Generalized Ulam-Hyers Stabilities of Quartic Derivations on Banach Algebras

M. Eshaghi Gordji
Semnan University, Iran

and

N. Ghalbipoor
Urmia University, Iran

Received: July 2010. Accepted: September 2010

Abstract

Let $A, B$ be two rings. A mapping $\delta : A \rightarrow B$ is called quartic derivation, if $\delta$ is a quartic function satisfies $\delta(ab) = a^4 \delta(b) + \delta(a)b^4$ for all $a, b \in A$. The main purpose of this paper is to prove the generalized Hyers-Ulam-Rassias stability of the quartic derivations on Banach algebras.

2000 Mathematics Subject Classification: Primary 39B52, Secondary 39B82.

Keywords: Banach algebras; quartic functional equation; quartic derivation; Hyer-Ulam-Rassias stability.
1. Introduction

The study of stability problems as just mentioned originated from a famous talk given by S.M. Ulam [65] in 1940: Under what condition does there exists a homomorphism near an approximate homomorphism? In 1941, D. H. Hyers [28] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \rightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then $T$ is linear. Finally in 1978, Th. M. Rassias [60] proved the following theorem.

**Theorem 1.1.** Let $f : E \rightarrow E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon (\|x\|^p + \|y\|^p) \quad (1.1)$$

for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \quad (1.2)$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for all $x, y \neq 0$, and (1.2) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into $E'$ is continuous in real $t$ for each fixed $x \in E$, then $T$ is linear.

In 1991, Z. Gajda [20] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers–Ulam–Rassias stability of functional equations.

In 1982–1994, J.M. Rassias (see [46]–[53]) solved the Ulam problem for different mappings and for many Euler–Lagrange type quadratic mappings, by involving a product of different powers of norms. In addition, J.M. Rassias considered the mixed product–sum of powers of norms control function
This concept is known as Ulam–Gavruta–Rassias stability of functional equations. For more details about the results concerning such problems and mixed product-sum stability (JMRassias Stability) the reader is referred to [1, 5, 6, 7, 8, 17, 19, 22, 24, 25, 26, 27, 30, 32, 34, 36, 37, 43, 45, 54, 55] and [56].

In 1994, a generalization of the Rassias, theorem was obtained by Gavruta as follows [21] (see also [23], [31]).

Suppose \((G, +)\) is an abelian group, \(E\) is a Banach space, and that the so-called admissible control function \(\varphi : G \times G \to \mathbb{R}\) satisfies

\[
\tilde{\varphi}(x, y) := 2^{-1} \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) < \infty
\]

for all \(x, y \in G\). If \(f : G \to E\) is a mapping with

\[
\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)
\]

for all \(x, y \in G\), then there exists a unique mapping \(T : G \to E\) such that \(T(x + y) = T(x) + T(y)\) and \(\|f(x) - T(x)\| \leq \tilde{\varphi}(x, x)\) for all \(x, y \in G\).

In [40], Won-Gil Park and Jea Hyeong Bae, considered the following functional equation:

\[
f(2x + y) + f(2x - y) = 4(f(x + y) + f(x - y)) + 24f(x) - 6f(y). \quad (1.3)
\]

In fact they proved that a function \(f\) between real vector spaces \(X\) and \(Y\) is a solution of (1.3) if and only if there exists a unique symmetric multi-additive function \(B : X \times X \times X \times X \to Y\) such that \(f(x) = B(x, x, x, x)\) for all \(x \in X\). It is easy to show that the function \(f(x) = x^4\) satisfies the functional equation (1.3), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function (see also [2]).

