β_a -cover $\mathcal{U}$ of G with $a \in \beta\left(\bigwedge_{F \in \mathcal{U}} \mathcal{T}(F)\right)$ has a finite subfamily $\mathcal{V}$ which is a (strong) β_a -cover of G .
- (3) For any $a \in M(L)$ and any countable strong β_a -cover $\mathcal{U}$ of G with $a \in \beta\left(\bigwedge_{F \in \mathcal{U}} \mathcal{T}(F)\right)$, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in M(L)$ with $a \in \beta^*(b)$ such that $\mathcal{V}$ is a (strong) β_b -cover of G .

Now in order to research properties of L -fuzzy countably compactness, we introduce the following definition.

Definition 3.6. Let $(X, \mathcal{T})$ be an L -topological space, $a \in M(L)$ and $G \in L^X$. G is said to be countably a -compact if and only if $\forall b \in \beta(a)$, each countable Q_a -open cover $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ which is a Q_b -open cover of G .

Theorem 3.7. Let $(X, \mathcal{T})$ be an L -topological space. $G \in L^X$ is countably compact if and only if $\forall a \in M(L)$, G is countably a -compact.

Theorem 3.8. Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $G \in L^X$. G is L -fuzzy countably compact in $(X, \mathcal{T})$ if and only if $\forall a \in M(L)$, G is countably a -compact in $(X, \mathcal{T}_{[a]})$.

Proof. (Necessity) Since G is L -fuzzy countably compact in $(X, \mathcal{T})$, by Definition 2.1 we know that for every countable family $\mathcal{U} \subseteq L^X$, it follows that

$$\bigwedge_{F \in \mathcal{U}} \mathcal{T}(F) \wedge \left(\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{U}} F(x) \right) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{V}} F(x) \right).$$

Hence $\forall a \in M(L)$ and for every countable family $\mathcal{U} \subseteq \mathcal{T}_{[a]}$, we have that

$$a \leq \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{U}} F(x) \right) \Rightarrow a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{V}} F(x) \right).$$

Thus $\forall b \in \beta(a)$, there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that $b \leq \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{V}} F(x) \right)$, i.e., $\forall a \in M(L)$, $\forall b \in \beta(a)$, each countable Q_a -cover $\mathcal{U}$ of G in $(X, \mathcal{T}_{[a]})$ has a finite subfamily $\mathcal{V}$ which is a Q_b -cover of G . Therefore $\forall a \in M(L)$, G is countably a -compact in $(X, \mathcal{T}_{[a]})$.

(Sufficiency) Suppose that $\forall a \in M(L)$, G is countably a -compact in $(X, \mathcal{T}_{[a]})$. Let $\mathcal{U} \subseteq L^X$ ($\mathcal{U}$ is countable family) and $a \leq \bigwedge_{F \in \mathcal{U}} \mathcal{T}(F) \wedge \left(\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{U}} F(x) \right) \right)$. Then $\mathcal{U} \subseteq \mathcal{T}_{[a]}$ and $a \leq \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{U}} F(x) \right)$. Thus $\forall b \in \beta(a)$, there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that $b \leq \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{V}} F(x) \right)$.

Hence $a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{V}} F(x) \right)$. Therefore G is L -fuzzy countably compact in $(X, \mathcal{T})$. $\square$

Analogous to Shi's proof in [8], we can obtain the following Lemma 2.7.

Lemma 3.9. Let $(X, \mathcal{T})$ be an L -topological space, $a \in M(L)$ and $G \in L^X$. If G is countably a -compact, then $G \wedge H$ is countably a -compact for each $H \in \mathcal{T}'$.

Theorem 3.10. Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $G \in L^X$. If G is L -fuzzy countably compact, then for each $H \in L^X$ with $\mathcal{T}^*(H) = \top$, $G \wedge H$ is L -fuzzy countably compact.

Proof. $\forall a \in M(L)$, since G is L -fuzzy countably compact in $(X, \mathcal{T})$, by Theorem 2.6, G is countably a -compact in $(X, \mathcal{T}_{[a]})$. By $\mathcal{T}^*(H) = \top$, we

know that $H \in \mathcal{T}'_{[a]}$. Further by Lemma 2.7, $G \wedge H$ is countably a -compact in $(X, \mathcal{T}_{[a]})$. Then by Theorem 2.8, $G \wedge H$ is L -fuzzy countably compact in $(X, \mathcal{T})$. $\square$

Analogous to Shi's proof in [8], we can obtain the following Lemma 2.9.

