A NOTE ON BÜCHI'S PROBLEM FOR P -ADIC NUMBERS

MARIANELA CASTILLO

UNIVERSIDAD DE CONCEPCIÓN, CHILE

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Abstract

We prove that for any prime p and any integer $k \geq 2$, there exist in the ring $\mathbf{Z}_p$ of p -adic integers arbitrarily long sequences whose sequence of k -th powers 1) has its k -th difference sequence equal to the constant sequence $(k!)$; and 2) is not a sequence of consecutive k -th powers. This shows that the analogue of Büchi's problem for higher powers has a negative answer over $\mathbf{Z}_p$. This result for $k = 2$ was recently obtained by J. Browkin.

MSC: 11D72, 11D88.

1. Introduction

Motivated by a mathematical logic problem, J.R. Büchi proposed the following problem in the early 1970's.

Problem 1.1 (Büchi's problem). $\mathbf{B}^2(\mathbf{Z})$. *Does there exist a positive integer M such that any sequence of M integer squares, with second difference constant equal to the constant sequence (2), is of the form $((x+n)^2)$, where $n = 1, \dots, M$, for some integer x ?*

Büchi's problem is open. In [6], Pheidas and Vidaux proposed a generalization of Büchi's problem to any unitary commutative ring and to higher powers.

Definition 1.2. *Let $k \geq 0$ be an integer. A sequence of elements of a unitary commutative ring A of characteristic 0 is called a k -Büchi sequence in A if the sequence of its k -th powers has k -th difference constant equal to $(k!)$. Every sequence whose sequence of k -th powers is of the form $((x+n)^k)$ for some x in A will be referred to as trivial k -Büchi sequence.*

Büchi's problem is generalized as follows (see [5] for a general survey).

Problem 1.3. $\mathbf{B}^k(A)$. *Let $k \geq 2$ be an integer and A a unitary commutative ring of characteristic 0. Does there exists an integer M such that every k -Büchi sequence in A of length M is trivial?*

We are interested in those rings for which Büchi's problem has a negative answer in a non-trivial way (intuitively, rings with not too many k -th powers). In [1], J. Browkin proved, in particular, that for $k = 2$, Büchi's problem for the ring of p -adic integers $\mathbf{Z}_p$ has a negative answer. This paper is an attempt to generalize to higher powers some of the results in Browkin's paper.

Theorem 1.4. *Let p be a prime number and $k \geq 2$ an integer.*

1. *For each $n \in \mathbf{Z}$, when $p > 2$ the quantity $n^k + p^{-k}$ is a k -th power in $\mathbf{Q}_p$. Moreover, any sequence (a_n) , $n \geq 1$, such that a_n has k -th power $n^k + p^{-k}$, is a non-trivial k -Büchi sequence of infinite length in $\mathbf{Q}_p$.*

2. For each $n \in \mathbf{Z}$, the quantity $n^k + 2^{-(2+k)}$ is a k -th power in $\mathbf{Q}_2$. Moreover, any sequence (a_n) , $n \geq 1$, such that a_n has k -th power $n^k + 2^{-(2+k)}$, is a non-trivial k -Büchi sequence of infinite length in $\mathbf{Q}_2$.
3. For each $m \in \mathbf{Z}$, the quantity $n^k + p^{mk+1}$ is a k -th power in $\mathbf{Z}_p$ when $n = 1, \dots, p^m - 1$. Moreover, if $p^m \geq k + 2$, then any sequence (a_n) , $n = 1, \dots, p^m - 1$, where a_n has k -th power $n^k + p^{mk+1}$, is a non-trivial k -Büchi sequence in $\mathbf{Z}_p$.

In Section 2, we give a characterization of the set of k -th powers in $\mathbf{Z}_p$, for each k and p . Some of the results seem to be well-known, but we were not able to find a proper reference. In Section 3, we will prove Items 1, 2 and 3 of our theorem.

For references on p -adic arithmetic, see for example [3].

