Bipartite theory of irredundant set

V. SWAMINATHAN
S. N. COLLEGE, INDIA

and

Y. B. VENKATAKRISHNAN
SASTRA UNIVERSITY, INDIA

Received : June 2010. Accepted : December 2010

Abstract

The bipartite version of irredundant set, edge-vertex irredundant set and vertex-edge irredundant set are introduced. Using the bipartite theory of graph, $\text{IRve}(G) + \gamma(G) \leq |V|$ and $\gamma_{ve}(G) + \text{IR}(G) \leq |V|$ are proved.

AMS classification : 05C69

Keywords : Bipartite graph, $X-$irredundant set, Hyper $Y-$irredundant set, edge-vertex and vertex-edge irredundant sets.
1. Introduction

All graphs considered here are simple and undirected. [4,5] suggests that given any problem, say P, on an arbitrary graph $G$, there is very likely a corresponding problem Q on a bipartite graph $G^1$, such that a solution for Q provides a solution for P. The bipartite theory of graphs was introduced in [4] and a parameter called $X$-domination number of a bipartite graph was defined. Let $G = (X,Y,E)$ be a bipartite graph with $|X| = p$ and $|Y| = q$. Two vertices $u$ and $v$ in $X$ are $X$-adjacent if they have a common adjacent vertex $y \in Y$. Let $y \in X$ and $\Delta_Y = \max\{|N_Y(u)| : y \in X\}$ where the $X$-neighbor set $N_Y(u)$ is defined as $N_Y(u) = \{v \in X : u \text{ and } v \text{ are } X-\text{adjacent}\}$.

A subset $X \subseteq X$ is an $X$-dominating set [4] if every $x \in X - D$ is $X$-adjacent to some vertex in $D$. The minimum cardinality of a $X$-dominating set is called $X$-domination number and is denoted by $\gamma_X(G)$.

We say a vertex $x \in X$ hyper $Y$-dominates $y \in Y$ if $y \in N(x)$ or $y \in N(N_Y(x))$. A subset $S \subseteq X$ is a hyper $Y$-dominating set [6] if every $y \in Y$ is hyper $Y$-dominated by a vertex of $S$. The minimum cardinality of a hyper $Y$-dominating set is called hyper $Y$-domination number and is denoted by $\gamma_{hY}(G)$.

Given an arbitrary graph $G = (V,E)$, a vertex $u \in V(G)$ ve-dominates an edge $vw \in E(G)$ if (a) $u = v$ or $u = w$ ($u$ incident to $vw$) or (b) $uv$ or $uw$ is an edge in $G$. A subset $S \subseteq V(G)$ is a vertex-edge dominating set [3] if for all edges $e \in E(G)$, there exists a vertex $v \in S$ such that $v$ dominates $e$. The minimum cardinality of a ve-dominating set of $G$ is called the vertex-edge domination number and is denoted as $\gamma_{ve}(G)$.

An edge $e = uv \in E(G)$ ev-dominates a vertex $w \in V(G)$ if (i) $u = w$ or $v = w$ ($w$ is incident to $e$) or (ii) $uw$ or $vw$ is an edge in $G$. ($w$ is adjacent to $u$ or $v$). A set $S \subseteq E(G)$ is an edge-vertex dominating set [3] if for all vertices $v \in V(G)$, there exists an edge $e \in S$ such that $e$ dominates $v$. The minimum cardinality of an ev-dominating set of $G$ is called the edge-vertex domination number and is denoted as $\gamma_{ev}(G)$.

**Observation: 1.** Let $G$ be an arbitrary graph. A vertex $u \in V(G)$ ve-dominates the edge $e \in E(G)$ if and only if the edge $e$ ev-dominates the vertex $u \in V(G)$. 
2. Bipartite Construction

The bipartite graph $VE(G)$ constructed from an arbitrary graph $G = (V, E)$ is defined as in [4]. $VE(G) = (V, E, F)$ is defined by the edges $F = \{(u, e) : e = (u, v) \in E\}$. $VE(G) \cong S(G)$, where $S(G)$ denotes the subdivision graph of $G$.

The bipartite graph $EV(G)$ [4] constructed from an arbitrary graph $G = (V, E)$ is defined as $EV(G) = (E, V, J)$ where $J = \{(e, u)(e, v) : e = (u, v) \in E\}$.

