COUNTABLE COMPACTNESS AND THE LINDELOF PROPERTY IN L-FUZZY TOPOLOGICAL SPACES *

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Abstract

In this paper, the concepts of L-fuzzy countable compactness and the L-fuzzy Lindelöf property are introduced in L-fuzzy topological spaces, where L is a completely distributive De Morgan algebra. An L-fuzzy compact L-fuzzy set is L-fuzzy countably compact and has the L-fuzzy Lindelöf property. An L-fuzzy set having the L-fuzzy Lindelöf property is L-fuzzy countably compact if and only if it is L-fuzzy compact. Many characterizations of L-fuzzy countable compactness and the L-fuzzy Lindelöf property are presented.

Keywords : L-fuzzy topology, L-fuzzy countable compactness, the L-fuzzy Lindelöf property.

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1. Introduction

In 1976, the concept of fuzzy compactness was introduced in [0, 1]-topological spaces by R. Lowen [5]. Subsequently its characterization was given by G.J. Wang in terms of α-net in [12]. In 1988, it was extended to L-topological spaces [13], where L is a completely distributive DeMorgan algebra. In [9], a new definition of fuzzy compactness was presented by means of open L-sets and their inequality in L-topological spaces. When L is a completely distributive DeMorgan algebra, it is equivalent to the notion of fuzzy compactness in [4, 7, 13]. Recently the concept of L-fuzzy compactness was introduced by Shi and Li [10] in L-fuzzy topological spaces.

In this paper, our aim is to continue the research of L-fuzzy countable compactness and the L-fuzzy Lindelöf property of L-fuzzy sets.

2. Preliminaries

Throughout this paper \((L, \vee, \wedge, \cdot)\) is a completely distributive DeMorgan algebra, \(X\) is a nonempty set. \(L^X\) is the set of all L-fuzzy sets on \(X\). The smallest element and the largest element in \(L^X\) are denoted respectively by \(\bot\) and \(\top\). An L-fuzzy set is briefly written as an L-set. We often do not distinguish a crisp subset \(A\) from its characteristic function \(\chi_A\).

The set of nonunit prime elements in \(L\) is denoted by \(P(L)\). The set of nonzero co-prime elements in \(L\) is denoted by \(M(L)\). The set of nonzero co-prime elements in \(L^X\) is denoted by \(M(L^X)\). The set all L-fuzzy points \(x_\lambda\) (i.e., an L-fuzzy set \(A \in L^X\) such that \(A(x) = \lambda \neq 0\) and \(A(y) = 0\) for \(y \neq x\)) is denoted by \(pt(L^X)\).

The binary relation \(<\) in \(L\) is defined as follows: for \(a, b \in L\), \(a < b\) if and only if for every subset \(D \subseteq L\), \(b \leq \sup D\) always implies the existence of \(d \in D\) with \(a \leq d\) [1]. In a completely distributive DeMorgan algebra \(L\), each member \(b\) is a sup of \(\{a \in L \mid a < b\}\). In the sense of [4, 13], \(\{a \in L \mid a < b\}\) is the greatest minimal family of \(b\), denoted by \(\beta(b)\), and \(\beta^*(b) = \beta(b) \cap M(L)\). Moreover for \(b \in L\), define \(\alpha(b) = \{a \in L \mid a' < b'\}\) and \(\alpha^*(b) = \alpha(b) \cap P(L)\).

For \(a \in L\) and \(A \in L^X\), we define \(A_{[a]} = \{x \in X \mid A(x) \geq a\}\).

**Definition 2.1 ([2, 3, 6, 11]).** An L-fuzzy topology on a set \(X\) is a map \(T : L^X \rightarrow L\) such that

1. \(T(\bot) = T(\top) = \top\);
2. \(\forall U, V \in L^X, T(U \wedge V) \geq T(U) \wedge T(V)\);
(3) \( \forall U_j \in L^X, j \in J, \mathcal{T}(\bigvee_{j \in J} U_j) \geq \bigwedge_{j \in J} \mathcal{T}(U_j). \)

\( \mathcal{T}(U) \) can be interpreted as the degree to which \( U \) is an open set. \( \mathcal{T}^*(U) = T(U') \) will be called the degree of closedness of \( U \). The pair \( (X, \mathcal{T}) \) is called an \( L \)-fuzzy topological space.