Let \(\mathcal{A}\) be an algebra over the real or complex field \(\mathbf{F}\) and \(\mathcal{X}\) a left \(\mathcal{A}\)-module (respectively \(\mathcal{A}\)-bimodule). An additive map \(\delta : \mathcal{A} \to \mathcal{X}\) said to be a module left derivation (respectively module derivation) if \(\delta(xy) = x.\delta(y) + y.\delta(x)\) (respectively \(\delta(xy) = x.\delta(y) + \delta(x).y\) holds for all \(x, y \in \mathcal{A}\) where . denotes the module multiplication on \(\mathcal{X}\). Since \(\mathcal{A}\) is a left \(\mathcal{A}\)-module (respectively \(\mathcal{A}\)-bimodule) with the product of \(\mathcal{A}\) giving the module multiplication (respectively two module multiplications), the module left derivation (respectively module derivation) \(\delta : \mathcal{A} \to \mathcal{A}\) is said to be a ring left derivation (respectively ring derivation) on \(\mathcal{A}\). Furthermore, if the identity \(\delta(kx) = k\delta(x)\) holds for all \(k \in \mathbf{F}\) and all \(x \in \mathcal{A}\), then \(\delta\) is a linear left derivation (respectively linear derivation).
Let us introduce the background of our investigation. Recently, T. Miura et al. [35] considered the stability of ring derivations on Banach algebras: Under suitable conditions, every approximate ring derivation \( f \) on a Banach algebra \( A \) is an exact ring derivation. In particular, if \( A \) is a commutative semisimple Banach algebra with the maximal ideal space without isolated points, then \( f \) is identically zero. The first stability result concerning derivations between operator algebras was obtained by P. Šemrl [62] (see also [4]–[18] and [38]–[44]). In this paper, we investigate the generalized Hyers–Ulam–Rassias stability of quartic derivations from a Banach algebra into its Banach modules.

2. Main result

In this section, we assume that \( A \) is a commutative Banach algebra and \( X \) a Banach \( A \)-module.

**Definition 2.1.** A mapping \( \delta : A \to X \) is called a quartic derivation if \( \delta \) is a quartic function satisfies \( \delta(ab) = \delta(a)b^4 + a^4\delta(b) \) for all \( a, b \in A \).

**Example 2.2.** We take

\[
T = \begin{bmatrix}
0 & A & A & A & A \\
0 & 0 & A & A & A \\
0 & 0 & 0 & A & A \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Then \( T \) is a Banach algebra equipped with the usual matrix-like operations and the following norm:

\[
\begin{bmatrix}
0 & a_1 & a_2 & a_3 & a_4 \\
0 & 0 & a_5 & a_6 & a_7 \\
0 & 0 & 0 & a_8 & a_9 \\
0 & 0 & 0 & 0 & a_{10}
\end{bmatrix}
\|
= \sum_{i=1}^{10} ||a_i|| \quad (a_i \in A).
\]

It is known that

\[
T^* = \begin{bmatrix}
0 & A^* & A^* & A^* \\
0 & 0 & A^* & A^* \\
0 & 0 & 0 & A^* \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
is the dual of $\mathcal{T}$ under the following norm
\[
\|f\| = \max\{\|f_i\|; \quad f_i \in T^*(i = 1, 2, \ldots, 10)\}.
\]

Let the left module action of $\mathcal{T}$ on $T^*$ be trivial and let the right module action of $\mathcal{T}$ on $T^*$ be defined as follows:
\[
\begin{bmatrix}
0 & f_1 & f_2 & f_3 & f_4 \\
0 & 0 & f_5 & f_6 & f_7 \\
0 & 0 & 0 & f_8 & f_9 \\
0 & 0 & 0 & 0 & f_{10}
\end{bmatrix}
\begin{bmatrix}
0 & a_1 & a_2 & a_3 & a_4 \\
0 & 0 & a_5 & a_6 & a_7 \\
0 & 0 & 0 & a_8 & a_9 \\
0 & 0 & 0 & 0 & a_{10}
\end{bmatrix}
\begin{bmatrix}
0 & b_1 & b_2 & b_3 & b_4 \\
0 & 0 & b_5 & b_6 & b_7 \\
0 & 0 & 0 & b_8 & b_9 \\
0 & 0 & 0 & 0 & b_{10}
\end{bmatrix}
\]
\[
= \sum_{i=1}^{10} f_i(a_i b_i)
\]

for all $f_i \in \mathcal{A}^*$, $a_i, b_i \in \mathcal{A}(i = 1, \ldots, 10)$. Then $T^*$ is a Banach $\mathcal{T}$-module.