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2. Powers in $\mathbf{Z}_p$ and in $\mathbf{Q}_p$

Fix an integer $k \geq 2$ for the whole section.

Lemma 2.1. *Let $n \geq 1$ be an integer and suppose that either $p \neq 2$ or $n \neq 1$. The following equality holds: $(1 + p^n \mathbf{Z}_p)^p = 1 + p^{1+n} \mathbf{Z}_p$.*

Proof. We use the fact that the exponential function is a group isomorphism between the additive group $p^n \mathbf{Z}_p$ and the multiplicative group $1 + p^n \mathbf{Z}_p$ whenever n is strictly bigger than $\frac{1}{p-1}$ - see [4, p. 38, Thm 1.2]. Therefore we have

$$\log((1 + p^n \mathbf{Z}_p)^p) = p^{n+1} \mathbf{Z}_p = \log(1 + p^{1+n} \mathbf{Z}_p).$$

□

Lemma 2.2. *The following equality holds: $(1 + 2\mathbf{Z}_2)^2 = 1 + 8\mathbf{Z}_2$*

Proof. Given $c \in \mathbf{Z}_2$, we have

$$(1 + 2c)^2 = 1 + 4c + 4c^2 = 1 + 4(c + c^2).$$

Since the first terms of c and of c^2 are either both 1 or both a positive power of 2, the quantity $c + c^2 \in 2\mathbf{Z}_2$. Therefore we have

$$(1 + 2\mathbf{Z}_2)^2 \subseteq 1 + 8\mathbf{Z}_2.$$

Now, given $c \in \mathbf{Z}_p$, consider the function $f(x) = x^2 - (1 + 8c)$ and its derivative $f'(x) = 2x$. We have

$$f'(1) = 2 \equiv 0 \pmod{2} \not\equiv 0 \pmod{2^2} \quad \text{and} \quad f(1) = -8c \equiv 0 \pmod{2^3}.$$

By Hensel's Lemma, there exists $a \in \mathbf{Z}_2$ such that $f(a) = 0$ and $a \equiv 1 \pmod{2^2}$. So, in particular, a belongs to $1 + 2\mathbf{Z}_2$. $\square$

Lemma 2.3. *Let $n \geq 0$ be an integer and suppose that $p \neq 2$. The following equality holds:*

$$(2.1) \quad (1 + p\mathbf{Z}_p)^{p^n} = 1 + p^{1+n}\mathbf{Z}_p.$$

Proof. The proof is analogous to that of Lemma 2.1. $\square$

Lemma 2.4. *For $n \geq 1$, the following equality holds:*

$$(2.2) \quad (1 + 2\mathbf{Z}_2)^{2^n} = 1 + 2^{2+n}\mathbf{Z}_2.$$

Proof. We prove the lemma by induction on n . By Lemma 2.2, we have

$$(1 + 2\mathbf{Z}_2)^2 = 1 + 8\mathbf{Z}_2$$

and Equality 2.2 holds for $n = 1$.

Suppose that Equation 2.2 holds up to n and let us prove that it holds for $n + 1$. Indeed, by induction hypothesis, we have

$$(1 + 2\mathbf{Z}_2)^{2^{1+n}} = \left(1 + 2^{2+n}\mathbf{Z}_2\right)^2 = 1 + 2^{3+n}\mathbf{Z}_2,$$

where the last equality comes from the fact that $2 + n \neq 1$ and we can therefore apply Lemma 2.1. $\square$

Corollary 2.5. *For all integers $k \geq 2$ and for p an odd prime, the following equalities hold: $(1 + p\mathbf{Z}_p)^k = 1 + p^{1+}$ and $(1 + 2\mathbf{Z}_2)^k = 1 + 2^{2+\text{ord}_2 k}\mathbf{Z}_2$.*