A mapping \( f : (X, \mathcal{T}) \to (Y, \mathcal{U}) \) is said to be \( L \)-fuzzy continuous if \( T(f^{-1}_L(B)) \geq \mathcal{U}(B) \) holds for all \( B \in L^Y \), where \( f^{-1}_L \) is defined by \( f^{-1}_L(x) = f(x) \) [6].

**Theorem 2.2 ([14]).** Let \( (X, \mathcal{T}) \) and \( (Y, \mathcal{U}) \) be \( L \)-fuzzy topological spaces. Then \( f : (X, \mathcal{T}_0) \to (Y, \mathcal{U}_0) \) be \( L \)-continuous if and only if \( \forall a \in M(L), f : (X, \mathcal{T}_a) \to (Y, \mathcal{U}_a) \) be \( L \)-continuous.

**Definition 2.3 ([8, 9]).** Let \( a \in L \setminus \{\top\} \) and \( G \in L^X \). A subfamily \( U \) in \( L^X \) is said to be

1. an \( a \)-shading of \( G \) if for any \( x \in X \), it follows that \( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \not\leq a. \)

2. a strong \( a \)-shading of \( G \) if \( \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \not\leq a. \)

**Definition 2.4 ([8, 9]).** Let \( a \in L \setminus \{\bot\} \) and \( G \in L^X \). A subfamily \( \mathcal{P} \) in \( L^X \) is said to be

1. an \( a \)-remote family of \( G \) if for any \( x \in X \), it follows that \( G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \not\geq a. \)

2. a strong \( a \)-remote family of \( G \) if \( \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{B \in \mathcal{P}} B(x) \right) \not\geq a. \)

**Definition 2.5 ([8, 9]).** Let \( a \in L \setminus \{\bot\} \) and \( G \in L^X \). A subfamily \( \mathcal{U} \) in \( L^X \) is called

1. a \( \beta_a \)-cover of \( G \) if for any \( x \in X \), it follows that \( a \in \beta \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right). \)

2. a strong \( \beta_a \)-cover of \( G \) if for any \( x \in X \), it follows that

\[
a \in \beta \left( \bigwedge_{x \in X} \left( G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \right).
\]
Definition 2.6 ([8, 9]). Let \( a \in L \setminus \{ \bot \} \) and \( G \in L^X \). A subfamily \( \mathcal{U} \) in \( L^X \) is called a \( Q_a \)-cover of \( G \) if \( a \leq \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{A \in \mathcal{U}} A(x) \right) \).

For a subfamily \( \Phi \subseteq L^X \), \( 2(\Phi) \) denotes the set of all finite subfamilies of \( \Phi \). \( 2[\Phi] \) denotes the set of countable subfamilies of \( \Phi \).

Definition 2.7 ([8]). Let \( (X, T) \) be an \( L \)-topological space. \( G \in L^X \) is said to be countably compact if for every countable family \( \mathcal{U} \subseteq L^X \), it follows that

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in \mathcal{U}} F(x) \right) \leq \bigvee_{V \in 2(\mathcal{U})} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in V} F(x) \right).
\]

Definition 2.8 ([8]). Let \( (X, T) \) be an \( L \)-topological space. \( G \in L^X \) is said to have the Lindelöf property if for every family \( \mathcal{U} \subseteq L^X \), it follows that

\[
\bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in \mathcal{U}} F(x) \right) \leq \bigvee_{V \in 2(\mathcal{U})} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in V} F(x) \right).
\]

