Difference sequence spaces defined by a sequence of modulus functions

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Abstract

In the present paper we study difference sequence spaces defined by a sequence of modulus functions and examine some topological properties of these spaces.

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1. Introduction and Preliminaries

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$,
3. $f$ is increasing,
4. $f$ is continuous from right at 0.

It follows that $f$ must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Subsequently, modulus function has been discussed in ([1], [7], [8]) and many others.

Let $X$ be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,
4. if $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and $(x_n)$ is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm $p$ for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair $(X, p)$ is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [9], Theorem 10.4.2, P-183).

Let $w$ be the set of all sequences, real or complex numbers and $l_\infty$, $c$ and $c_0$ be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$, normed by $||x|| = \sup_k |x_k|$, where $k \in \mathbb{N}$, the set of positive integers.
Let $\Lambda = (\lambda_n)$ be a non decreasing sequence of positive reals tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. The generalized de la Vallee-Poussin means is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be $(V, \lambda)$-summable to a number $l$ if $t_n(x) \to l$ as $n \to \infty$ (see [5]). If $\lambda_n = n$, $(V, \lambda)$-summability and strong $(V, \lambda)$-summability are reduced to $(C, 1)$-summability and $[C, 1]$-summability, respectively.

In [4], Kizmaz defined the sequence spaces

$$X(\Delta) = \left\{ x = (x_k) : (\Delta x_k) \in X \right\}$$

for $X = l_\infty$, $c$ or $c_0$, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ for all $k \in \mathbb{N}$. Et and Colak [2] generalized the above sequence spaces to the sequence spaces

$$X(\Delta^m) = \left\{ x = (x_k) : (\Delta^m x_k) \in X \right\}$$

for $X = l_\infty$, $c$ or $c_0$, where $m \in \mathbb{N}, \Delta^0 x = (x_k), \Delta x = (x_k - x_{k+1}), \Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ for all $k \in \mathbb{N}$.

The generalized difference operator has the following binomial representation,

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$$

for all $k \in \mathbb{N}$.

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $D = \max(1, 2H - 1)$ then

\begin{equation}
|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}
\end{equation}

for all $k$ and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$. 

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Let $E$ be a Banach space, we define $w(E)$ to be the vector space of all $E$-valued sequences that is

$$w(E) = \{ x = (x_k) : x_k \in E \}.$$  

Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then we define the following sequence spaces:

$$[V, \lambda, F, p]_1(\Delta^m, E, u) = \left\{ x \in w(E) : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k \left( ||\Delta^m u_k x_k - L e || \right) \right]^{p_k} = 0, \text{ for some } L \right\},$$

$$[V, \lambda, F, p]_0(\Delta^m, E, u) = \left\{ x \in w(E) : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k \left( ||\Delta^m u_k x_k || \right) \right]^{p_k} = 0 \right\}$$

and

$$[V, \lambda, F, p]_\infty(\Delta^m, E, u) = \left\{ x \in w(E) : \sup_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k \left( ||\Delta^m u_k x_k || \right) \right]^{p_k} < \infty \right\},$$

where $e = (1, 1, 1, \cdots)$.

If $u = e$ and $f_k = f$, then these spaces reduce to those which were studied by Et, M., Altin, Y. and Altinok, H. [3].

For $f_k(x) = x$, we have

$$[V, \lambda, p]_1(\Delta^m, E, u) = \left\{ x \in w(E) : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ ||\Delta^m u_k x_k - L e || \right]^{p_k} = 0, \text{ for some } L \right\},$$

$$[V, \lambda, p]_0(\Delta^m, E, u) = \left\{ x \in w(E) : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ ||\Delta^m u_k x_k || \right]^{p_k} = 0 \right\}$$

and

$$[V, \lambda, p]_\infty(\Delta^m, E, u) = \left\{ x \in w(E) : \sup_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ ||\Delta^m u_k x_k || \right]^{p_k} < \infty \right\}.$$
For $p_k = 1$, we have

