On graphs whose chromatic transversal number is two

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Abstract

In this paper we characterize the class of trees, block graphs, cactus graphs and cubic graphs for which the chromatic transversal domination number is equal to two.

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1. Introduction

Let \( G = (V, E) \) be a simple graph of order \( p \). A vertex \( v \) of \( G \) is a critical vertex if \( \chi(G - v) < \chi(G) \), where \( \chi(G) \) is the chromatic number of \( G \). If every vertex of \( G \) is a critical vertex, then \( G \) is called a vertex critical graph.

A subset \( D \subset V \) is a dominating set, if every \( v \in V - D \) is adjacent to some \( u \in D \). The domination number \( \gamma = \gamma(G) \) is the minimum cardinality of a dominating set of \( G \). A dominating set \( D \) is called a chromatic transversal dominating set (ctd-set) if \( D \) has non-empty intersection with every color class of every chromatic partition of \( G \). The chromatic transversal domination number \( \gamma_{ct} = \gamma_{ct}(G) \) is the minimum cardinality of a ctd-set of \( G \).

The parameter \( \gamma_{ct} \) for a few well known graphs was computed by L. Benedict et al. [1]. For example, if \( G \) is a vertex critical graph of order \( p \), then \( \gamma_{ct}(G) = p \).

By a double star we mean a tree obtained by joining the centers of two stars \( K_{1, m} \) and \( K_{1, n} \) by an edge. If we subdivide the edge connecting the centers of two stars, then it is called a double star with one subdivision. Similarly, a double star with two subdivisions is defined. The diameter of a graph \( G \) is the length of the longest path in \( G \) and is denoted by \( \text{diam}(G) \). A vertex \( v \) of a connected graph \( G \) is said to be a cutvertex if \( G - v \) is no longer connected. A connected subgraph \( B \) of \( G \) is a block, if \( B \) has no cutvertex and every subgraph \( B' \subset G \) with \( B \subset B' \) and \( B \neq B' \) has at least one cutvertex. A block \( B \) of \( G \) is called an end block, if \( B \) contains at most one cutvertex of \( G \); such a cutvertex is called an end block cutvertex. A graph \( G \) is called a block graph, if every block \( G \) is a complete graph. A graph \( G \) is called a cactus graph if every edge of \( G \) is in at most one cycle of \( G \). A graph \( G \) is said to be a cubic graph if it is 3-regular. \( \Delta(G) = \max\{\deg(v) : v \in V(G)\} \). A support vertex in \( G \) is one which is adjacent to a leaf.

**Theorem 1.0.1.** [2] Let \( G \) be a connected bipartite graph of order \( p \geq 3 \) with partition \((V_1, V_2)\) of \( V(G) \), where \(|V_1| \leq |V_2|\). Then \( \gamma_{ct}(G) = \gamma(G) + 1 \) if and only if every vertex in \( V_1 \) has at least two pendant neighbors.

**Theorem 1.0.2.** [2] For a tree \( T \), \( \gamma_{ct}(T) = \gamma(T) + 1 \) if and only if either \( T \) is \( K_2 \) or \( T \) satisfies the condition that whenever \( v \) is a support vertex, then each vertex \( w \) with \( d(v, w) \) even is also a support vertex and each support vertex has at least two pendant neighbors. Otherwise \( \gamma_{ct}(T) = \gamma(T) \).
2. Characterization

2.1. Trees

Lemma 2.1.1. For a tree $T$, $\gamma(T) = 2$ if and only if $T$ is one of the following:

(i) a double star

(ii) a double star with one subdivision

(iii) a double star with two subdivisions.

Proof: Assume that $\gamma(T) = 2$.

Claim: $diam(T) \leq 5$.

If not, let $P$ be the largest path in $T$ with length greater than 5. Then $\gamma(P) \geq 3$ where $\gamma(P)$ refers to the domination number of the path $P$. Without loss of generality assume that $\gamma(P) = 3$ and let $D = \{x_1, x_2, x_3\}$ be a $\gamma$-set of $P$.

Now, take any $\gamma$-set $S = \{x, y\}$ of $T$. If $x$ or $y$ is not in $P$, then a cycle will be formed with one of the vertices of $D$. In fact, if $x = x_1$ and $y$ is not in $P$, then $x_3$ must be adjacent to $y$ and at least one of the neighbors of $x_3$, say $u$, will be adjacent to $y$ so that the vertices $x_3, y, u$ form a cycle. Thus $x, y \in P$. But if $x, y \in P$ then at least one of the vertices of $D$ will not be dominated by $S$, contradicting the assumption that $\gamma(T) = 2$.

Case 1. Let $diam(T) = 3$ and $P_4 : u_1 u_2 u_3 u_4$ be the longest path in $T$. If $S = \{x, y\}$ is a $\gamma$-set of $T$, then as argued earlier, $S \subset V(P_4)$. As $P_4$ is the longest path in $T$ it follows that $u_1$ and $u_4$ are pendant vertices in $T$. If $x = u_1$ and $y = u_4$, then $T$ is a path $P_4$. If $x = u_1$ and $y = u_3$, then $T$ is a double star with $K_{1,1}$ at $u_3$. Similarly, we get a double star if $x = u_2$ and $y = u_4$ (or if $x = u_2$ and $y = u_3$).