Definition 2.9 ([10]). Let \( (X, T) \) be an \( L \)-fuzzy topological space. \( G \in L^X \) is said to be \( L \)-fuzzy compact if for every family \( \mathcal{U} \subseteq L^X \), it follows that

\[
\bigwedge_{F \in \mathcal{U}} T(F) \land \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in \mathcal{U}} F(x) \right) \right) \leq \bigvee_{V \in 2(\mathcal{U})} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in V} F(x) \right).
\]

3. \( L \)-fuzzy countable compactness

Definition 3.1. Let \( (X, T) \) be an \( L \)-fuzzy topological space. \( G \in L^X \) is said to be \( L \)-fuzzy countably compact if for every countable family \( \mathcal{U} \subseteq L^X \), it follows that

\[
\bigwedge_{F \in \mathcal{U}} T(F) \land \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in \mathcal{U}} F(x) \right) \right) \leq \bigvee_{V \in 2(\mathcal{U})} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in V} F(x) \right).
\]

Obviously \( L \)-fuzzy compactness implies \( L \)-fuzzy countable compactness.

Let \( (X, T) \) be an \( L \)-topological space. Let \( \chi_T : L^X \to L \)

\[
\chi_T = \begin{cases} 
1, & A \in T, \\
0, & A \notin T.
\end{cases}
\]
Obviously, \((X, \chi_T)\) is a special \(L\)-fuzzy topological spaces. So we can easily prove the following theorem.

**Theorem 3.2.** Let \((X, T)\) be an \(L\)-topological space and \(G \in L^X\). \(G\) is \(L\)-fuzzy countably compact in \((X, \chi_T)\) if and only if \(G\) is countably compact \([8]\) in \((X, T)\).

From Definition 2.1 we easily obtain the following theorem by simply using quasi-complement.

**Theorem 3.3.** Let \((X, T)\) be an \(L\)-fuzzy topological space. \(G \in L^X\) is \(L\)-fuzzy countably compact if and only if for every countably family \(\mathcal{P} \subseteq L^X\) it follows that

\[
\bigvee_{F \in \mathcal{P}} T'(F) \vee \left( \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{F \in \mathcal{P}} F(x) \right) \right) \geq \bigwedge_{H \in 2^{(\mathcal{P})}} \bigvee_{x \in X} \left( G(x) \wedge \bigwedge_{F \in H} F(x) \right).
\]

By Definition 2.1 and Theorem 2.2 and analogous to \([8]\) we immediately obtain the following result.

**Theorem 3.4.** Let \((X, T)\) be an \(L\)-fuzzy topological space and \(G \in L^X\). Then the following conditions are equivalent to each other.

1. \(G\) is \(L\)-fuzzy countably compact.
2. For any \(a \in M(L)\), each countable strong \(a\)-remote family \(\mathcal{P}\) of \(G\) with \(\bigwedge_{F \in \mathcal{P}} T^*(F) \not\leq a'\) has a finite subfamily \(\mathcal{H}\) which is a (strong) \(a\)-remote family of \(G\).
3. For any \(a \in M(L)\), and any countable strong \(a\)-remote family \(\mathcal{P}\) of \(G\) with \(\bigwedge_{F \in \mathcal{P}} T^*(F) \not\leq a'\), there exists a finite subfamily \(\mathcal{H}\) of \(\mathcal{P}\) and \(b \in \beta^*(a)\) such that \(\mathcal{H}\) is a (strong) \(b\)-remote family of \(G\).
4. For any \(a \in P(L)\), each countable strong \(a\)-shading \(U\) of \(G\) with \(\bigwedge_{F \in U} T(F) \not\leq a\) has a finite subfamily \(\mathcal{V}\) which is a (strong) \(a\)-shading of \(G\).
5. For any \(a \in P(L)\) and any countable strong \(a\)-shading \(U\) of \(G\) with \(\bigwedge_{F \in U} T(F) \not\leq a\), there exists a finite subfamily \(\mathcal{V}\) of \(U\) and \(b \in \alpha^*(a)\) such that \(\mathcal{V}\) is a (strong) \(b\)-shading of \(G\).
(6) For any $a \in M(L)$ and any $b \in \beta^*(a)$, each countable $Q_a$-cover $U$ of $G$ with $T(F) \geq a \ (\forall F \in U)$ has a finite subfamily $V$ which is a $Q_b$-cover of $G$.