Let
\[
\begin{bmatrix}
0 & f_1 & f_2 & f_3 & f_4 \\
0 & 0 & f_5 & f_6 & f_7 \\
0 & 0 & 0 & f_8 & f_9 \\
0 & 0 & 0 & 0 & f_{10}
\end{bmatrix}
\in T^*.
\]

We define $\delta : \mathcal{T} \to T^*$ by
\[
\delta(\begin{bmatrix}
0 & a_1 & a_2 & a_3 & a_4 \\
0 & 0 & a_5 & a_6 & a_7 \\
0 & 0 & 0 & a_8 & a_9 \\
0 & 0 & 0 & 0 & a_{10}
\end{bmatrix}) = \begin{bmatrix}
0 & f_1 & f_2 & f_3 & f_4 \\
0 & 0 & f_5 & f_6 & f_7 \\
0 & 0 & 0 & f_8 & f_9 \\
0 & 0 & 0 & 0 & f_{10}
\end{bmatrix}
\begin{bmatrix}
0 & a_1 a_2 & a_3 a_4 & a_5 a_6 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Then we can see that $\delta$ is a quartic derivation from $\mathcal{T}$ into $T^*$.

Now, we investigate the generalized Hyers–Ulam–Rassias stability of quartic derivations from $\mathcal{A}$ into $\mathcal{X}$. For convenience, we use the following abbreviation for a given mapping $f : \mathcal{A} \to \mathcal{X}$;

\[
D_f(x, y) = f(2x + y) + f(2x - y) - 4[f(x + y) + f(x - y)] - 24 f(x) + 6 f(y)
\]
214  

M. Eshaghi Gordji and N. Ghobadipour

for all $x, y \in A$.

**Theorem 2.3.** Let $f : A \to X$ with $f(0) = 0$ be a mapping for which there exists function $\varphi : A \times A \times A \times A \to [0, \infty)$ such that

$$\|D_f(x, y) + f(zt) - z^4 f(t) - f(z)t^4\| \leq \varphi(x, y, z, t), \quad (2.1)$$

$$\tilde{\varphi}(x) := \sum_{i=0}^{\infty} \frac{1}{16^i} \varphi(2^i x, 0, 0, 0) < \infty, \quad (2.2)$$

$$\lim_{i \to \infty} \frac{1}{16^i} \varphi(2^i x, 2^i y, 2^i z, 2^i t) = 0 \quad (2.3)$$

for all $x, y, z, t \in A$. Then there exists a unique quartic derivation $\delta : A \to X$ such that

$$\|\delta(x) - f(x)\| \leq \frac{1}{32} \tilde{\varphi}(x) \quad (2.4)$$

for all $x \in A$.

**Proof.** Letting $z = t = y = 0$ in (2.1), we get

$$\|\frac{1}{16} f(2x) - f(x)\| \leq \frac{1}{32} \varphi(x, 0, 0, 0) \quad (2.5)$$

for all $x \in A$. By induction, we have

$$\|\frac{1}{16^n} f(2^n x) - f(x)\| \leq \frac{1}{32} \sum_{i=0}^{n-1} \frac{1}{16^i} \varphi(2^i x, 0, 0, 0) \quad (2.6)$$

for all $x \in A$. In order to show that functions $\delta_n(x) = \frac{1}{16^n} f(2^n x)$ form a Convergent sequence, we used Cauchy convergence criterion. In deed, replace $x$ by $2^m x$ in (2.6) and result divide by $16^m$, where $m$ is an arbitrary positive integer, we find that

$$\|\frac{1}{16^{n+m}} f(2^{n+m} x) - \frac{1}{16^m} f(2^m x)\| \leq \frac{1}{32} \sum_{i=m}^{m+n-1} \frac{1}{16^i} \varphi(2^i x, 0, 0, 0) \quad (2.7)$$

for all $x \in A$. By (2.2) and since $X$ is complete then by $n \to \infty, \lim_{n \to \infty} \delta_n(x)$ exists for all $x \in A$. 