$$[V, \lambda, F]_1(\Delta^m, E, u) = \left\{ x \in w(E) : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k \left( \| \Delta^m u_k x_k - L \| \right) \right] = 0, \right. $$

for some $L$, 

$$[V, \lambda, F]_0(\Delta^m, E, u) = \left\{ x \in w(E) : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k \left( \| \Delta^m u_k x_k \| \right) \right] = 0 \right\}$$

and

$$[V, \lambda, F]_\infty(\Delta^m, E, u) = \left\{ x \in w(E) : \sup_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k \left( \| \Delta^m u_k x_k \| \right) \right] < \infty \right\}.$$ 

For $f_k(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$, we have

$$[V, \lambda]_1(\Delta^m, E, u) = \left\{ x \in w(E) : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ \| \Delta^m u_k x_k - L \| \right] = 0, \right. $$

for some $L$, 

$$[V, \lambda]_0(\Delta^m, E, u) = \left\{ x \in w(E) : \lim_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ \| \Delta^m u_k x_k \| \right] = 0 \right\}$$

and

$$[V, \lambda]_\infty(\Delta^m, E, u) = \left\{ x \in w(E) : \sup_{n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ \| \Delta^m u_k x_k \| \right] < \infty \right\}.$$ 

Throughout this paper, $X$ will denote any one of the notations 0, 1 or $\infty$.

In this paper we study some topological properties and inclusion relations between above defined sequence spaces.

2. Main Results

**Theorem 2.1** Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the sequence spaces $[V, \lambda, F, p]_1(\Delta^m, E, u)$, $[V, \lambda, F, p]_0(\Delta^m, E, u)$ and $[V, \lambda, F, p]_\infty(\Delta^m, E, u)$ are linear spaces.
Proof. Let \( x, y \in [V, \lambda, F, p]_0(\Delta^m, E, u) \) and \( \alpha, \beta \in \mathbb{C} \). Then there exist positive number \( M_\alpha \) and \( N_\beta \) such that \( |\alpha| \leq M_\alpha \) and \( |\beta| \leq N_\beta \). Since \( f_k \) is subadditive and \( \Delta^m \) is linear, we have

\[
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k \left( \| \Delta^m (\alpha u_k x_k + \beta u_k y_k) \| \right) \right]^{p_k}
\]

\[
\leq \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k (|\alpha| \| \Delta^m u_k x_k \|) + f_k (|\beta| \| \Delta^m u_k y_k \|) \right]^{p_k}
\]

\[
\leq D(M_\alpha) \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k (\| \Delta^m u_k x_k \|) \right]^{p_k} + D(N_\beta) \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k (\| \Delta^m u_k y_k \|) \right]^{p_k}
\]

\[
\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

This proves that \( [V, \lambda, F, p]_0(\Delta^m, E, u) \) is a linear space. Similarly we can prove that \( [V, \lambda, F, p]_1(\Delta^m, E, u) \) and \( [V, \lambda, F, p]_\infty(\Delta^m, E, u) \) are linear spaces in view of the above proof.

**Theorem 2.2** Let \( F = (f_k) \) be a sequence of modulus functions. Then \( [V, \lambda, F, p]_0(\Delta^m, E, u) \subset [V, \lambda, F, p]_1(\Delta^m, E, u) \subset [V, \lambda, F, p]_\infty(\Delta^m, E, u) \).