(7) For any $a \in M(L)$ and any $b \in \beta^*(a)$, each countable $Q_a$-cover $U$ of $G$ with $T(F) \geq a \ (\forall F \in U)$ has a finite subfamily $V$ which is a (strong) $\beta_b$-cover of $G$.

Theorem 3.5. Let $(X, T)$ be an $L$-fuzzy topological space and $G \in L^X$. If $\beta(c \wedge d) = \beta(c) \cap \beta(d) \ (\forall c, d \in L)$, then the following conditions are equivalent to each other.

1. $G$ is $L$-fuzzy countably compact.

2. For any $a \in M(L)$, each countable strong $\beta_a$-cover $U$ of $G$ with $a \in \beta \left( \bigwedge_{F \in U} T(F) \right)$ has a finite subfamily $V$ which is a (strong) $\beta_a$-cover of $G$.

3. For any $a \in M(L)$ and any countable strong $\beta_a$-cover $U$ of $G$ with $a \in \beta \left( \bigwedge_{F \in U} T(F) \right)$, there exists a finite subfamily $V$ of $U$ and $b \in M(L)$ with $a \in \beta^*(b)$ such that $V$ is a (strong) $\beta_b$-cover of $G$.

Now in order to research properties of $L$-fuzzy countably compactness, we introduce the following definition.

Definition 3.6. Let $(X, T)$ be an $L$-topological space, $a \in M(L)$ and $G \in L^X$. $G$ is said to be countably $a$-compact if and only if $\forall b \in \beta(a)$, each countable $Q_a$-open cover $U$ of $G$ has a finite subfamily $V$ which is a $Q_b$-open cover of $G$.

Theorem 3.7. Let $(X, T)$ be an $L$-topological space. $G \in L^X$ is countably compact if and only if $\forall a \in M(L)$, $G$ is countably $a$-compact.

Theorem 3.8. Let $(X, T)$ be an $L$-fuzzy topological space and $G \in L^X$. $G$ is $L$-fuzzy countably compact in $(X, T)$ if and only if $\forall a \in M(L)$, $G$ is countably $a$-compact in $(X, T_{[a]})$. 
Proof. (Necessity) Since $G$ is $L$-fuzzy countably compact in $(X, T)$, by Definition 2.1 we know that for every countable family $\mathcal{U} \subseteq L^X$, it follows that

$$\bigwedge_{F \in \mathcal{U}} T(F) \wedge \left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in \mathcal{V}} F(x) \right) \right) \leq \bigvee_{V \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in \mathcal{V}} F(x) \right).$$

Hence $\forall a \in M(L)$ and for every countable family $\mathcal{U} \subseteq T[a]$, we have that

$$a \leq \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in \mathcal{V}} F(x) \right) \Rightarrow a \leq \bigvee_{V \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in \mathcal{V}} F(x) \right).$$

Thus $\forall b \in \beta(a)$, there exists $V \in 2^{\mathcal{U}}$ such that $b \leq \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in \mathcal{V}} F(x) \right)$, i.e., $\forall a \in M(L)$, $\forall b \in \beta(a)$, each countable $Q_a$-cover $\mathcal{U}$ of $G$ in $(X, T[a])$ has a finite subfamily $\mathcal{V}$ which is a $Q_b$-cover of $G$. Therefore $\forall a \in M(L)$, $G$ is countably a-compact in $(X, T[a])$.