Let \( m = 0 \) and \( n \to \infty \) in (2.7), we have

\[
\|\delta(x) - f(x)\| \leq \frac{1}{32} \sum_{i=0}^{\infty} \frac{1}{16^i} \varphi(2^i x, 0, 0, 0) = \frac{1}{32} \tilde{\varphi}(x)
\]

such that \( \delta \) is defined \( \delta : \mathcal{A} \to \mathcal{X} \) by \( \delta(x) = \lim_{n \to \infty} \frac{1}{16^n} f(2^n x) \) for all \( x \in \mathcal{A} \). Letting \( z = t = 0 \) and replacing \( x, y \) by \( 2^n x, 2^n y \), respectively, in the inequality (2.1), we get

\[
\|Df(2^n x, 2^n y)\| \leq \varphi(2^n x, 2^n y, 0, 0)
\]

for all \( x, y \in \mathcal{A} \), that is,

\[
\left\| \frac{1}{16^n} Df(2^n x, 2^n y) \right\| \leq \frac{1}{16^n} \varphi(2^n x, 2^n y, 0, 0)
\]

for all \( x, y \in \mathcal{A} \). Passing the limit \( n \to \infty \), we have

\[
D\delta(x, y) = 0
\]

for all \( x, y \in \mathcal{A} \). Hence \( \delta \) is a quartic functional equation. On the other hand, letting \( x = y = 0 \) and replacing \( z, t \) by \( 2^n z, 2^n t \), respectively, in (2.1), we obtain

\[
\|f(2^n z t) - 16^n zf(2^n t) - f(2^n z 16^n t)\| \leq \varphi(0, 0, 2^n z, 2^n t)
\]

for all \( z, t \in \mathcal{A} \). Hence

\[
\left\| \frac{1}{16^{2n}} f(2^{2n} z t) - \frac{1}{16^n} z f(2^n t) - f(2^n z) \frac{1}{16^n} t \right\| \leq \frac{1}{16^{2n}} \varphi(0, 0, 2^n z, 2^n t)
\]

for all \( z, t \in \mathcal{A} \). Passing the limit \( n \to \infty \), we obtain

\[
\delta(zt) = z^4 \delta(t) + \delta(z)t^4
\]

for all \( z, t \in \mathcal{A} \).

Now, suppose there exists a function \( \delta' : \mathcal{A} \to \mathcal{X} \) with

\[
D\delta'(x, y) = 0
\]

for all \( x, y \in \mathcal{A} \) and

\[
\|\delta'(x) - f(x)\| \leq \frac{1}{32} \tilde{\varphi}(x)
\]

for all \( x \in \mathcal{A} \).
We have
\[
\|\delta(x) - \delta'(x)\| = \frac{1}{16^n} \|\delta(2^n x) - \delta'(2^n x)\| = \frac{1}{16^n} (\|\delta(2^n x) - f(2^n x)\| + \|\delta'(2^n x) - f(2^n x)\|) \leq \frac{1}{16^n} \sum_{i=n}^{\infty} \frac{1}{16^n} \varphi(2^i x, 0)
\]
for all \( x \in \mathcal{A} \). Passing the limit \( n \to \infty \), we obtain \( \delta(x) = \delta'(x) \) for all \( x \in \mathcal{A} \).

Now, we establish the Ulam–Gavruta–Rassias stability of quadratic derivations as follows:

**Corollary 2.4.** Let \( p > 0, q_j > 0, (j = 1, 2, 3, 4) \) and \( \theta \) be positive real numbers with

\[
\text{Max}\{p, \sum_{j=1}^{4} q_j\} < 4.
\]

If \( f : \mathcal{A} \to \mathcal{X} \) with \( f(0) = 0 \) is a mapping such that

\[
\|D_f(x, y) + f(z t) - z^4 f(t) - f(z) t^4\| \leq \theta (\|x\|^p + \|y\|^p + \|z\|^p + \|t\|^p + \|x\|^{q_1} \|y\|^{q_2} \|z\|^{q_3} \|t\|^{q_4})
\]

for all \( x, y, z, t \in \mathcal{A} \), then there is a unique quartic derivation \( \delta : \mathcal{A} \to \mathcal{X} \) such that

\[
\|\delta(x) - f(x)\| \leq \frac{\theta}{32 - 2p+1} \|x\|^p
\]

for all \( x \in \mathcal{A} \).

**Proof.** The proof follows from Theorem 2.1 taking

\[
\varphi(x, y, z, t) := \theta (\|x\|^p + \|y\|^p + \|z\|^p + \|t\|^p + \|x\|^{q_1} \|y\|^{q_2} \|z\|^{q_3} \|t\|^{q_4})
\]

for all \( x, y, z, t \in \mathcal{A} \).

