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Some Mathieu–type series for the I –function occurring in the Fokker–Planck equation

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Abstract

Closed form expressions are obtained for a family of convergent Mathieu type $\mathbf{a}$ –series and its alternating variants, whose terms contain an I –function which is a generalization of the Fox’s H –function. The results derived are of general character and provide an elegant generalization for the closed form expressions of these series associated with the H –function by Pogány [9], for Fox–Wright functions by Pogány and Srivastava [10] and for ${}_pF_q$ and Meijer’s G –function by Pogány and Tomovski [13], and others.

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1. Introduction and preliminaries

In order to unify and extend the results for the convergent Mathieu-type $\mathbf{a}$ -series and its alternative form series and alternating Mathieu-type $\mathbf{a}$ -series whose terms contain the familiar transcendental functions, such as Gauss hypergeometric function ${}_2F_1$, generalized hypergeometric function ${}_pF_q$, the Fox–Wright function ${}_p\Psi_q$, the Meijer's G -function and Fox's H -function, published in a series of papers by Pogány [7, 8, 9], Pogány *et al.* [10, 11, 12, 13], Srivastava and Tomovski [21], Tomovski [23] and Tomovski and Tuan [26], the authors introduce the Mathieu-type a -series and its alternative variant, whose terms contain an I -function. Inequalities for Mathieu-type series are discussed by Cerone [1], Pogány and Tomovski [14], Srivastava and Tomovski [21], Tomovski and Hilfer [24] and Tomovski and Pogány [25]. The results obtained by the authors serve as the key formulas for numerous potentially useful special functions of Science, Engineering and Technology scattered in the literature.

The I -function like the H -function, is defined in terms of a Mellin–Barnes type integral in the following form [19]:

$$(1.1) \quad I_{p_\ell, q_\ell, \kappa}^{m, n}[z] = I_{p_\ell, q_\ell, \kappa}^{m, n} \left[z \left| \begin{array}{c} (a_j, A_j)_{\overline{1, n}}, \quad (a_{j\ell}, A_{j\ell})_{j=\overline{n+1, p_\ell}}^{\ell=\overline{1, \kappa}} \\ (b_j, B_j)_{\overline{1, m}}, \quad (b_{j\ell}, B_{j\ell})_{j=\overline{m+1, q_\ell}}^{\ell=\overline{1, \kappa}} \end{array} \right. \right]$$

$$:= \frac{1}{2\pi i} \int_{L_\kappa} \frac{\chi_\kappa(s)}{z^s} ds,$$

where $\kappa \in \mathbf{N}$ and $(a_j, A_j)_{\overline{1, n}}$ denotes the parameters sequence $(a_1, A_1), \dots, (a_n, A_n)$, while $(a_{j\ell}, A_{j\ell})_{j=\overline{1, p_\ell}}^{\ell=\overline{1, \kappa}}$ stands for the parameters sequence $(a_{11}, A_{11}), \dots, (a_{p_\kappa \kappa}, A_{p_\kappa \kappa})$ and

$$\chi_\kappa(s) = \frac{\prod_{j=1}^m \Gamma(b_j + B_j s) \cdot \prod_{j=1}^n \Gamma(1 - a_j - A_j s)}{\sum_{\ell=1}^{\kappa} \prod_{j=n+1}^{p_\ell} \Gamma(a_{j\ell} + A_{j\ell} s) \cdot \prod_{j=m+1}^{q_\ell} \Gamma(1 - b_{j\ell} - B_{j\ell} s)}.$$

Here m, n, p_ℓ, q_ℓ are nonnegative integers satisfying $0 \leq n \leq p_\ell, 0 \leq m \leq q_\ell, \ell = \overline{1, \kappa}$ and $(a_j, A_j), (b_j, B_j) \in \mathbf{C} \times \mathbf{R}^+$ such that $A_j(b_h + \ell) \neq B_h(a_j - k - 1)$, for $\ell, k \in \mathbf{N}_0; h = \overline{1, m}, j = \overline{1, n}$. The parameters $a_{j\ell}, A_{j\ell}, b_{j\ell}, B_{j\ell} \in \mathbf{C}$. The contour L_κ in the complex s -plane extends from $w - i\infty$ to $w + i\infty, w + \max_{1 \leq h \leq m} |\Im\{b_h\}|/B_h > 0$ separating the points $-B_h^{-1}(b_h + \ell), h = \overline{1, m}, \ell \in \mathbf{N}_0$, which are the poles of $\Gamma(b_j + B_j s), j = \overline{1, m}$, from the points $-A_j^{-1}(a_j - k - 1), j = \overline{1, n}, k \in \mathbf{N}_0$ which are the poles of $\Gamma(1 - a_j - A_j s), j = \overline{1, n}$.