Lemma 3.11. *Let $(X, \mathcal{T})$ be an L -topological space, $G, H \in L^X$ and $a \in M(L)$. If G and H are countably a -compact, then $G \vee H$ is countably a -compact as well.*

Theorem 3.12. *Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $H, G \in L^X$. If G and H are L -fuzzy countably compact, then $G \vee H$ is L -fuzzy countably compact as well.*

Proof. Since both G and H are L -fuzzy countably compact in $(X, \mathcal{T})$, by Theorem 2.6, $\forall a \in M(L)$, we know that both G and H are countably a -compact in $(X, \mathcal{T}_{[a]})$. By Lemma 2.9, $G \vee H$ is countably a -compact in $(X, \mathcal{T}_{[a]})$. So $G \vee H$ is L -fuzzy countably compact in $(X, \mathcal{T})$. $\square$

Analogous to Shi's proof in [8], we can obtain the following Lemma 2.11.

Lemma 3.13. *Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be two L -topological spaces and $a \in M(L)$. If G is countably a -compact in $(X, \mathcal{T})$ and $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ is an L -continuous mapping, then $f_L^{\rightarrow}(G)$ is countably a -compact in $(Y, \mathcal{U})$.*

Theorem 3.14. *Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be two L -fuzzy topological spaces, and $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be an L -fuzzy continuous mapping. If $G \in L^X$ is L -fuzzy countably compact in $(X, \mathcal{T})$, then so is $f_L^{\rightarrow}(G)$ in $(Y, \mathcal{U})$.*

Proof. Since G is L -fuzzy countably compact in $(X, \mathcal{T})$, by Theorem 2.6, $\forall a \in M(L)$, G is countably a -compact in $(X, \mathcal{T}_{[a]})$. By Theorem 1.2, $f : (X, \mathcal{T}_{[a]}) \rightarrow (Y, \mathcal{U}_{[a]})$ is an L -continuous mapping. Hence $f_L^{\rightarrow}(G)$ is countably a -compact in $(Y, \mathcal{U}_{[a]})$. Therefore $f_L^{\rightarrow}(G)$ is L -fuzzy countably compact in $(Y, \mathcal{U})$. $\square$

4. The L -fuzzy Lindelöf property

Definition 4.1. *Let $(X, \mathcal{T})$ be an L -fuzzy topological space. $G \in L^X$ is said to have the L -fuzzy Lindelöf property if for every family $\mathcal{U} \subseteq L^X$, it*

follows that

$$\bigwedge_{F \in \mathcal{U}} \mathcal{T}(F) \wedge \left(\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{U}} F(x) \right) \right) \leq \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{F \in \mathcal{V}} F(x) \right).$$

Obviously we have the following theorem.

Theorem 4.2. *Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $G \in L^X$ has the L -fuzzy Lindelöf property. Then G is L -fuzzy compact if and only if it is L -fuzzy countably compact.*

Analogous to L -fuzzy countable compactness, we have the following results.

Theorem 4.3. *Let $(X, \mathcal{T})$ be an L -topological space and $G \in L^X$. G has the L -fuzzy Lindelöf property in $(X, \chi_{\mathcal{T}})$ if and only if G has the Lindelöf property in $(X, \mathcal{T})$.*

Theorem 4.4. *Let $(X, \mathcal{T})$ be an L -fuzzy topological space. $G \in L^X$ has the L -fuzzy Lindelöf property if and only if for every family $\mathcal{P} \subseteq L^X$, it follows that*

$$\bigvee_{F \in \mathcal{P}} \mathcal{T}'(F) \vee \left(\bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{F \in \mathcal{P}} F(x) \right) \right) \geq \bigwedge_{\mathcal{H} \in 2^{[\mathcal{P}]}} \bigvee_{x \in X} \left(G(x) \wedge \bigwedge_{F \in \mathcal{H}} F(x) \right).$$

Theorem 4.5. *Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $G \in L^X$. Then the following conditions are equivalent to each other.*

- (1) G has the L -fuzzy Lindelöf property.
- (2) For any $a \in M(L)$, each strong a -remote family $\mathcal{P}$ of G with $\bigwedge_{F \in \mathcal{P}} \mathcal{T}^*(F) \not\leq a'$ has a countable subfamily $\mathcal{H}$ which is a (strong) a -remote family of G .
- (3) For any $a \in M(L)$, and any strong a -remote family $\mathcal{P}$ of G with $\bigwedge_{F \in \mathcal{P}} \mathcal{T}^*(F) \not\leq a'$, there exists a countable subfamily $\mathcal{H}$ of $\mathcal{P}$ and $b \in \beta^*(a)$ such that $\mathcal{H}$ is a (strong) b -remote family of G .
- (4) For any $a \in P(L)$, each strong a -shading $\mathcal{U}$ of G with $\bigwedge_{F \in \mathcal{U}} \mathcal{T}(F) \not\leq a$ has a countable subfamily $\mathcal{V}$ which is a (strong) a -shading of G .