Moreover, we investigate the superstability of quartic derivations as follows:

**Corollary 2.5.** Let \( q_j > 0, (j = 1, 2, 3, 4) \) with \( \sum_{j=1}^{4} q_j < 4 \), and \( \theta \) be positive real numbers. If \( f : \mathcal{A} \to \mathcal{X} \) with \( f(0) = 0 \) is a mapping

\[
\|D_f(x, y) + f(z t) - z^4 f(t) - f(z) t^4\| \leq \theta (\|x\|^{q_1} \|y\|^{q_2} \|z\|^{q_3} \|t\|^{q_4})
\]

for all \( x, y, z, t \in \mathcal{A} \), then \( f \) is a quartic derivation.
Proof. It follows from Theorem 2.1 by putting
\[ \varphi(x, y, z, t) := \theta(\|x\|^{q_1}\|y\|^{q_2}\|z\|^{q_3}\|t\|^{q_4}) \]
for all \( x, y, z, t \in A \).

\[ \square \]

Theorem 2.6. Let \( p_1 + p_2 < 4, q_1 + q_2 < 8 \) and \( \theta \) be positive real numbers. If \( f : A \to X \) is a mapping
\[ \|D_f(x, y) + f(zt) - z^4f(t) - f(z)t^4\| \leq \theta(\|x\|^{p_1}\|y\|^{p_2} + \|z\|^{q_1}\|t\|^{q_2}) \] (2.8)
for all \( x, y, z, t \in A \), then there is a unique quartic derivation \( \delta : A \to X \) such that
\[ \|\delta(x) - f(x)\| \leq \frac{\theta}{3^4 - 3^{p_1+p_2}}\|x\|^{p_1+p_2} \] (2.9)
for all \( x \in A \).

Proof. In the inequality (2.8), let \( y = x = z = t = 0 \), then \( 23\|f(0)\| \leq 0 \). Hence \( f(0) = 0 \). Letting \( y = z = t = 0 \) in (2.8), we see that \( 2f(x) = 2^4f(x) \) for all \( x \in A \). In the inequality (2.8), put \( z = t = 0 \) and replace \( y \) with \( x \). Then we obtain
\[ \|f(3x) - 81f(x)\| \leq \theta\|x\|^{p_1+p_2} \] (2.10)
for all \( x \in A \). Hence
\[ \|\frac{f(3x)}{81} - f(x)\| \leq \frac{\theta}{81}\|x\|^{p_1+p_2} \] (2.11)
for all \( x \in A \). By using the induction, we can get that
\[ \|\frac{f(3^n x)}{81^n} - f(x)\| \leq \frac{\theta\|x\|^{p_1+p_2}}{81^n}\sum_{i=0}^{n-1} \frac{3^i(p_1+p_2)}{81^i} \] (2.12)
for all \( x \in A \). It follows from \( p_1 + p_2 < 4 \) that the sequence \( \{\frac{1}{81^n}f(3^n x)\} \) is Cauchy sequence and so it is convergent since \( X \) is complete. Thus we can define a function \( \delta : A \to X \) given by
\[ \delta(x) := \lim_{n \to \infty} \frac{1}{81^n}f(3^n x) \] (2.13)
for all \( x \in A \). In (2.12), passing the limit \( n \to \infty \), we obtain the inequality (2.9). The proof of the uniqueness of \( \delta \), is similar to the proof of Theorem 2.1. \( \square \)
Theorem 2.7. Let \( p_1 + p_2 > 4 \), \( q_1 + q_2 > 8 \) and \( \theta \) be positive real numbers. If \( f : A \to X \) is a mapping satisfying (2.8), then there is a unique quartic derivation \( \delta : A \to X \) such that

\[
\|f(x) - \delta(x)\| \leq \frac{\theta 3^{-(p_1 + p_2)}}{1 - 3^{-(p_1 + p_2)}} \|x\|^{p_1 + p_2}
\] (2.14)

for all \( x \in A \).