As set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set [2] if every $v \in V$ is either an element of $S$ or is adjacent to an element of $S$. The minimum cardinality of a dominating set of a graph $G$ is called the domination number and is denoted by $\gamma(G)$.

A set $F \subseteq E(G)$ of edges in a graph $G = (V, E)$ is called an edge dominating set [2] if every $e \in E(G)$ is either an element of $F$ or is adjacent to an element of $E - F$. The minimum cardinality of an edge dominating set of a graph $G$ is called the edge domination number and is denoted by $\gamma_1(G)$.

**Theorem:2.1** [4] For any graph $G$,
(a) $\gamma_X(VE(G)) = \gamma(G)$
(b) $\gamma_X(EV(G)) = \gamma_1(G)$.

**Theorem:2.2** [6] For any graph $G$,
(a) $\gamma_{hv}(VE(G)) = \gamma_{ve}(G)$
(b) $\gamma_{hv}(EV(G)) = \gamma_{ev}(G)$.

3. Irredundant sets

3.1. Vertex-edge irredundant set

A vertex $v \in S \subseteq V(G)$ has a private edge $e = uw \in E(G)$ (with respect to a set $S$), if: 1. $v$ is incident to $e$ or $v$ is adjacent to either $u$ or $w$, and 2. for every vertices $x \in S - \{v\}$, $x$ is not incident to $e$ and $x$ is not adjacent to either $u$ or $w$.

A set $S$ is a vertex-edge irredundant set [3] (simply a ve-irredundant set) if every vertex $v \in S$ has a private edge. The vertex-edge irredundance of a graph $G$ is the cardinality of a maximal ve-irredundant set with minimum number of vertices and is denoted by $ir_{ve}(G)$. The upper vertex-edge irredundance number of a graph $G$ is the cardinality of a maximum
ve-irredundant set of vertices and is denoted by $IR_{ve}(G)$.

**Theorem: 3.1.1** [3] Every minimal ve-dominating set is a maximal ve-irredundant set.

### 3.2. Edge-vertex irredundant set

An edge $e = uv \in F \subseteq E(G)$ has a private vertex $w \in V(G)$ (with respect to a set $F$), if: 1. $e$ is incident to $w$, and 2. for all edges $f = xy \in F - \{e\}$, $f$ is not incident to $w$ and neither $x$ nor $y$ is adjacent to $w$.

A set $F$ is an edge-vertex irredundant set [3](simply a ev-irredundant set) if every edge $e \in F$ has a private vertex. The edge-vertex irredundance of a graph $G$ is the cardinality of a maximal ev-irredundant set with minimum number of vertices and is denoted by $ir_{ev}(G)$. The upper edge-vertex irredundance number of a graph $G$ is the cardinality of a maximum ev-irredundant set of vertices and is denoted by $IR_{ev}(G)$.

**Theorem 3.2.1:**[3] Every minimal ev-dominating set of $G$ is a maximal ev-irredundant set.

### 3.3. Hyper $Y$– Irredundant set

Let $G = (X,Y,E)$ be a bipartite graph. Let $S \subseteq X$. A vertex $x \in S$ has a private hyper $Y$–neighbor $y \in Y$ if 1. $x$ is adjacent to $y$ or $y \in N(N_Y(x))$ and 2. for all vertices $x_1 \in S - \{x\}$, $x_1$ is not adjacent to $y$ and $y \notin N(N_Y(x_1))$.

A set $S$ is hyper $Y$–irredundant set if every $v \in S$ has a private hyper $Y$–neighbor. The hyper $Y$–irredundance number of a graph $G$ is the minimum cardinality of a maximal hyper $Y$–irredundant set of vertices and is denoted by $ir_{hY}(G)$. The upper hyper $Y$–irredundance number of a graph $G$ is the maximum cardinality of a maximal hyper $Y$–irredundant set of vertices and is denoted by $IR_{hY}(G)$.

**Theorem: 3.3.1** A hyper $Y$–dominating set $S$ is a minimal hyper $Y$–dominating set if and only if it is hyper $Y$–dominating set and hyper $Y$–irredundant set.
**Proof:** Let $S$ be a hyper $Y$-dominating set. Then $S$ is a minimal hyper $Y$-dominating set if and only if $\forall u \in S$, $\exists y \in Y$ which is not hyper $Y$-dominated by $S - \{u\}$. Equivalently, $S$ is a minimal hyper $Y$-dominating set if and only if $\forall u \in S$, $u$ has at least one private hyper $Y$-neighbour. Thus $S$ is minimal hyper $Y$-dominating set if and only if it is hyper $Y$-irredundant set.

