On f -clean rings and f -clean elements

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Abstract

An associative ring R with identity is called f -clean ring if every element in R is the sum of an idempotent and a full element. In this paper, various basic properties of f -clean rings and f -clean elements are proved. Also, we give some new characterizations of f -clean rings. In addition, we prove that the ring of skew Hurwitz series $T = (HR, \sigma)$ where σ is an automorphism of R is f -clean if and only if R is f -clean.

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1. Introduction

A clean ring is one in which every element is the sum of an idempotent and a unit, and this definition dates back to a paper by Nicholson [6] in 1977. Later, Camillo and Yu proved in [5] that all semiperfect rings and unit-regular rings are clean. Nicholson and Zhou [9] extend the definition of a clean ring to general rings (without unity). For the study of clean rings and their generalizations, we refer to [3], [5], [7] and [8].

Following Li and Feng [2], an element $x \in R$ is said to be a full element if there exist $s, t \in R$ such that $sxt = 1$, an element in R is said to be f -clean if it can be written as the sum of an idempotent and a full element. Also, a ring R is called a f -clean ring if each element in R is a f -clean element. Purely infinite simple rings which introduced by Ara, Goodearl, and Pardo [4] is an example of a f -clean ring. Li and Feng in [2] showed that the $n \times n$ matrix ring over the ring R is f -clean for any f -clean ring R and they got a condition under which the definitions of cleanness and f -cleanness are equivalent. Moreover, the class of f -clean rings is closed under homomorphic images and a polynomial ring is never a f -clean ring.

In this paper, we continue to study f -clean rings and f -clean elements. We prove that the ring of skew Hurwitz series $T = (HR, \sigma)$ where σ is an automorphism of R is f -clean if and only if R is f -clean. Also, we give some new characterizations of f -clean rings.

Throughout this paper, R denotes an associative ring with identity, and all R -modules are unitary. We denote by $U(R)$, $K(R)$, $Aut(R)$ and $Id(R)$ the set of all units of R , the set of all full elements of R , the set of all automorphisms of R , the set of all idempotent elements of R , respectively.

2. Results

We start this section with the following theorem

Theorem 2.1. *Let R be a ring. Then R is f -clean if and only if for every $x \in R$ there exist $e \in Id(R)$, $t \in K(R)$ such that $ex = et$ and $(e - 1)(x - 1) = (e - 1)t$.*

Proof. Suppose R is f -clean and $x \in R$, then $x = t + g$ with $t \in K(R)$ and $g \in Id(R)$. Let $e = 1 - g$. Then $ex = e(t + g) = et$ and $(e - 1)(x - 1) = (e - 1)t$.

Conversely, assume that for every $x \in R$ there exist $e \in Id(R)$, $t \in K(R)$ such that $ex = et$ and $(e - 1)(x - 1) = (e - 1)t$. Note that $et - t = (e - 1)(x - 1) = ex - e - x + 1$. So we have $x = t + (1 - e)$. Hence, x is f -clean. $\square$

Proposition 2.2. *If $e \in Id(R)$ and $a \in eRe$ is a f -clean in eRe , then a is a f -clean in R .*

Proof. Suppose $a = s + d$ where $d \in Id(eRe)$, $s \in eRe$ and $tsr = e$ for $t, r \in eRe$. Then $(t - (1 - e))(s - (1 - e))(r + (1 - e)) = (tsr + (1 - e)) = (e + (1 - e)) = 1$. This shows that $s - (1 - e) \in K(R)$. Certainly, $d + (1 - e) \in Id(R)$. Therefore, $a = (s - (1 - e)) + (d + (1 - e))$ is f -clean in R . $\square$

Proposition 2.3. *Let R be a ring. $a \in R$ is f -clean if and only if $1 - a$ is f -clean.*

The ring R is called abelian if every idempotent is central, that is, $ae = ea$ for any $e^2 = e, a \in R$.