Proof. It follows from (2.10) that

\[
\|f(x) - 81f\left(\frac{x}{3}\right)\| \leq \frac{\theta \|x\|^{p_1 + p_2}}{3^{p_1 + p_2}}
\] (2.15)

for all \( x \in X \). By using the induction, we can get that

\[
\|f(x) - 81^n f\left(\frac{x}{3^n}\right)\| \leq \frac{\theta \|x\|^{p_1 + p_2}}{81^n} \sum_{i=1}^{n} \frac{81^i}{3^{i(p_1 + p_2)}}
\] (2.16)

for all \( x \in A \). It follows from \( p_1 + p_2 > 4 \) that the sequence \( 81^n f\left(\frac{x}{3^n}\right) \) is Cauchy sequence and so it is convergent since \( X \) is complete. Thus we can define a function \( \delta : A \to X \) given by

\[
\delta(x) := \lim_{n \to \infty} 81^n f\left(\frac{x}{3^n}\right)
\]

for all \( x \in A \). The rest of the proof is similar to the proof of Theorem 2.3.

\[
\square
\]

Theorem 2.8. Let \( f : A \to X \) with \( f(0) = 0 \) be a mapping for which there exists function \( \varphi : A \times A \times A \times A \to [0, \infty) \) such that

\[
\|Df(x,y) + f(zt) - z^4f(t) - f(z)t^4\| \leq \varphi(x, y, z, t),
\] (2.17)

\[
\tilde{\varphi}(x) := \sum_{i=1}^{\infty} 16i \varphi(2^{-i}x, 0, 0, 0) < \infty,
\] (2.18)

\[
\lim_{i \to \infty} 16^i \varphi(2^{-i}x, 2^{-i}y, 2^{-i}z, 2^{-i}t) = 0
\] (2.19)

for all \( x, y, z, t \in A \). Then there exists a unique quartic derivation \( \delta : A \to X \) such that

\[
\|f(x) - \delta(x)\| \leq \frac{1}{32} \tilde{\varphi}(x)
\] (2.20)

for all \( x \in A \).
Proof. It follows from (2.5) that
\[
\|f(x) - 16f(2^{-1}x)\| \leq 2^{-1} \varphi(2^{-1}x, 0, 0, 0) \tag{2.21}
\]
for all \(x \in \mathcal{A}\). In (2.21), multiply the both sides by 16 and replace \(x\) with \(2^{-1}x\), we have
\[
\|16f(2^{-1}x) - 16^2f(2^{-2}x)\| \leq 2^{-1}16\varphi(2^{-2}x, 0, 0, 0) \tag{2.22}
\]
for all \(x \in \mathcal{A}\). From two inequalities (2.21) and (2.22), we get
\[
\|f(x) - 16^2f(2^{-2}x)\| \leq 2^{-1}\varphi(2^{-1}x, 0, 0, 0) + 2^{-1}16\varphi(2^{-2}x, 0, 0, 0) \tag{2.23}
\]
for all \(x \in \mathcal{A}\). Continuing this way, we get
\[
\|f(x) - 16^n f(2^{-n}x)\| \leq \frac{1}{32} \sum_{i=1}^{n} 16^i \varphi(2^{-i}x, 0, 0, 0) \tag{2.24}
\]
for all \(x \in \mathcal{A}\). For any positive integer \(m\), multiply the both sides by \(16^m\) and replace \(x\) by \(2^{-m}x\) in (2.24), then we have
\[
\|16^m f(2^{-m}x) - 16^{n+m} f(2^{-(n+m)}x)\| \leq \frac{1}{32} \sum_{i=1}^{n} 16^{i+m} \varphi(2^{-(i+m)}x, 0, 0, 0) \tag{2.25}
\]
for all \(x \in \mathcal{A}\). Passing the limit \(m \to \infty\), the sequence \(\{16^n f(2^{-n}x)\}\) is a Cauchy sequence in \(\mathcal{X}\). By the completeness of \(\mathcal{X}\), the sequence \(\{16^n f(2^{-n}x)\}\) converges and so we can define a function \(\delta: \mathcal{A} \to \mathcal{X}\) given by
\[
\delta(x) = \lim_{n \to \infty} 16^n f(2^{-n}x)
\]
for all \(x \in \mathcal{A}\). The rest of the proof is similar to the proof of Theorem 2.1. \(\Box\)