Proof. Write $k = p^{\text{ord}_p k} r$, where p does not divide r . Since p does not divide r , the map $x \mapsto x^r$ defines an automorphism of the multiplicative group of one-units $1 + p\mathbf{Z}_p$ (see Hasse [2, Ch. 15, Section 2, p. 215-217]) - note that the existence of an r -th root comes immediately from Hensel's Lemma, hence the map $x \mapsto x^r$ is a surjective morphism. Indeed, Hasse defines a map

$$\begin{array}{ccc} 1 + p\mathbf{Z}_p & \longrightarrow & 1 + p\mathbf{Z}_p \\ x & \longmapsto & x^c \end{array}$$

for any $c \in \mathbf{Z}_p$ and shows that it is an homomorphism of groups. Since r is invertible in $\mathbf{Z}_p$, the above map with $c = \frac{1}{r}$ is the reciprocal of $x \mapsto x^r$, which, therefore, is injective.

For $p > 2$, Lemma 2.3 gives

$$(1 + p\mathbf{Z}_p)^k = ((1 + p\mathbf{Z}_p)^r)^{p^{\text{ord}_p k}} = (1 + p\mathbf{Z}_p)^{p^{\text{ord}_p k}} = 1 + p^{1+\text{ord}_p k}\mathbf{Z}_p,$$

and for $p = 2$ we apply Lemma 2.4 and find

$$(1 + 2\mathbf{Z}_2)^k = ((1 + 2\mathbf{Z}_2)^r)^{2^{\text{ord}_2 k}} = (1 + 2\mathbf{Z}_2)^{2^{\text{ord}_2 k}} = 1 + 2^{2+\text{ord}_2 k}\mathbf{Z}_2.$$

□

Notation 2.6. 1. Let ε be 1 if $p = 2$ and 0 if $p \neq 2$.

2. We will denote by S_p^k the set of non-zero k -th powers in $\mathbf{Z}_p$.

Lemma 2.7. Let $g \in \mathbf{N}$ be a fixed primitive root modulo p . We have $S_p^k = \{p^{km} g^{k\ell} (1 + p^{\varepsilon+1+\text{ord}_p k} c) : m \geq 0, \ell \geq 0, c \in \mathbf{Z}_p\}$

Proof. Let α be a non-zero k -th power in $\mathbf{Z}_p$, i.e. there is an integer $m \geq 0$ and there exists $\beta \in \mathbf{Z}_p$ invertible such that

$$\alpha = p^{km} \beta^k.$$

Since $\beta \in \mathbf{Z}_p^\times$, there are integers $a_i \in \{0, 1, \dots, p-1\}$ such that

$$\beta = a_0 + a_1 p + a_2 p^2 + \dots$$

where $a_0 \neq 0$. Since the residue class modulo p of a_0 is not 0, it is generated by the residue class of g in $\mathbf{F}_p^\times$, so there exists an integer $\ell \geq 0$ and an integer z such that $a_0 = g^\ell + pz$. We have

$$\beta = g^\ell \left(1 + p \left(\frac{1}{g^\ell} (z + a_1 + a_2 p + a_3 p^2 + \dots) \right) \right).$$

Write $c = \frac{1}{g^\ell}(z + a_1 + a_2p + a_3p^2 + \dots)$. Since $\text{ord}_p(c) \geq 0$, we conclude that, for all non-zero k -th power α , there are integers $m \geq 0$ and $\ell \geq 0$, and a p -adic integer c such that

$$\alpha = p^{km}g^{k\ell}(1 + pc)^k$$

On the other hand, for all $c \in \mathbf{Z}_p$, the quantity $p^{km}g^{k\ell}(1 + pc)^k$ is trivially a k -th power. Finally, by Corollary 2.5, we have

$$S_p^k = p^{km}g^{k\ell}(1 + p\mathbf{Z}_p)^k = p^{km}g^{k\ell}(1 + p^{\varepsilon+1+\text{ord}_p k}\mathbf{Z}_p).$$

□

3. Büchi sequences in $\mathbf{Z}_p$ and $\mathbf{Q}_p$ for any power

Fix an integer $k \geq 2$. The notation $\sqrt[k]{x}$ will refer to any specific k -th root of x .