Existence conditions for the integral (1.1) are given below

$$\alpha_\ell > 0, \quad |\arg\{z\}| < \frac{\pi}{2} \alpha_\ell, \quad \ell = \overline{1, \kappa}$$

and

$$\alpha_\ell \geq 0, \quad |\arg\{z\}| \leq \frac{\pi}{2} \alpha_\ell, \quad \text{and} \quad \Re\{\beta_\ell\} + 1 < 0, \quad \ell = \overline{1, \kappa},$$

where

$$\begin{aligned} \alpha_\ell &= \sum_{j=1}^n A_j - \sum_{j=n+1}^{p_\ell} A_{j\ell} + \sum_{j=1}^m B_j - \sum_{j=m+1}^{q_\ell} B_{j\ell}, \quad \ell = \overline{1, \kappa}, \\ \beta_\ell &= \sum_{j=1}^m b_j + \sum_{j=n+1}^{q_\ell} b_{j\ell} - \sum_{j=1}^n a_j - \sum_{j=n+1}^{p_\ell} a_{j\ell} + \frac{1}{2}(p_\ell - q_\ell), \quad \ell = \overline{1, \kappa}. \end{aligned}$$

Remark 1. For $\kappa = 1$ in (1.1), the I -function reduces to the familiar H -function, defined by Fox [4] in the following manner:

$$\begin{aligned} I_{p_1, q_1, 1}^{m, n}[z] &= H_{p_1, q_1}^{m, n} \left[z \left| \begin{array}{cc} (a_j, A_j)_{\overline{1, n}}, & (a_{j1}, A_{j1})_{j=\overline{n+1, p_1}} \\ (b_j, B_j)_{\overline{1, m}}, & (b_{j1}, B_{j1})_{j=\overline{m+1, q_1}} \end{array} \right. \right] \\ &:= \frac{1}{2\pi i} \int_{L_1} \frac{\chi_1(s)}{z^s} ds. \end{aligned}$$

A comprehensive account of the H -function is available from the monographs written by Mathai and Saxena [5], Srivastava et al. [20], Kilbas and Saigo [3] and Mathai et al. [6].

Remark 2. I -function naturally occurs in certain problems associated with driftless Fokker-Planck equations with power-law diffusion, see Südland et al. [22].

Note 1. We note that integral operators involving I -function are defined and studied by Saxena and Singh [17]. A basic analogue of the I -function is given by Saxena and Kumar in [15]. Saigo-Maeda operators of the product of I -function and a general class of polynomials are discussed by Saxena et al. [16].

Now, let us define the Mathieu-type $\mathbf{a}$ -series $\Theta_{\lambda, \mu}$ and its alternating variant $\tilde{\Theta}_{\lambda, \mu}$ by the following series:

$$\begin{aligned} &\Theta_{\lambda, \mu} \left\{ I_{p_\ell+1, q_\ell, \kappa}^{m, n+1}; \mathbf{c}, r \right\} \\ (1.2) \quad &:= \sum_{j=1}^{\infty} \frac{I_{p_\ell+1, q_\ell}^{m, n+1} \left[\frac{r}{c_j} \left| \begin{array}{cc} (\alpha, \beta), (a_j, A_j)_{\overline{1, n}}, (a_{j\ell}, A_{j\ell})_{j=\overline{n+1, p_\ell}}^{\ell=\overline{1, \kappa}} \\ (b_j, B_j)_{\overline{1, m}}, (b_{j\ell}, B_{j\ell})_{j=\overline{m+1, q_\ell}}^{\ell=\overline{1, \kappa}} \end{array} \right. \right]}{c_j^\lambda (c_j + r)^\mu}, \end{aligned}$$