Theorem 2.4. *Let R be an abelian ring. Let $a \in R$ be a f -clean in R and $e \in Id(R)$, then*

- (1) ae is f -clean.
- (2) if $-a$ is also f -clean, then $a + e$ is f -clean.

Proof. (1). Since a is f -clean in R , then $a = r + g$ with $r \in K(R)$ and $g \in Id(R)$. Hence, $ae = re + ge$. Clearly, $re \in K(eRe)$ and $ge \in Id(eRe)$ since R is abelian. Therefore, ae is f -clean in eRe . By Proposition 2.2, ae is f -clean.

(2). Since $-a$ is f -clean, a is f -clean. It follows from Proposition 2.3, that $1 + a$ is f -clean. Let $a = s_1 + g_1$ and $1 + a = s_2 + g_2$, where $s_1, s_2 \in K(R)$ and $g_1, g_2 \in Id(R)$. Then $a + e = ae + a(1 - e) + e = (1 + a)e + a(1 - e) = (s_2 + g_2)e + (s_1 + g_1)(1 - e) = s_2e + s_1(1 - e) + g_2e + g_1(1 - e)$. Clearly, $g_2e + g_1(1 - e) \in Id(R)$. We will prove that $s_2e + s_1(1 - e)$ is a full element in R . Since $s_1, s_2 \in K(R)$, $t_1s_1v_1 = 1$ and $t_2s_2v_2 = 1$ for some $t_1, v_1, t_2, v_2 \in R$. Let $w_1 = t_2e + t_1(1 - e)$ and $w_2 = v_2e + v_1(1 - e)$, then we have

$$\begin{aligned}
w_1(s_2e + s_1(1-e))w_2 &= (t_2es_2 + t_1(1-e)s_1)(v_2e + v_1(1-e)) \\
&= t_2s_2v_2e + t_1s_1v_1(1-e) \\
&= e + (1-e) \\
&= 1.
\end{aligned}$$

Hence, $s_2e + s_1(1-e) \in K(R)$. Therefore, $a + e$ is f -clean. $\square$

Lemma 2.5. *Let R be a ring. If $e_i \in Id(R)$ ($1 \leq i \leq n$) such that $e_ie_j = 0$ if $i \neq j$, then $e = \sum_i e_i \in Id(R)$.*

Theorem 2.6. *Let e be a central idempotent of R . For an element $x \in R$, x is f -clean in R if and only if ex is f -clean in eRe and $(1-e)x$ is f -clean in $(1-e)R(1-e)$.*

Proof. Since x is f -clean in R , $x = s + g$, where $s \in K(R)$ and $g \in Id(R)$. It is easy to show that $es \in K(eRe)$, $eg \in Id(eRe)$, $(1-e)s \in K((1-e)R(1-e))$ and $(1-e)g \in Id((1-e)R(1-e))$. Thus $ex = es + eg$ is f -clean in eRe and $(1-e)x = (1-e)s + (1-e)g$ is f -clean in $(1-e)R(1-e)$. Conversely, since ex is f -clean in eRe and $(1-e)x$ is f -clean in $(1-e)R(1-e)$, $ex = s_1 + g_1$ and $(1-e)x = s_2 + g_2$, where $s_1 \in K(eRe)$, $g_1 \in Id(eRe)$, $s_2 \in K((1-e)R(1-e))$ and $g_2 \in Id((1-e)R(1-e))$. Hence, $g_1g_2 = g_2g_1$. So $g_1 + g_2$ is an idempotent. Now if $s_1 = eae \in K(eRe)$ and $s_2 = (1-e)b(1-e) \in K((1-e)R(1-e))$, then $(ea_1e)(eae)(ea_2e) = e$ for some $a_1, a_2 \in R$ and $((1-e)b_1(1-e))((1-e)b(1-e))((1-e)b_2(1-e)) = 1-e$ for some $b_1, b_2 \in R$. Thus, $ea_1aa_2 + (1-e)b_1bb_2 = 1$. Let $w_1 = a_1e + (1-e)b_1$ and $w_2 = a_2e + (1-e)b_2$, then we have