(Sufficiency) Suppose that $\forall a \in M(L)$, $G$ is countably a-compact in $(X, T[a])$. Let $\mathcal{U} \subseteq L^X$ ( $\mathcal{U}$ is countable family ) and $a \leq \bigwedge_{F \in \mathcal{U}} T(F) \wedge$\

$$\left( \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in \mathcal{V}} F(x) \right) \right).$$

Then $\mathcal{U} \subseteq T[a]$ and $a \leq \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in \mathcal{V}} F(x) \right)$.

Thus $\forall b \in \beta(a)$, there exists $V \in 2^{\mathcal{U}}$ such that $b \leq \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in \mathcal{V}} F(x) \right)$.

Hence $a \leq \bigvee_{V \in 2^{\mathcal{U}}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in \mathcal{V}} F(x) \right)$. Therefore $G$ is $L$-fuzzy countably compact in $(X, T)$. □

Analogous to Shi’s proof in [8], we can obtain the following Lemma 2.7.

Lemma 3.9. Let $(X, T)$ be an $L$-topological space, $a \in M(L)$ and $G \in L^X$. If $G$ is countably a-compact, then $G \wedge H$ is countably a-compact for each $H \in T'$.

Theorem 3.10. Let $(X, T)$ be an $L$-fuzzy topological space and $G \in L^X$. If $G$ is $L$-fuzzy countably compact, then for each $H \in L^X$ with $T^*(H) = T$, $G \wedge H$ is $L$-fuzzy countably compact.

Proof. $\forall a \in M(L)$, since $G$ is $L$-fuzzy countably compact in $(X, T)$, by Theorem 2.6, $G$ is countably a-compact in $(X, T[a])$. By $T^*(H) = T$, we
know that $H \in T_a'$. Further by Lemma 2.7, $G \wedge H$ is countably $a$-compact in $(X, T[a])$. Then by Theorem 2.8, $G \wedge H$ is $L$-fuzzy countably compact in $(X, T)$. □

Analogous to Shi’s proof in [8], we can obtain the following Lemma 2.9.

**Lemma 3.11.** Let $(X, T)$ be an $L$-topological space, $G, H \in L^X$ and $a \in M(L)$. If $G$ and $H$ are countably $a$-compact, then $G \vee H$ is countably $a$-compact as well.

**Theorem 3.12.** Let $(X, T)$ be an $L$-fuzzy topological space and $H, G \in L^X$. If $G$ and $H$ are $L$-fuzzy countably compact, then $G \vee H$ is $L$-fuzzy countably compact as well.

**Proof.** Since both $G$ and $H$ are $L$-fuzzy countably compact in $(X, T)$, by Theorem 2.6, $\forall a \in M(L)$, we know that both $G$ and $H$ are countably $a$-compact in $(X, T[a])$. By Lemma 2.9, $G \vee H$ is countably $a$-compact in $(X, T[a])$. So $G \vee H$ is $L$-fuzzy countably compact in $(X, T)$. □

Analogous to Shi’s proof in [8], we can obtain the following Lemma 2.11.

**Lemma 3.13.** Let $(X, T), (Y, U)$ be two $L$-topological spaces and $a \in M(L)$. If $G$ is countably $a$-compact in $(X, T)$ and $f : (X, T) \to (Y, U)$ is an $L$-continuous mapping, then $f_L^-(G)$ is countably $a$-compact in $(Y, U)$.

**Theorem 3.14.** Let $(X, T), (Y, U)$ be two $L$-fuzzy topological spaces, and $f : (X, T) \to (Y, U)$ be an $L$-fuzzy continuous mapping. If $G \in L^X$ is $L$-fuzzy countably compact in $(X, T)$, then so is $f_L^-(G)$ in $(Y, U)$.