Conversely, let $S$ be both hyper $Y$-dominating and hyper $Y$-irredundant.

**Claim:** $S$ is a minimal hyper $Y$-dominating set.

If $S$ is not minimal hyper $Y$-dominating set, there exists $v \in S$ for which $S - \{v\}$ is hyper $Y$-dominating. Since $S$ is hyper $Y$-irredundant, $v$ has a private hyper $Y$-neighbour of $u$. By definition $u$ is not hyper $Y$-adjacent to any vertex in $S - \{v\}$. That is, $S - \{v\}$ is not hyper $Y$-dominating set, a contradiction. Hence, $S$ is a minimal hyper $Y$-dominating set.

**Theorem: 3.3.2** Every minimal hyper $Y$-dominating set is a maximal hyper $Y$-irredundant set.

**Proof:** Every minimal hyper $Y$-dominating set $S$ is hyper $Y$-irredundant set.

**Claim:** $S$ is a maximal hyper $Y$-irredundant set.

Suppose $S$ is not maximal hyper $Y$-irredundant set. Then there exists a vertex $u \in X - S$ for which $S \cup \{u\}$ is hyper $Y$-irredundant. There exists at least one vertex $y \in Y$ which is a private hyper $Y$-neighbour of $u$ with respect to $S \cup \{u\}$. That is no vertex in $S$ is hyper $Y$-adjacent to $y$. Hence, $S$ is not a hyper $Y$-dominating set, a contradiction. Hence, $S$ is a maximal hyper $Y$-irredundant set.

**Theorem: 3.3.3** For any graph $G$,

(a) $\text{ir}_{hY}(V(E(G))) = \text{ir}_{ve}(G)$

(b) $\text{ir}_{hY}(E(V(G))) = \text{ir}_{ve}(G)$.

**Proof:** Let $S$ be a $\text{ir}_{hY}$-set of $V(E(G)) = (X, Y, E)$. Every $x \in S$ has a private hyper $Y$-neighbour $y \in Y$. $x$ is adjacent to $y$ or $y \in N(N_Y(x))$ and for all vertices $x_1 \in S - \{x\}$, $x_1$ is not adjacent to $y$ and $y \notin N(N_Y(x_1))$. In graph $G$, $x \in S \subseteq V$ is incident with $y \in E$ or $x$ is adjacent to either $u$ or $v$ where $y = uv$ and for every $x_1 \in S - \{x\}$, $y \in E$ is not incident with
$x_1$ and $x_1$ is not adjacent to either $u$ or $v$. $S$ is a vertex edge irredundant set.

$$ir_{ve}(G) \leq |S| = ir_{hv}(VE(G)).$$

Let $U$ be a $ir_{ve}$-set of $G$. Every vertex $v \in S$ has a private edge $e = uw$ with respect to $U$. Equivalently, $v$ is incident with $e$ or $v$ is adjacent to either $u$ or $w$ and for every $x \in U - \{v\}$, $x$ is not incident with $e$ and $x$ is not adjacent to either $u$ or $w$. In $VE(G)$, every $v \in S$ has private hyper $Y$-neighbor $e$. Therefore, $U \subseteq X$ is a hyper $Y$-irredundant set of $VE(G)$. Hence, $ir_{hv}(VE(G)) \leq |U| = ir_{ve}(G)$.

Similarly (b) can be proved.

### 3.4. X-Irredundant set

Let $G = (X, Y, E)$ be a bipartite graph. Let $S \subseteq X$. Let $u \in S$. A vertex $v$ is a private $X$-neighbor of $u$ with respect to $S$ if $u$ is the only point of $S$, $X$-adjacent to $v$.

A set $S$ is a X-irredundant set if every $u \in S$ has a private $X$-neighbor. The $X$-irredundance number of a graph $G$ is the cardinality of a maximal $X$-irredundant set of vertices with minimum cardinality and is denoted by $ir_X(G)$. The upper $X$-irredundance number of a graph $G$ is the cardinality of a $X$-irredundant set of vertices with maximum cardinality and is denoted by $IR_X(G)$.