$$\begin{aligned} & \tilde{\Theta}_{\lambda,\mu} \left\{ I_{p+1,q}^{m,n+1}; \mathbf{c}, r \right\} \\ &:= \sum_{j=1}^{\infty} \frac{(-1)^{j-1} I_{p_\ell+1,q_\ell}^{m,n+1} \left[\frac{r}{c_j} \middle| \begin{array}{c} (\alpha, \beta), (a_j, A_j)_{\overline{1,n}}, (a_{j\ell}, A_{j\ell})_{j=n+1,p_\ell}^{\ell=\overline{1,\kappa}} \\ (b_j, B_j)_{\overline{1,m}}, (b_{j\ell}, B_{j\ell})_{j=m+1,q_\ell}^{\ell=\overline{1,\kappa}} \end{array} \right]}{c_j^\lambda (c_j + r)^\mu}, \end{aligned}$$

where the following convention is followed that the real sequence $\mathbf{c} = (c_n)_{n \in \mathbf{N}}$ increases and tends to infinity; equivalently

$$(1.3) \quad \mathbf{c}: 0 < c_1 < c_2 < \cdots < c_n \uparrow \infty.$$

2. Integral representations

The Laplace transform of the I -function is given by

$$(2.1) \quad \int_0^\infty x^{\lambda-1} e^{-sx} I_{p_\ell,q_\ell,\kappa}^{m,n} [\omega x^\rho] dx = s^{-\lambda} I_{p_\ell+1,q_\ell,\kappa}^{m,n+1} \left[\frac{\omega}{s^\rho} \middle| \begin{array}{c} (1-\lambda, \rho), (a_j, A_j)_{\overline{1,n}}, (a_{j\ell}, A_{j\ell})_{j=n+1,p_\ell}^{\ell=\overline{1,\kappa}} \\ (b_j, B_j)_{\overline{1,m}}, (b_{j\ell}, B_{j\ell})_{j=m+1,q_\ell}^{\ell=\overline{1,\kappa}} \end{array} \right],$$

where $\lambda, s, \omega \in \mathbf{C}$; $\Re\{s\} > 0$, $\rho > 0$, and

$$\Re\{\lambda\} + \rho \min_{1 \leq j \leq m} \frac{\Re\{b_j\}}{B_j} > 0, \quad |\arg \omega| < \frac{\pi}{2} \Omega_\ell,$$

being

$$\Omega_\ell = \sum_{j=1}^m B_j + \sum_{j=1}^n A_j - \sum_{j=m+1}^{q_\ell} B_{j\ell} - \sum_{j=n+1}^{p_\ell} A_{j\ell} > 0 \quad \ell = \overline{1,\kappa}.$$

The formula (2.1) can be easily established with the help of the definition (1.1) of I -function and using the gamma function formula

$$(2.2) \quad \Gamma(\mu) \zeta^{-\mu} = \int_0^\infty x^{\mu-1} e^{-\zeta x} dx \quad \min(\Re\{\mu\}, \Re\{\zeta\}) > 0.$$

Theorem. Let $\lambda > 0, \mu > 0, r > 0, \alpha = 1 - \lambda, \beta = \rho = 1$ and let the sequence $\mathbf{c}$ satisfies (1.3). Then there holds the following results:

$$(2.3) \quad \Theta_{\lambda,\mu} \left\{ I_{p_\ell+1,q_\ell,\kappa}^{m,n+1}; \mathbf{c}, r \right\} = \mathbf{I}_{\mathbf{c}}^I(\lambda + 1, \mu) + \mu \mathbf{I}_{\mathbf{c}}^I(\lambda, \mu + 1)$$

$$(2.4) \quad \tilde{\Theta}_{\lambda,\mu} \left\{ I_{p_\ell+1,q_\ell,\kappa}^{m,n+1}; \mathbf{c}, r \right\} = \tilde{\mathbf{I}}_{\mathbf{c}}^I(\lambda + 1, \mu) + \mu \tilde{\mathbf{I}}_{\mathbf{c}}^I(\lambda, \mu + 1),$$