- (5) For any $a \in P(L)$ and any strong a -shading $\mathcal{U}$ of G with $\bigwedge_{F \in \mathcal{U}} \mathcal{T}(F) \not\leq a$, there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha^*(a)$ such that $\mathcal{V}$ is a (strong) b -shading of G .
- (6) For any $a \in M(L)$ and any $b \in \beta^*(a)$, each Q_a -cover $\mathcal{U}$ of G with $\mathcal{T}(F) \geq a$ ($\forall F \in \mathcal{U}$) has a countable subfamily $\mathcal{V}$ which is a Q_b -cover of G .
- (7) For any $a \in M(L)$ and any $b \in \beta^*(a)$, each Q_a -cover $\mathcal{U}$ of G with $\mathcal{T}(F) \geq a$ ($\forall F \in \mathcal{U}$) has a countable subfamily $\mathcal{V}$ which is a (strong) β_b -cover of G .

Theorem 4.6. Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $G \in L^X$. If $\beta(c \wedge d) = \beta(c) \cap \beta(d)$ ($\forall c, d \in L$), then the following conditions are equivalent to each other.

- (1) G has the L -fuzzy Lindelöf property.
- (2) For any $a \in M(L)$, each strong β_a -cover $\mathcal{U}$ of G with $a \in \beta \left(\bigwedge_{F \in \mathcal{U}} \mathcal{T}(F) \right)$ has a countable subfamily $\mathcal{V}$ which is a (strong) β_a -cover of G .
- (3) For any $a \in M(L)$ and any strong β_a -cover $\mathcal{U}$ of G with $a \in \beta \left(\bigwedge_{F \in \mathcal{U}} \mathcal{T}(F) \right)$, there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in M(L)$ with $a \in \beta^*(b)$ such that $\mathcal{V}$ is a (strong) β_b -cover of G .

Definition 4.7. Let $(X, \mathcal{T})$ be an L -topological space, $a \in M(L)$ and $G \in L^X$. G has the a -Lindelöf property if and only if $\forall b \in \beta(a)$, each Q_a -open cover $\mathcal{U}$ of G has a countable subfamily $\mathcal{V}$ which is a Q_b -open cover of G .

Theorem 4.8. Let $(X, \mathcal{T})$ be an L -topological space. Then $G \in L^X$ has the Lindelöf property if and only if $\forall a \in M(L)$, G has the a -Lindelöf property.

Theorem 4.9. Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $G \in L^X$. Then G has the L -fuzzy Lindelöf property in $(X, \mathcal{T})$ if and only if $\forall a \in M(L)$, G has the a -Lindelöf property in $(X, \mathcal{T}_{[a]})$.

Lemma 4.10. Let $(X, \mathcal{T})$ be an L -topological space, $a \in M(L)$ and $G \in L^X$. If G has the a -Lindelöf property, then $G \wedge H$ has the a -Lindelöf property for each $H \in \mathcal{T}'$.

Theorem 4.11. Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $G \in L^X$. If G has the L -fuzzy Lindelöf property, then for each $H \in L^X$ with $\mathcal{T}^*(H) = \top$, $G \wedge H$ has the L -fuzzy Lindelöf property.

Lemma 4.12. Let $(X, \mathcal{T})$ be an L -topological space, $G, H \in L^X$ and $a \in M(L)$. If G and H have the a -Lindelöf property, then $G \vee H$ has the a -Lindelöf property as well.

Theorem 4.13. Let $(X, \mathcal{T})$ be an L -fuzzy topological space and $H, G \in L^X$. If G and H have the L -fuzzy Lindelöf property, then $G \vee H$ has the L -fuzzy Lindelöf property as well.

Lemma 4.14. Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be two L -topological spaces and $a \in M(L)$. If G has the a -Lindelöf property in $(X, \mathcal{T})$ and $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ is an L -continuous mapping, then $f_L^\rightarrow(G)$ has the a -Lindelöf property in $(Y, \mathcal{U})$.

Theorem 4.15. Let $(X, \mathcal{T})$, $(Y, \mathcal{U})$ be two L -fuzzy topological spaces, and $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U})$ be an L -fuzzy continuous mapping. If $G \in L^X$ has the L -fuzzy Lindelöf property in $(X, \mathcal{T})$, then $f_L^\rightarrow(G)$ has the L -fuzzy Lindelöf property in $(Y, \mathcal{U})$.

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