Case 2. Let $diam(T) = 4$ and $P_5 : u_1 u_2 u_3 u_4 u_5$ be the longest path in $T$ with $u_1$ and $u_5$ as pendant vertices in $T$. Then for any $\gamma$-set $S = \{x, y\}$ of $T$, we have $S \subset V(P_5)$. We claim that $x = u_2$ and $y = u_4$. Suppose $x = u_3$ and $y = u_4$. Then $u_1$ will not be dominated by $S$. Similarly, the other possibilities for $x$ and $y$ except $x = u_2$ and $y = u_4$. Thus $T$ is a double star.
with one subdivision.

**Case 3.** Let $\text{diam}(T) = 5$ and $P_6 : u_1u_2u_3u_4u_5u_6$ be the longest path in $T$. Then any $\gamma$-set $S = \{x, y\}$ of $T$ is a subset of $V(P_6)$ and as argued earlier $x = u_2$ and $y = u_4$ making $T$ a double star with two subdivisions.

The converse is obvious.

**Theorem 2.1.1.** For a tree $T$, $\gamma_{ct}(T) = 2$ if and only if $T$ is one of the following:

(i) a double star

(ii) a double star with two subdivisions

(iii) a star graph.

**Proof:** Assume that $\gamma_{ct}(T) = 2$. According to Theorem 1.0.2, $\gamma_{ct}(T)$ is either $\gamma(T)$ or $\gamma(T) + 1$. If $\gamma_{ct}(T) = \gamma(T)$, then $\gamma(T) = 2$ and so by Lemma 2.1.1, $T$ is a double star or a double star with one subdivision or a double star with two subdivisions. But $T$ cannot be a double star with one subdivision in view of Theorem 1.0.2.

If $\gamma_{ct}(T) = \gamma(T) + 1$, then $\gamma(T) = 1$ and so $T$ is a star graph.

The converse is obvious.

2.2. Block graphs

**Proposition: 2.2.1.** For a block graph $G$, $\gamma_{ct}(G) = 2$ if and only if $G$ is a star graph.

**Proof:** Assume that $\gamma_{ct}(G) = 2$. Let $K_n$ be a maximal clique with maximum number of cutvertices. Then one can show that $\gamma_{ct}(G) = n + \gamma(G')$ where $G' = G - V(K_n) - L$ and $L$ is the set of all leaves with supports at some vertices of $K_n$. Then $\gamma_{ct}(G) = 2$ if and only if $n = 2$ and $\gamma(G') = 0$ or $n = 1$ and $\gamma(G') = 1$. In either case $G$ is a star graph.

The converse is obvious.

**Note:** For a block graph $G$, one can easily verify that $\gamma(G) = 2$ if and only if $G$ has exactly two end vertices and at most two internal cut vertices.
2.3. Cubic graphs

**Proposition: 2.3.1.** For a connected cubic graph $G$ of order $p$, $\gamma_{ct}(G) = 2$ if and only if $p \leq 8$.

**Proof:** Let $G$ be a cubic graph with $\gamma_{ct}(G) = 2$. Then since $\gamma_{ct}(G) \geq \chi(G)$, $\chi(G) = 2$. That is $G$ is a bipartite graph. Therefore by Theorem 1.0.1, $\gamma_{ct}(G) = \gamma(G)$ and so $\gamma(G) = 2$. But then we have $\frac{p}{1+\Delta(G)} \leq 2$ which implies $\frac{p}{4} \leq 2$. That is $p \leq 8$. Conversely, if $G$ is a cubic graph with $p \leq 8$, one can easily verify that $\gamma_{ct}(G) = 2$.

This proves the result.

2.4. Cactus graphs

**Proposition: 2.4.1.** If $G$ is a cactus graph with at least one cycle, then $\gamma_{ct}(G) = 2$ if and only if $G$ is either $C_4$ with at most two support vertices that are adjacent or $C_6$ with a pair of support vertices $u_i$ and $u_j$ where $j = i + 3 \ (mod \ 6)$ if $V(C_6) = \{u_0, u_1, \ldots, u_5\}$.

**Proof:** Let us assume that $\gamma_{ct}(G) = 2$. If $G$ has an odd cycle, $\chi(G) = 3$ and so $\gamma_{ct}(G) \geq \gamma(G) \geq 3$. Therefore $G$ cannot have an odd cycle.

Suppose $G$ has an even cycle of length greater than or equal to 8. Then $\gamma_{ct}(G) = \lceil \frac{p}{3} \rceil = 3$, which is a contradiction. Therefore $G$ has an even cycle of length 4 or 6. Furthermore $G$ is unicyclic. If not, $\gamma_{ct}(G) \geq \gamma(G) \geq 3$.

**Case 1.** Let $G$ be a unicyclic graph with $C_4$, a cycle of length 4. Let $X$ be the set of all vertices of degree 2 in $C_4$. Now $\gamma_{ct}(G) = 2$ implies $|X| \geq 2$. If $|X| = 4$, $G$ is just $C_4$. If $|X| = 3$, $G$ is $C_4$ with one support vertex. Similarly, if $|X| = 2$, $G$ is $C_4$ with two support vertices and as $\gamma_{ct}(G) = 2$, these support vertices are adjacent.

**Case 2.** Let $G$ be a unicyclic graph with $C_6$, cycle of length 6.

The proof of this case is just similar to Case 1 except that two support vertices require to be of distance 3 to form a $\gamma_{ct}$-set of $G$.

The converse is obvious.
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