$$\begin{aligned}
w_1(s_1 + s_2)w_2 &= (a_1e + (1-e)b_1)(ea + (1-e)b)(a_2e + (1-e)b_2) \\
&= (a_1ae + b_1b(1-e))(a_2e + (1-e)b_2) \\
&= a_1aa_2e + b_1bb_2(1-e) \\
&= ea_1aa_2 + (1-e)b_1bb_2 \\
&= 1.
\end{aligned}$$

It follows that $(s_1 + s_2) \in K(R)$. Hence, $x = ex + (1-e)x = (s_1 + s_2) + (g_1 + g_2)$ is f -clean. $\square$

Corollary 2.7. *Let e be a central idempotent in R . R is f -clean if and only if eRe and $(1-e)R(1-e)$ are both f -cleans.*

Corollary 2.8. *For each element $x \in R$. Let $1 = e_1 + e_2 + \dots + e_n$ in R , $n \geq 1$, where e_i are orthogonal central idempotents. x is f -clean in R if and only if $e_i x$ is f -clean in $e_i R e_i$ for each i .*

Corollary 2.9. *Let $1 = e_1 + e_2 + \dots + e_n$ in R , $n \geq 1$, where e_i are orthogonal central idempotents. R is f -clean if and only if $e_i R e_i$ is f -clean.*

Proposition 2.10. [2]. *Any homomorphic image of a f -clean ring is f -clean ring.*

The ring $T = (HR, \sigma)$ of skew Hurwitz series over a ring R and $\sigma \in \text{Aut}(R)$ is defined as follows: the elements of $T = (HR, \sigma)$ are functions, $f : \mathbf{N} \rightarrow R$, where $\mathbf{N}$ is the set of all natural numbers, the operation of addition in $T = (HR, \sigma)$ is component wise and the operation of multiplication for each $f, g \in T$ is defined by

$$(fg)(n) = \sum_{k=0}^n \binom{n}{k} f(k) \sigma^k(g(n-k))$$

for all $n \in \mathbf{N}$, where $\binom{n}{k}$ is the binomial coefficient.

It can be easily shown that T is a ring with identity h_1 , defined by $h_1(0) = 1$ and $h_1(n) = 0$ for all $n \geq 1$. It is clear that R is canonically embedded as a subring of T via $0 \neq r \in R \mapsto h_r \in T$, where $h_r(0) = r$ and $h_r(n) = 0$ for every $n \geq 1$. See [1] for more details.

Proposition 2.11. *Let R be a ring. Then f is a full element in $T = (HR, \sigma)$ if and only if $f(0)$ is a full element in R .*

Proof. If $f \in T = (HR, \sigma)$ is a full element, then there exist $h, g \in T = (HR, \sigma)$ such that $hfg = h_1$. This implies that $(hfg)(0) = h(0)f(0)g(0) = 1$. Hence, $f(0)$ is a full element in R .

Conversely, suppose that $f \in T$ such that $f(0) = s$ is a full element in R . So there exist $t, r \in R$ such that $tsr = 1$. Let $h \in T$ such that $h(0) = t$. Define an element $g \in T$ by

$$g(n) = \begin{cases} r, & n = 0 \\ -r \sum_{k=0}^n \binom{n}{k} (hf)(k) \sigma^k(g(n-k)), & n > 0 \end{cases}.$$

It is easy to verify that $hfg = h_1$. Thus, f is a full element in $T = (HR, \sigma)$.

□

Theorem 2.12. [1] Let R be a ring and σ be an automorphism of R . Then $T = (HR, \sigma)$ is a clean ring if and only if R is.

For f -clean rings we have the following

Theorem 2.13. Let R be a ring and σ be an automorphism of R . Then $T = (HR, \sigma)$ is f -clean ring if and only if R is.