where

$$\begin{aligned}
\mathbf{I}_{\mathbf{c}}^I(u, v) &:= \int_{c_1}^{\infty} I_{p_{\ell}+1, q_{\ell}, \kappa}^{m, n+1} \left[\frac{r}{x} \left| \begin{array}{c} (1-u, 1), (a_j, A_j)_{\overline{1, n}}, (a_{j\ell}, A_{j\ell})_{j=n+1, p_{\ell}}^{\ell=\overline{1, \kappa}} \\ (b_j, B_j)_{\overline{1, m}}, (b_{j\ell}, B_{j\ell})_{j=m+1, q_{\ell}}^{\ell=\overline{1, \kappa}} \end{array} \right. \right] \\
&\quad \times \frac{[c^{-1}(x)]}{x^u (r+x)^v} dx, \\
\tilde{\mathbf{I}}_{\mathbf{c}}^I(u, v) &:= \int_{c_1}^{\infty} I_{p_{\ell}+1, q_{\ell}, \kappa}^{m, n+1} \left[\frac{r}{x} \left| \begin{array}{c} (1-u, 1), (a_j, A_j)_{\overline{1, n}}, (a_{j\ell}, A_{j\ell})_{j=n+1, p_{\ell}}^{\ell=\overline{1, \kappa}} \\ (b_j, B_j)_{\overline{1, m}}, (b_{j\ell}, B_{j\ell})_{j=m+1, q_{\ell}}^{\ell=\overline{1, \kappa}} \end{array} \right. \right] \\
&\quad \times \frac{\sin^2\left(\frac{\pi}{2}[c^{-1}(x)]\right) dx}{x^u (r+x)^v}.
\end{aligned}$$

Here $c: \mathbf{R}_+ \mapsto \mathbf{R}_+$ is an increasing function such that $c(x)|_{x \in \mathbf{N}} = \mathbf{c}$, $c^{-1}(x)$ denotes the inverse of $c(x)$, $[c^{-1}(x)]$ stands for the integer part of the quantity $c^{-1}(x)$.

Proof. Taking $\zeta = c_n + r$ in (2.2), setting $s = c_j$; $\rho = 1, \omega = r$ and inserting $\lambda = 1 - \alpha, \beta = 1$ in (1.2), that is *a fortiori* in (2.1), we find that

$$\begin{aligned}
&\Theta_{\lambda, \mu} \left\{ I_{p_{\ell}+1, q_{\ell}, \kappa}^{m, n+1}; \mathbf{c}, r \right\} \\
&= \sum_{j=1}^{\infty} \frac{I_{p_{\ell}+1, q_{\ell}, \kappa}^{m, n+1} \left[\frac{r}{c_j} \left| \begin{array}{c} (1-\lambda, 1), (a_j, A_j)_{\overline{1, n}}, (a_{j\ell}, A_{j\ell})_{j=n+1, p_{\ell}}^{\ell=\overline{1, \kappa}} \\ (b_j, B_j)_{\overline{1, m}}, (b_{j\ell}, B_{j\ell})_{j=m+1, q_{\ell}}^{\ell=\overline{1, \kappa}} \end{array} \right. \right]}{c_j^{\lambda} (c_j + r)^{\mu}} \\
&= \sum_{j=1}^{\infty} \int_0^{\infty} s^{\lambda-1} e^{-c_j s} I_{p_{\ell}, q_{\ell}, \kappa}^{m, n} [rs] ds \int_0^{\infty} \frac{t^{\mu-1}}{\Gamma(\mu)} e^{-(c_j+r)t} dt \\
(2.5) \quad &= \frac{1}{\Gamma(\mu)} \int_0^{\infty} \int_0^{\infty} \left(\sum_{j=1}^{\infty} e^{-c_j(s+t)} \right) s^{\lambda-1} t^{\mu-1} e^{-rt} I_{p_{\ell}, q_{\ell}, \kappa}^{m, n} [rs] ds dt,
\end{aligned}$$

where $\Re\{\mu\} > 0$ is already assumed. The inside Dirichlet series

$$\mathcal{D}_{\mathbf{c}}(s+t) = \sum_{j=1}^{\infty} e^{-c_j(s+t)}$$

has a Laplace type integral representation [7, 9] such that it can be expressed in the form

$$\begin{aligned} \mathcal{D}_{\mathbf{c}}(s+t) &= (s+t) \int_0^\infty e^{-(s+t)x} \left(\sum_{j: c_j \leq x} 1 \right) dx \\ (2.6) \quad &= (s+t) \int_{c_1}^\infty e^{-(s+t)x} [c^{-1}(x)] dx, \end{aligned}$$

with $[c^{-1}(x)] = 0$ for $x \in [0, c_1)$. By virtue of (2.6), the equation (2.5) becomes

$$\begin{aligned} &\Theta_{\lambda, \mu} \left\{ I_{p_\ell+1, q_\ell, \kappa}^{m, n+1}; \mathbf{c}, r \right\} \\ &= \frac{1}{\Gamma(\mu)} \int_0^\infty \int_0^\infty \int_{c_1}^\infty s^\lambda t^{\mu-1} e^{-(r+x)t-xs} I_{p_\ell, q_\ell, \kappa}^{m, n}[rs] [c^{-1}(x)] ds dt dx \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^\infty \int_0^\infty \int_{c_1}^\infty s^{\lambda-1} t^\mu e^{-(r+x)t-xs} I_{p_\ell, q_\ell, \kappa}^{m, n}[rs] [c^{-1}(x)] ds dt dx. \end{aligned}$$