**Theorem 3.4.1** A $X$-dominating set $S$ is a minimal $X$-dominating set if and only if it is $X$-dominating and $X$-irredundant.

**Proof:** Let $S$ be a $X$-dominating set. Then $S$ is a minimal $X$-dominating set if and only if for every $u \in S$ there exists $v \in X - (S - \{u\})$ which is not $X$-dominated by $S - \{u\}$. Equivalently, $S$ is a minimal $X$-dominating set if and only if $\forall u \in S$, $u$ has at least one private $X$-neighbor with respect to $S$. Thus $S$ is minimal $X$-dominating set if and only if it is $X$-irredundant.

Conversely, Let $S$ be both $X$-dominating and $X$-irredundant.

**Claim:** $S$ is a minimal $X$-dominating set.

If $S$ is not a minimal $X$-dominating set, then there exists $v \in S$ for which $S - \{v\}$ is $X$-dominating. Since $S$ is $X$-irredundant, $v$ has a private $X$-neighbor of with respect to $S$ say $u$ ($u$ may be equal to $v$). By definition, $u$ is not $X$-adjacent to any vertex in $S - \{v\}$. Therefore, $S - \{v\}$ is not a $X$-dominating set, a contradiction. Hence, $S$ is a minimal $X$-dominating set.
Theorem: 3.4.2 Every minimal $X$-dominating set is a maximal $X$-irredundant set.

Proof: Every minimal $X$-dominating set $S$ is $X$-irredundant set.

Claim: $S$ is a maximal $X$-irredundant set.

Suppose $S$ is not a maximal $X$-irredundant set. Then there exists a vertex $u \in X - S$ for which $S \cup \{u\}$ is $X$-irredundant. Therefore, there exists atleast one vertex $x$ which is a private $X$-neighbor of $u$ with respect to $S \cup \{u\}$. Hence, no vertex in $S$ is $X$-adjacent to $x$. Thus $S$ is not $X$-dominating set, a contradiction. Hence, $S$ is maximal $X$-irredundant set.

A vertex $v$ is a private neighbor of a vertex $u$ in a set $S \subseteq V(G)$ with respect to $S$ if $N[v] \cap S = \{u\}$. The private neighbor set of $u$ with respect to $S$ is defined as $pn[u,S] = \{v : N[v] \cap S = \{u\}\}$. A set $S$ is called irredundant set [2] if for every vertex $u \in S$, $pn[u,S] \neq \emptyset$. The irredundance number of a graph $G$ is the cardinality of a maximal irredundant set with minimum number of vertices and is denoted by $ir(G)$. The upper irredundance number of a graph $G$ is the cardinality of a maximum irredundant set of vertices and is denoted by $IR(G)$.

Theorem: 3.4.3 For any graph $G$,

(a) $ir_X(VE(G)) = ir(G)$

(b) $ir_X(EV(G)) = ir^1(G)$

Proof: Let $S$ be a $ir_X$ set of $VE(G) = (X,Y,E^1)$. Every $v$ has a private $X$-neighbor $u$. Equivalently, $v$ is $X$-adjacent to $u$ and no other vertex in $S$ is $X$-adjacent to $u$. In $G$, $v \in S$ is the only vertex adjacent to $u$ and no other vertex in $S$ is adjacent to $u$. Therefore, $S$ is an irredundant set of $G$.

$ir(G) \leq |S| = ir_X(VE(G))$.

Let $U$ be an $ir-$ set of $G$. For every vertex $v \in U$, $pn[v,U] \neq \emptyset$. Every vertex $v \in U$ has at least one private neighbor with respect to $u$. In $VE(G)$, that is every vertex $v \in U$ has at least one private $X$-neighbor. Therefore, $U$ is an $X$-irredundant set. Hence, $ir_X(VE(G)) \leq |U| = ir(G)$. Hence, $ir_X(VE(G)) = ir(G)$. 


(b) Let $S$ be an $ir_x$ set of $EV(G) = (X, Y, E^{1})$. Every $e$ has a private $X$-neighbor $f$. Equivalently, $e$ is $X$-adjacent to $f$ and no other vertex in $S$ is $X$-adjacent to $f$. In $G$, $e \in S$ is the only edge adjacent to $f$ and no other edge in $S$ is adjacent to $f$. Therefore, $S$ is an edge irredundant set of $G$. Hence, $ir^1(G) \leq |S| = ir_x(EV(G))$.