Proof. The first inclusion is obvious. For the second inclusion, let \( x \in [V, \lambda, F, p]_1(\Delta^m, E, u) \). Then by definition, we have

\[
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k (\| \Delta^m u_k x_k \|) \right]^{p_k}
\]

\[
= \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k (\| \Delta^m u_k x_k - Le \|) \right]^{p_k}
\]

\[
\leq D \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k (\| \Delta^m u_k x_k - Le \|) \right]^{p_k} + D \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k (\| Le \|) \right]^{p_k}
\]

Now, there exists a positive number \( A \) such that \( \| Le \| \leq A \). Hence we have

\[
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k (\| \Delta^m u_k x_k \|) \right]^{p_k} \leq \frac{D}{\lambda_n} \sum_{k \in I_n} \left[ f_k (\| \Delta^m u_k x_k - Le \|) \right]^{p_k} +
\]
Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $[V, \lambda, F, p](\Delta^m, E, u)$ is a paranormed space with

$$g_\Delta(x) = \sup_n \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k(||\Delta^m u_k x_k||) \right]^{p_k} \right)^{\frac{1}{p_k}}$$

where $K = \max(1, \sup p_k)$.

**Proof.** Clearly $g_\Delta(x) = g_\Delta(-x)$. It is trivial that $\Delta^m u_k x_k = 0$ for $x = 0$. Since $f(0) = 0$, we get $g_\Delta(x) = 0$ for $x = 0$. Since $\frac{p_k}{K} \leq 1$, using the Minkowski’s inequality, for each $n$, we have

$$\left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k(||\Delta^m u_k x_k + \Delta^m u_k y_k||) \right]^{p_k} \right)^{\frac{1}{p_k}}$$

$$\leq \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k(||\Delta^m u_k x_k||) \right]^{p_k} \right)^{\frac{1}{p_k}} + \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k(||\Delta^m u_k y_k||) \right]^{p_k} \right)^{\frac{1}{p_k}}.$$ 

Hence $g_\Delta(x)$ is subadditive. For, the continuity of multiplication, let us take any complex number $\alpha$. By definition, we have

$$g_\Delta(\alpha x) = \sup_n \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k(||\Delta^m \alpha u_k x_k||) \right]^{p_k} \right)^{\frac{1}{p_k}}$$

$$\leq C^{H/K}_\alpha g_\Delta(x),$$
where $C_\alpha$ is a positive integer such that $|\alpha| \leq C_\alpha$. Now, let $\alpha \to 0$ for any fixed $x$ with $g_\Delta(x) \neq 0$. By definition for $|\alpha| < 1$, we have

\begin{equation}
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k(||\alpha \Delta^m u_k x_k||) \right]^{p_k} < \epsilon \quad \text{for} \quad n > n_0(\epsilon)
\end{equation}

Also, for $1 \leq n \leq n_0$, taking $\alpha$ small enough, since $f_k$ is continuous, we have

\begin{equation}
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k(||\alpha \Delta^m u_k x_k||) \right]^{p_k} < \epsilon.
\end{equation}

Now, eqn. (2.2) and (2.3) together imply that

\[ g_\Delta(\alpha x) \to 0 \text{ as } \alpha \to 0. \]

**Theorem 2.4** Let $F = (f_k)$ be a sequence of modulus functions and $m \geq 1$, then the inclusion

\[ [V, \lambda, F]_X(\Delta^{m-1}, E, u) \subset [V, \lambda, F]_X(\Delta^m, E, u) \]

is strict. In general

\[ [V, \lambda, F]_X(\Delta^i, E, u) \subset [V, \lambda, F]_X(\Delta^m, E, u) \]

for all $i = 1, 2, \ldots, m - 1$ and the inclusion is strict.

**Proof.** Let $x \in [V, \lambda, F]_\infty(\Delta^{m-1}, E, u)$. Then we have

\[
\sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k(||\Delta^{m-1} u_k x_k||) \right] < \infty.
\]

By definition, we have

\[
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k(||\Delta^m u_k x_k||) \right] = \frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k(||\Delta^{m-1} u_k x_k||) \right] + \\
\frac{1}{\lambda_n} \sum_{k \in I_n} \left[ f_k(||\Delta^{m-1} u_{k+1} x_{k+1}||) \right] \leq \infty.
\]

Thus $[V, \lambda, F]_\infty(\Delta^{m-1}, E, u) \subset [V, \lambda, F]_\infty(\Delta^m, E, u)$. 
Proceeding in this way, we have

\[ [V, \lambda, F]_\infty(\Delta^i, E, u) \subset [V, \lambda, F]_\infty(\Delta^m, E, u) \]

for all \( i = 1, 2, \ldots, m - 1 \). Let \( E = \mathbb{C} \) and \( \lambda_n = n \) for each \( n \in \mathbb{N} \).