Since

$$\begin{aligned} J_s &= \int_{c_1}^\infty \left(\int_0^\infty s^\lambda e^{-sx} I_{p_\ell, q_\ell, \kappa}^{m, n}[rs] ds \right) \left(\int_0^\infty t^{\mu-1} e^{-(r+x)t} dt \right) \frac{[c^{-1}(x)]}{\Gamma(\mu)} dx \\ &= \int_{c_1}^\infty I_{p_\ell+1, q_\ell, \kappa}^{m, n+1} \left[\frac{r}{x} \middle| \begin{array}{c} (-\lambda, 1), (a_j, A_j)_{\overline{1, n}}, (a_{j\ell}, A_{j\ell})_{j=\overline{n+1, p_\ell}}^{\ell=\overline{1, \kappa}} \\ (b_j, B_j)_{\overline{1, m}}, (b_{j\ell}, B_{j\ell})_{j=\overline{m+1, q}}^{\ell=\overline{1, \kappa}} \end{array} \right] \\ &\quad \times \frac{[c^{-1}(x)]}{x^{\lambda+1}(r+x)^\mu} dx, \end{aligned}$$

introducing the auxiliary integral

$$\begin{aligned} \mathbf{I}_{\mathbf{c}}^I(u, v) &:= \int_{c_1}^\infty I_{p_\ell+1, q_\ell, \kappa}^{m, n+1} \left[\frac{r}{x} \middle| \begin{array}{c} (1-u, 1), (a_j, A_j)_{\overline{1, n}}, (a_{j\ell}, A_{j\ell})_{j=\overline{n+1, p_\ell}}^{\ell=\overline{1, \kappa}} \\ (b_j, B_j)_{\overline{1, m}}, (b_{j\ell}, B_{j\ell})_{j=\overline{m+1, q}}^{\ell=\overline{1, \kappa}} \end{array} \right] \\ &\quad \times \frac{[c^{-1}(x)]}{x^u(r+x)^v} dx, \end{aligned}$$

it readily follows that

$$J_s = \mathbf{I}_{\mathbf{c}}^I(\lambda+1, \mu) \quad \text{and} \quad J_t = \mu \cdot \mathbf{I}_{\mathbf{c}}^I(\lambda, \mu+1).$$

This establishes the first assertion of the Theorem.

The proof of (2.4) is similar to that of (2.3), if we employ the definition of the new alternating inner Dirichlet series $\widehat{\mathcal{D}}_{\mathbf{c}}(\cdot)$ [11, p. 77, Section 4] given

below:

$$\begin{aligned}
 \tilde{\mathcal{D}}_{\mathbf{c}}(s+t) &= \sum_{j=1}^{\infty} (-1)^{j-1} e^{-c_j(s+t)} \\
 &= (s+t) \int_0^{\infty} e^{-(s+t)x} \sum_{j: c_j \leq x} (-1)^{j-1} dx \\
 &= \frac{s+t}{2} \int_0^{\infty} e^{-(s+t)x} \left(1 - (-1)^{[c^{-1}(x)]} \right) dx \\
 (2.7) \quad &= (s+t) \int_{c_1}^{\infty} e^{-(s+t)x} \sin^2 \left(\frac{\pi}{2} [c^{-1}(x)] \right) dx.
 \end{aligned}$$

The application of (2.7) completes the proof of (2.4). $\square$

3. Special cases

As I -function is most generalized function, numerous special cases associated with various transcendental functions, such Mittag-Leffler functions, Bessel functions, Whittaker functions, hypergeometric functions, generalized hypergeometric function, Meijer's G -function, Fox-Wright Ψ function and Fox's H -function and their special cases can be deduced by making use of the special cases of the I -function given in the book [18].

Now, some interesting special case results are presented here.

3.1.