Let $U$ be a $ir^1-$ set of $G$. For every edge $e \in U$, $pn[e, U] \neq \phi$. Hence, every edge $e \in U$ has at least one private neighbor. That is, in $EV(G)$, every vertex $e \in U$ has at least one private $X$-neighbor. Therefore, $U$ is an $X$-irredundant set in $EV(G)$. Thus, $ir_x(EV(G)) \leq |U| = ir^1(G)$. Hence, $ir_x(EV(G)) = ir^1(G)$.

4. Main Result

For any graph $G$, $I_{Xe}(G) + \gamma(G) \leq |V|$ and $\gamma_{te}(G) + IR(G) \leq |V|$ are proved using bipartite theory of graphs, which are open problem in [3].

**Theorem 4.1** Let $G = (X, Y, E)$ be a bipartite graph with $N_Y(x) \neq \phi$ for every $x \in X$. Then $I_{hY}(G) + \gamma_X(G) \leq |X|$.

**Proof:** Let $S$ be a $I_{hY}$ set of $G$. Then, $S$ is a maximal hyper $Y$-irredundant set. Therefore, $S$ is a hyper $Y$-irredundant set. That is every $x \in S$ has a private hyper $Y$-neighbor $y \in Y$. Then $x$ is adjacent to $y$ or $y \in N(N_Y(x))$ and for all vertices $x_1 \in S - \{x\}$; $x_1$ is not adjacent to $y$ and $y \notin N(N_Y(x))$.

**Case (i):** $x$ is adjacent with $y$.

Since $N_Y(y) \neq \phi$, $x$ has $X$-neighbours. Let $z$ be any $X$-neighbour of $x$. Suppose $z \in S$. Then $z$ is not adjacent to $y$ and $y \notin N(N_Y(z))$. But $y \in N(N_Y(x))$, since $x$ is a $X$-neighbour of $z$, a contradiction. Therefore, any $X$-neighbour of $x$ is in $X - S$.

**Case (ii):** $y \in N(N_Y(x))$.

Vertices in $N(y)$ are in $X - S$. Then $N(y) \subseteq X - S$. Other wise, we get a contradiction to $y \in Y$ is a private hyper $Y$-neighbor of $x \in S$. Hence, for every $x \in S$ there exists $x_1 \in X - S$ such that $x$ and $x_1$ are $X$-adjacent. That is, $X - S$ is a $X$-dominating set. Therefore, $\gamma_X(G) \leq |X - S| = |X| - I_{hY}(G)$. Hence, $I_{hY}(G) + \gamma_X(G) \leq |X|$.
Corollary: 4.2 For any graph $G$,
(a) $IR_{ve}(G) + \gamma(G) \leq |V|$
(b) $IR_{ev}(G) + \gamma_1(G) \leq |E|.$

Theorem: 4.3 Let $G = (X, Y, E)$ be a bipartite graph with $N_Y(x) \neq \emptyset$ for every $x \in X$ then $IR_X(G) + \gamma_Y(G) \leq |X|.$

Proof: Let $S$ be a $IR_X$ set of $G$. Every element $x \in S$ has a private $X$-neighbor. Consider the set $X - S$. Since $X - S$ is a $X$-dominating set elements of $Y$ are either adjacent to $X - S$ or adjacent to vertices which are $X$-adjacent to elements of $X - S$. Therefore, $X - S$ is a hyper $Y$-dominating set. Therefore, $\gamma_Y \leq |X - S| = |X| - IR_X$. Hence, $IR_X + \gamma_Y \leq |X|.$

Corollary: 4.4 For any graph $G$,
(a) $\gamma_{ve}(G) + IR(G) \leq |V|$
(b) $\gamma_{ev}(G) + IR^1(G) \leq |E|.$

Acknowledgement: We are thankful to the anonymous referee for helpful suggestions, which led to substantial improvement in the paper.

References


**V. Swaminathan**
Research Coordinator,  
Ramanujan Research Centre, 
S. N. College, 
Madurai, 
India  
e-mail : sulanesri@yahoo.com

and

**Y. B. Venkatakrishnan**
Department of Mathematics,  
SASTRA University,  
Tanjore, 
India  
e-mail : venkatakrish2@maths.sastra.edu