**Corollary 2.9.** Let \(p > 0, q_j > 0, (j = 1, 2, 3, 4)\) and \(\theta\) be positive real numbers with
\[
\text{Min}\{p, \sum_{j=1}^{4} q_j \} > 4.
\]
If \(f: \mathcal{A} \to \mathcal{X}\) with \(f(0) = 0\) is a mapping such that
\[
\|D_f(x, y) + f(zt) - z^4 f(t) - f(z) t^4\|
\]
\[
\leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|t\|^p + \|x\|^{q_1}\|y\|^{q_2}\|z\|^{q_3}\|t\|^{q_4})
\]
for all \(x, y, z, t \in A\), then there is a unique quartic derivation \(\delta : A \to X\) such that
\[
\|
\delta(x) - f(x)\| \leq \frac{\theta}{32 - 2^{p+1}}\|x\|^p
\]
for all \(x \in A\).

**Proof.** The proof follows from Theorem 2.5 taking
\[
\varphi(x, y, z, t) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|t\|^p + \|x\|^{q_1}\|y\|^{q_2}\|z\|^{q_3}\|t\|^{q_4})
\]
for all \(x, y, z, t \in A\). \(\square\)

Also, we obtain a superstability result for quartic derivations as follows:

**Corollary 2.10.** Let \(q_j > 0\) \((j = 1, 2, 3, 4)\) with \(\sum_{j=1}^4 q_j > 4\), and \(\theta\) be positive real numbers. If \(f : A \to X\) with \(f(0) = 0\) is a mapping
\[
\|D_f(x, y) + f(zt) - z^4f(t) - f(z)t^4\| \leq \theta(\|x\|^{q_1}\|y\|^{q_2}\|z\|^{q_3}\|t\|^{q_4})
\]
for all \(x, y, z, t \in A\), then \(f\) is a quartic derivation.

In the following example, we show that the superstability of quartic derivations does not hold in general case.

**Example 2.11.** Let \(x, y, z, t \in X\) be fixed. We define \(f : A \to X\) by \(f(a) := a^4x - xa^4 + y\) for all \(a \in A\),
\[
\varphi(a, b, c, d) := \|D_f(x, y) - z^4f(t) - f(z)t^4\| = \|y\|\|24 + z^4 + t^4\|.
\]
Then we have
\[
\sum_{i=0}^{\infty}\frac{\varphi(2^i a, 0, 0, 0)}{16^i} = \sum_{i=0}^{\infty} \frac{\|y\|\|24 + z^4 + t^4\|}{16^i} = \frac{16}{15}\|y\|\|24 + z^4 + t^4\|,
\]
\[
\lim_{n \to \infty} \frac{1}{16^n} \varphi(2^n a, 2^n b, 2^n c, 2^n d) = 0
\]
for all \(a, b, c, d \in A\). Hence \(\delta(a) = \lim_{n \to \infty} \frac{f(2^a)}{16^n} = a^4x - xa^4\) for all \(a \in A\).
On the other hand we have
\[
\delta(ab) = (ab)^4x - x(ab)^4 = a^4b^4x - xa^4b^4,
\]
\[ a^4 \delta(b) + \delta(a)b^4 = a^4(b^4x - xb^4) + (a^4x - xa^4)b^4 = a^4b^4x - xa^4b^4. \]

Thus
\[ \delta(ab) = a^4 \delta(b) + \delta(a)b^4 \]

for all \( a, b \in A \). Furthermore,

\[ \delta(2a + b) + \delta(2a - b) = [(2a + b)^4x - x(2a + b)^4] + [(2a - b)^4x - x(2a - b)^4]. \]

On the other hand we have
\[
4[\delta(a + b) + \delta(a - b)] + 24\delta(a) - 6\delta(b) \\
= 4[((a + b)^4x - x(a + b)^4) + ((a - b)^4x - x(a - b)^4)] \\
+ 24[a^4x - xa^4] - 6[b^4x - xb^4].
\]

Then \( \delta \) is quartic, that is, \( D_\delta(a, b) = 0 \) for all \( a, b \in A \).

References


M. Eshaghi Gordji
Department of Mathematics,
Semnan University,
P. O. Box 35195-363, Semnan,
Iran
e-mail: madjid.eshaghi@gmail.com

and

N. Ghobadipour
Department of Mathematics,
Urmia University, Urmia,
Iran
e-mail: ghobadipour.n@gmail.com