For $\kappa = 1$, the I -function reduces to Fox's H -function (see Remark 1.) and we obtain the following result given by Pogány [9, Theorem 1].

Corollary 1. *Assume that*

$$\lambda, \mu, r > 0, a_{p_1+1,1} = 1 - \lambda, A_{p_1+1,1} = 1$$

and let the sequence $\mathbf{c}$ satisfies the condition (1.3). Then there hold the following formulas:

$$\begin{aligned}
 &\Theta_{\lambda,\mu} \left\{ H_{p_1+1,q_1}^{m,n+1}; \mathbf{c}, r \right\} \\
 &:= \sum_{j=1}^{\infty} \frac{H_{p_1+1,q_1}^{m,n+1} \left[\frac{r}{c_j} \mid \begin{matrix} (a_j, A_j)_{\overline{1,n}}, (a_{j1}, A_{j1})_{\overline{n+1,p_1+1}} \\ (b_j, B_j)_{\overline{1,m}}, (b_{j1}, B_{j1})_{\overline{m+1,q_1}} \end{matrix} \right]}{c_j^{\lambda} (c_j + r)^{\mu}} \\
 &= \mathbf{I}_{\mathbf{c}}^H(\lambda + 1, \mu) + \mu \mathbf{I}_{\mathbf{c}}^H(\lambda, \mu + 1),
 \end{aligned}$$

and

$$\begin{aligned} \tilde{\Theta}_{\lambda, \mu} \left\{ H_{p_1+1, q_1}^{m, n+1}; \mathbf{c}, r \right\} &:= \\ \sum_{j=1}^{\infty} (-1)^{j-1} &\frac{H_{p_1+1, q_1}^{m, n+1} \left[\frac{r}{c_j} \middle| \begin{matrix} (a_j, A_j)_{\overline{1, n}}, (a_{j1}, A_{j1})_{\overline{n+1, p_1+1}} \\ (b_j, B_j)_{\overline{1, m}}, (b_{j1}, B_{j1})_{\overline{m+1, q_1}} \end{matrix} \right]}{c_j^\lambda (c_j + r)^\mu} \\ &= \tilde{\mathbf{I}}_{\mathbf{c}}^H(\lambda + 1, \mu) + \mu \tilde{\mathbf{I}}_{\mathbf{c}}^H(\lambda, \mu + 1), \end{aligned}$$

where

$$\begin{aligned} \mathbf{I}_{\mathbf{c}}^H(u, v) &:= \int_{c_1}^{\infty} H_{p_1+1, q_1}^{m, n+1} \left[\frac{r}{x} \middle| \begin{matrix} (1-u, 1), (a_j, A_j)_{\overline{1, n}}, (a_{j1}, A_{j1})_{\overline{n+1, p_1+1}} \\ (b_j, B_j)_{\overline{1, m}}, (b_{j1}, B_{j1})_{\overline{m+1, q_1}} \end{matrix} \right] \\ &\quad \times \frac{[c^{-1}(x)]}{x^u (r+x)^v} dx, \\ \tilde{\mathbf{I}}_{\mathbf{c}}^H(u, v) &:= \int_{c_1}^{\infty} H_{p_1+1, q_1}^{m, n+1} \left[\frac{r}{x} \middle| \begin{matrix} (1-u, 1), (a_j, A_j)_{\overline{1, n}}, (a_{j1}, A_{j1})_{\overline{n+1, p_1+1}} \\ (b_j, B_j)_{\overline{1, m}}, (b_{j1}, B_{j1})_{\overline{m+1, q_1}} \end{matrix} \right] \\ &\quad \times \frac{\sin^2 \left(\frac{\pi}{2} [c^{-1}(x)] \right)}{x^u (r+x)^v} dx, \end{aligned}$$

and $c : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is an increasing function such that $c(x)|_{x \in \mathbb{N}} = \mathbf{c}$, $c^{-1}(x)$ is the inverse of $c(x)$, $[c^{-1}(x)]$ stands for the integer part of $c^{-1}(x)$.