Then the sequence \( x = (x^m) \in [V, \lambda, F]_\infty(\Delta^m, E, u) \) but does not belong to \( [V, \lambda, F]_\infty(\Delta^{m-1}, E, u) \) for \( f_k(x) = x \).

Similarly, we can prove for the case \([V, \lambda, F]_0(\Delta^m, E, u)\) and \([V, \lambda, F]_1(\Delta^m, E, u)\) in view of the above proof.

**Corollary 2.5** Let \( F = (f_k) \) be a sequence of modulus functions. Then

\[ [V, \lambda, F, p]_1(\Delta^{m-1}, E, u) \subset [V, \lambda, F]_0(\Delta^m, E, u). \]

**Theorem 2.5** Let \( F = (f_k), \ F' = (f'_k) \) and \( F'' = (f''_k) \) are sequence of modulus functions. Then we have

(i) \([V, \lambda, F', p]_X(\Delta^m, E, u) \subset [V, \lambda, F \circ F', p]_X(\Delta^m, E, u)\),

(ii) \([V, \lambda, F', p]_X(\Delta^m, E, u) \cap [V, \lambda, F'', p]_X(\Delta^m, E, u) \subset [V, \lambda, F + F', p]_X(\Delta^m, E, u)\).

**Proof.** (i) Let \( \epsilon > 0 \) and choose \( \delta \) with \( 0 < \delta < 1 \) such that \( f(t) < \epsilon \) for \( 0 \leq t \leq \delta \). Write \( y_k = f_k(||\Delta^m u_k x_k||) \) and consider

\[
\sum_{k \in I_n} [f_k(y_k)]^{p_k} = \sum_1 [f_k(y_k)]^{p_k} + \sum_2 [f_k(y_k)]^{p_k},
\]

where the first summation is over \( y_k \leq \delta \) and second summation is over \( y_k \geq \delta \). Since \( f_k \) is continuous, we have

\[
(2.4) \quad \sum_1 [f_k(y_k)]^{p_k} < \lambda_n \epsilon^H
\]

and for \( y_k > \delta \), we use the fact that

\[
y_k < \frac{y_k}{\delta} \leq 1 + \frac{y_k}{\delta}.
\]

By the definition, we have for \( y_k > \delta \),

\[
f_k(y_k) < 2f_k(1) \frac{y_k}{\delta}.
\]
Hence

\[(2.5) \quad \frac{1}{\lambda_n} \sum_{k} \left[ f_k(y_k) \right]^{p_k} \leq \max \left( 1, (2f_k(1)\delta^{-1})^H \right) \frac{1}{\lambda_n} \sum_{k} y_k. \]

From eqn. (2.4) and (2.5), we have

\[ [V,\lambda,F,p]_0(\Delta^m,E,u) \subset [V,\lambda,F\circ F',p]_0(\Delta^m,E,u). \]

This completes the proof of (i).

The proof of (ii) follows from the following inequality:

\[ \left[ (f'_k + f''_k)(||\Delta^m u_k x_k||) \right]^{p_k} \leq D \left[ f'_k(||\Delta^m u_k x_k||) \right]^{p_k} + D \left[ f''_k(||\Delta^m u_k x_k||) \right]^{p_k}. \]

**Corollary 2.6** Let \( F = (f_k) \) be a sequence of modulus functions. Then

\[ [V,\lambda,p]_X(\Delta^m,E,u) \subset [V,\lambda,F,p]_X(\Delta^m,E,u). \]

**References**


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