Proof. Let $W = \{f \in T \mid f(0) = 0\}$. Then $R \cong T/W$ is a homomorphic image of T . Hence, if T is f -clean ring, so is R .

Conversely, suppose that R is a f -clean. For any $h \in T$, write $h(0) \in R$ as $h(0) = x + e$ where $x \in K(R)$ and $e \in Id(R)$. Define an element $g \in T$ by

$$g(n) = \begin{cases} x, & n = 0 \\ h(n), & n > 0 \end{cases}.$$

Then $h = g + h_e$, where g is a full element of T and h_e is an idempotent element of T . This proves that h is f -clean, and hence $T = (HR, \sigma)$ is f -clean ring. $\square$

Camillo and Yu [5], showed that every element of a clean ring which 2 is invertible is a sum of no more than two units. For f -clean ring, we have the following result.

Theorem 2.14. Every f -clean ring R in which 2 is invertible is a sum of no more than three full elements.

Proof. Let $x \in R$. Since 2 is invertible in R , $\frac{x+1}{2}$ is in R . R is f -clean ring implies that $\frac{x+1}{2} = s + g$, where $s \in K(R)$ and $g \in Id(R)$. So we obtain

$$x = 2s + (1 - 2g) + 2(2g - 1),$$

where $(1 - 2g), 2(2g - 1) \in U(R) \subseteq K(R)$ and $2s \in K(R)$. Hence R is a sum of three full elements. $\square$

Proposition 2.15. Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \dots$, be a graded ring. If R is f -clean ring, then R_0 is f -clean ring.

Proof. Let $r_0 \in R_0$. Since R is f -clean, $r_0 = s + g$ where $s \in K(R)$ and $g \in Id(R)$. Since $Id(R) = Id(R_0)$, $g \in Id(R_0)$. Now, $g, r_0 \in R_0$ gives $s \in R_0$. So, $s \in R_0 \cap K(R)$. Hence, $s \in K(R_0)$. Therefore, R_0 is f -clean ring. $\square$

Theorem 2.16. $M = P_1 \oplus P_2 \oplus \dots \oplus P_n$ for some $n \geq 1$ where P_i is α -invariant and $\alpha|_{P_i}$ is f -clean in $end(P_i)$ for each i . Then α is f -clean in $E = end({}_R M)$.

Proof. Extend maps λ_i in $end(P_i)$ to $\widehat{\lambda}_i$ in $end(M)$ by defining $(p_1 + p_2 + \dots + p_n)\widehat{\lambda}_i = p_i\lambda_i$. Then $\widehat{\lambda}_i\widehat{\lambda}_j = 0$ if $i \neq j$ while $\widehat{\lambda}_i\widehat{\mu}_i = \widehat{\lambda}_i\mu_i$ and $\widehat{\lambda}_i + \widehat{\mu}_i = \widehat{\lambda_i + \mu_i}$ for all μ_i in $end(P_i)$. By hypothesis, $\alpha|_{P_i} = s_i + g_i$ in $end(P_i)$ where $s_i \in Id(end(P_i))$ and $g_i \in K(end(P_i))$. Let $s = \sum_i \widehat{s_i}$ and $g = \sum_i \widehat{g_i}$. It follows from Lemma 2.5 that $s \in Id(E)$. Since $g_i \in K(end(P_i))$, for each i there exist $t_i, v_i \in end(P_i)$ such that $t_i g_i v_i = 1_{P_i}$. Then $tgv = \sum_i \widehat{t_i g_i v_i} = \sum_i \widehat{t_i g_i} v_i = \sum_i \widehat{1_{P_i}} = 1_E$. So $g \in K(E)$. Hence $\alpha = \sum_i \widehat{\alpha|_{P_i}} = \sum_i \widehat{s_i + g_i} = s + g$ is the f -clean expression of α in $E = end({}_R M)$. $\square$

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