3.2. If we employ further the identity

$$(3.1) \quad {}_p\Psi_q \left[\begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \middle| -z \right] = H_{p, q+1}^{1, p} \left[z \middle| \begin{matrix} (1-a_p, A_p) \\ (0, 1), (1-b_q, B_q) \end{matrix} \right],$$

where ${}_p\Psi_q$ is the Fox–Wright function defined by [5, p. 11]

$${}_p\Psi_q \left[\begin{matrix} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{matrix} \middle| z \right] := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(a_j + A_j n)}{\prod_{j=1}^q \Gamma(b_j + B_j n)} \frac{z^n}{n!}.$$

Here $a_j, b_j \in \mathbf{C}$, $A_j, B_j > 0$, $(i = \overline{1, p}, j = \overline{1, q})$ and $\sum_{j=1}^q B_j - \sum_{j=1}^p A_j > -1$.

Remark 3. The relation connecting Fox–Wright ${}_p\Psi_q$ function and the H -function has been given by Mathai and Saxena for the first time in the book [5, p. 11, Eq. (1.7.8)].

By virtue of Theorem and the above result (3.1), one can deduce results for Fox–Wright function. Suppose

$$\Theta_{\lambda,\mu}\{_{p+1}\Psi_q; \mathbf{c}, r\} := \sum_{j=1}^{\infty} \frac{{}_{p+1}\Psi_q \left[\begin{matrix} (\alpha, \beta), (a_p, A_p) \\ (b_q, B_q) \end{matrix} \middle| -\frac{r}{c_j} \right]}{c_j^\lambda (c_j + r)^\mu},$$

and

$$\tilde{\Theta}_{\lambda,\mu}\{_{p+1}\Psi_q; \mathbf{c}, r\} := \sum_{j=1}^{\infty} \frac{(-1)^{j-1} {}_{p+1}\Psi_q \left[\begin{matrix} (\alpha, \beta), (a_p, A_p) \\ (b_q, B_q) \end{matrix} \middle| -\frac{r}{c_j} \right]}{c_j^\lambda (c_j + r)^\mu}.$$

Now one can easily arrive at

Corollary 2. *Let $\lambda \notin \mathbf{N}, \mu > 0, r > 0, (\alpha, \beta) = (1 - \lambda, 1), (b_q, B_q) = (1, 1)$ and let the sequence $\mathbf{c}$ satisfies (1.3). Then we have*

$$\begin{aligned} \Theta_{\lambda,\mu}\{_{p+1}\Psi_q; \mathbf{c}, r\} &= \mathbf{I}_{\mathbf{c}}^\Psi(\lambda + 1, \mu) + \mu \mathbf{I}_{\mathbf{c}}^\Psi(\lambda, \mu + 1) \\ \tilde{\Theta}_{\lambda,\mu}\{_{p+1}\Psi_q; \mathbf{c}, r\} &= \tilde{\mathbf{I}}_{\mathbf{c}}^\Psi(\lambda + 1, \mu) + \mu \tilde{\mathbf{I}}_{\mathbf{c}}^\Psi(\lambda, \mu + 1), \end{aligned}$$

where

$$\mathbf{I}_{\mathbf{c}}^\Psi(u, v) := \int_{c_1}^{\infty} \frac{[c^{-1}(x)]}{x^u (r+x)^v} {}_{p+1}\Psi_q \left[\begin{matrix} (1-u, 1), (a_p, A_p) \\ (1, 1), (b_{q-1}, B_{q-1}) \end{matrix} \middle| -\frac{r}{x} \right] dx$$

and

$$\tilde{\mathbf{I}}_{\mathbf{c}}^\Psi(u, v) := \int_{c_1}^{\infty} \frac{\sin^2 \left(\frac{\pi}{2} [c^{-1}(x)] \right)}{x^u (r+x)^v} {}_{p+1}\Psi_q \left[\begin{matrix} (1-u, 1), (a_p, A_p) \\ (1, 1), (b_{q-1}, B_{q-1}) \end{matrix} \middle| -\frac{r}{x} \right] dx.$$

Here $c(x), c^{-1}(x), [c^{-1}(x)]$ retain the same meanings than in previous corollary.

Remark 4. Finally, it is interesting to observe that, by virtue of the relation

$$E_{\alpha,\beta}(z) = H_{1,2}^{1,1} \left[-z \middle| \begin{matrix} (0, 1) \\ (0, 1), (1 - \beta, \alpha) \end{matrix} \right],$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function [2, Chapter 18] and [4, p. 80], defined by

$$E_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},$$

where

$$\alpha, \beta \in \mathbf{C}; \Re\{\alpha\}, \Re\{\beta\} > 0,$$

the results for the Mittag-Leffler function can be easily deduced from Corollary 1.

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