**Proof.** Since $G$ is $L$-fuzzy countably compact in $(X, T)$, by Theorem 2.6, $\forall a \in M(L)$, $G$ is countably $a$-compact in $(X, T[a])$. By Theorem 1.2, $f : (X, T[a]) \to (Y, U[a])$ is an $L$-continuous mapping. Hence $f_L^-(G)$ is countably $a$-compact in $(Y, U[a])$. Therefore $f_L^-(G)$ is $L$-fuzzy countably compact in $(Y, U)$. □

4. The $L$-fuzzy Lindelöf property

**Definition 4.1.** Let $(X, T)$ be an $L$-fuzzy topological space. $G \in L^X$ is said to have the $L$-fuzzy Lindelöf property if for every family $\mathcal{U} \subseteq L^X$, it
follows that
\[ \bigwedge_{F \in U} T(F) \land \left( \bigvee_{x \in X} \left( G'(x) \lor \bigvee_{F \in U} F(x) \right) \right) \leq \bigvee_{V \in \mathcal{P}} \bigwedge_{x \in X} \left( G'(x) \lor \bigvee_{F \in V} F(x) \right). \]

Obviously we have the following theorem.

**Theorem 4.2.** Let \((X, T)\) be an \(L\)-fuzzy topological space and \(G \in L^X\) has the \(L\)-fuzzy Lindelöf property. Then \(G\) is \(L\)-fuzzy compact if and only if it is \(L\)-fuzzy countably compact.

Analogous to \(L\)-fuzzy countable compactness, we have the following results.

**Theorem 4.3.** Let \((X, T)\) be an \(L\)-topological space and \(G \in L^X\). \(G\) has the \(L\)-fuzzy Lindelöf property in \((X, \chi T)\) if and only if \(G\) has the Lindelöf property in \((X, T)\).

**Theorem 4.4.** Let \((X, T)\) be an \(L\)-fuzzy topological space. \(G \in L^X\) has the \(L\)-fuzzy Lindelöf property if and only if for every family \(\mathcal{P} \subseteq L^X\), it follows that
\[ \bigvee_{F \in \mathcal{P}} T'(F) \lor \left( \bigvee_{x \in X} (G(x) \land \bigwedge_{F \in \mathcal{P}} F(x)) \right) \geq \bigwedge_{H \in \mathcal{P}} \bigvee_{x \in X} \left( G(x) \land \bigwedge_{F \in H} F(x) \right). \]

**Theorem 4.5.** Let \((X, T)\) be an \(L\)-fuzzy topological space and \(G \in L^X\). Then the following conditions are equivalent to each other.

1. \(G\) has the \(L\)-fuzzy Lindelöf property.
2. For any \(a \in M(L)\), each strong \(a\)-remote family \(\mathcal{P}\) of \(G\) with \(\bigwedge_{F \in \mathcal{P}} T^*(F) \not\leq a'\) has a countable subfamily \(\mathcal{H}\) which is a (strong) \(a\)-remote family of \(G\).
3. For any \(a \in M(L)\), and any strong \(a\)-remote family \(\mathcal{P}\) of \(G\) with \(\bigwedge_{F \in \mathcal{P}} T^*(F) \not\leq a'\), there exists a countable subfamily \(\mathcal{H}\) of \(\mathcal{P}\) and \(b \in \beta^*(a)\) such that \(\mathcal{H}\) is a (strong) \(b\)-remote family of \(G\).
4. For any \(a \in P(L)\), each strong \(a\)-shading \(U\) of \(G\) with \(\bigwedge_{F \in U} T(F) \not\leq a\) has a countable subfamily \(\mathcal{V}\) which is a (strong) \(a\)-shading of \(G\).
For any $a \in P(L)$ and any strong $a$-shading $\mathcal{U}$ of $G$ with $\bigwedge_{F \in \mathcal{U}} T(F) \not\leq a$, there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha^*(a)$ such that $\mathcal{V}$ is a (strong) $b$-shading of $G$.

(6) For any $a \in M(L)$ and any $b \in \beta^*(a)$, each $Q_a$-cover $\mathcal{U}$ of $G$ with $T(F) \geq a$ ($\forall F \in \mathcal{U}$) has a countable subfamily $\mathcal{V}$ which is a $Q_b$-cover of $G$.