Lemma 3.1. *For any prime number p and for any non-zero $b \in \mathbf{Q}_p$, the sequence (a_n) defined by $a_n = \sqrt[k]{n^k - b}$, $1 \leq n \leq M$ is a non-trivial Büchi sequence over the algebraic closure of $\mathbf{Q}_p$, whenever $M \geq k$.*

Proof. If (a_n) is trivial then, by definition, there exists $x \in \bar{\mathbf{Q}}_p$ such that

$$n^k - b = (x + n)^k$$

for each $n \geq 1$. If $x = 0$ then trivially $b = 0$, and if x is non-zero then b is a polynomial in n of degree $k - 1 \geq 1$, which is impossible since the sequence has length greater than $k - 1$. □

Proof. [Proof of Item 1 of the main theorem] By Lemma 3.1 we need only prove that for each integer $n \in \mathbf{Z}$, the quantity $n^k + p^{-k} \in \mathbf{Q}_p$ is a k -th power in $\mathbf{Q}_p$, which is equivalent to prove that $p^k n^k + 1$ is a k -th power in $\mathbf{Z}_p$. For each k and p we have $k \geq 1 + \text{ord}_p k$, hence the quantity $c = p^{k-1-\text{ord}_p k} n^k$ is a p -adic integer and we deduce that

$$1 + p^k n^k = 1 + p^{1+\text{ord}_p k} \left(p^{k-1-\text{ord}_p k} n^k \right)$$

is a k -th power by Lemma 2.7. □

Proof. [Proof of Item 2 of the main theorem] Again, we prove that for each integer $n \in \mathbf{Z}$, the quantity $n^k + 2^{-(2+k)}$ is a k -th power in $\mathbf{Q}_2$, which is equivalent to prove that $2^{2+k}n^k + 1$ is a k -th power in $\mathbf{Z}_2$. Since for each k we have $k \geq \text{ord}_2 k$, we deduce that $c = 2^{k-\text{ord}_2 k}n^k$ is a p -adic integer and therefore

$$1 + 2^{2+k}n^k = 1 + 2^{2+\text{ord}_2 k} \left(2^{k-\text{ord}_2 k}n^k \right)$$

is a k -th power by Lemma 2.7. $\square$

Proof. [Proof of Item 3 of the main theorem] Similarly, for every $m \geq 1$, we want to prove that the quantity $n^k + p^{km+1}$ is a k -th power in $\mathbf{Z}_p$ for $n = 1, \dots, p^m - 1$. By Lemma 2.7, since

$$n^k + p^{km+1} = n^k \left(1 + p^{2+\text{ord}_p k} \frac{p^{km-1-\text{ord}_p k}}{n^k} \right)$$

we need only prove that the quantity

$$\frac{p^{km-1-\text{ord}_p k}}{n^k}$$

is a p -adic integer. Since $n < p^m$, we have $\text{ord}_p n \leq m - 1$, hence

$$\text{ord}_p n^k \leq km - k \leq km - (1 + \text{ord}_p k).$$

Finally we obtain

$$\text{ord}_p \left(\frac{p^{km-1-\text{ord}_p k}}{n^k} \right) = km - (1 + \text{ord}_p k) - \text{ord}_p n^k \geq 0.$$

$\square$

Note that the sequence used for the above proof can not be extended to an infinite sequence. Actually, if $n = p^{m+1}$, then $\text{ord}_p (n^k + p^{km+1}) = \min \{ \text{ord}_p (p^{km+k}), \text{ord}_p (p^{km+1}) \} = km + 1$ which is not a multiple of k , therefore, $n^k + p^{km+1}$ is not a k -th power in $\mathbf{Z}_p$.

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Marianela Castillo
Universidad de Concepción
Casilla 160-C
Concepción
e-mail : mcastillo@udec.cl