(7) For any $a \in M(L)$ and any $b \in \beta^*(a)$, each $Q_a$-cover $\mathcal{U}$ of $G$ with $T(F) \geq a$ ($\forall F \in \mathcal{U}$) has a countable subfamily $\mathcal{V}$ which is a (strong) $\beta_b$-cover of $G$.

**Theorem 4.6.** Let $(X, T)$ be an $L$-fuzzy topological space and $G \in LX$. If $\beta(c \land d) = \beta(c) \cap \beta(d)$ ($\forall c, d \in L$), then the following conditions are equivalent to each other.

(1) $G$ has the $L$-fuzzy Lindelöf property.

(2) For any $a \in M(L)$, each strong $\beta_a$-cover $\mathcal{U}$ of $G$ with $a \in \beta \left( \bigwedge_{F \in \mathcal{U}} T(F) \right)$ has a countable subfamily $\mathcal{V}$ which is a (strong) $\beta_a$-cover of $G$.

(3) For any $a \in M(L)$ and any strong $\beta_a$-cover $\mathcal{U}$ of $G$ with $a \in \beta \left( \bigwedge_{F \in \mathcal{U}} T(F) \right)$, there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in M(L)$ with $a \in \beta^*(b)$ such that $\mathcal{V}$ is a (strong) $\beta_b$-cover of $G$.

**Definition 4.7.** Let $(X, T)$ be an $L$-topological space, $a \in M(L)$ and $G \in LX$. $G$ has the $a$-Lindelöf property if and only if $\forall b \in \beta(a)$, each $Q_a$-open cover $\mathcal{U}$ of $G$ has a countable subfamily $\mathcal{V}$ which is a $Q_b$-open cover of $G$.

**Theorem 4.8.** Let $(X, T)$ be an $L$-topological space. Then $G \in LX$ has the Lindelöf property if and only if $\forall a \in M(L)$, $G$ has the $a$-Lindelöf property.

**Theorem 4.9.** Let $(X, T)$ be an $L$-fuzzy topological space and $G \in LX$. Then $G$ has the $L$-fuzzy Lindelöf property in $(X, T)$ if and only if $\forall a \in M(L)$, $G$ has the $a$-Lindelöf property in $(X, T[a])$.

**Lemma 4.10.** Let $(X, T)$ be an $L$-topological space, $a \in M(L)$ and $G \in LX$. If $G$ has the $a$-Lindelöf property, then $G \land H$ has the $a$-Lindelöf property for each $H \in T'$. 
Theorem 4.11. Let $(X, T)$ be an L-fuzzy topological space and $G \in L^X$. If $G$ has the L-fuzzy Lindelöf property, then for each $H \in L^X$ with $T^*(H) = \top$, $G \land H$ has the L-fuzzy Lindelöf property.

Lemma 4.12. Let $(X, T)$ be an L-topological space, $G, H \in L^X$ and $a \in M(L)$. If $G$ and $H$ have the $a$-Lindelöf property, then $G \lor H$ has the $a$-Lindelöf property as well.

Theorem 4.13. Let $(X, T)$ be an L-fuzzy topological space and $H, G \in L^X$. If $G$ and $H$ have the L-fuzzy Lindelöf property, then $G \lor H$ has the L-fuzzy Lindelöf property as well.

Lemma 4.14. Let $(X, T)$, $(Y, U)$ be two L-topological spaces and $a \in M(L)$. If $G$ has the $a$-Lindelöf property in $(X, T)$ and $f : (X, T) \to (Y, U)$ is an L-continuous mapping, then $f_L^*(G)$ has the $a$-Lindelöf property in $(Y, U)$.

Theorem 4.15. Let $(X, T)$, $(Y, U)$ be two L-fuzzy topological spaces, and $f : (X, T) \to (Y, U)$ be an L-fuzzy continuous mapping. If $G \in L^X$ has the L-fuzzy Lindelöf property in $(X, T)$, then $f_L^*(G)$ has the L-fuzzy Lindelöf property in $(Y, U)$.

References


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