On certain isotopic maps of central loops

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Abstract

It is shown that the Holomorph of a C-loop is a C-loop if each element of the automorphism group of the loops is left nuclear. Condition under which an element of the Bryant-Schneider group of a C-loop will form an automorphism is established. It is proved that elements of the Bryant-Schneider group of a C-loop can be expressed a product of pseudo-automorphisms and right translations of elements of the nucleus of the loop. The Bryant-Schneider group of a C-loop is also shown to be a kind of generalized holomorph of the loop.

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1. Introduction

Central loops (C-loops) are loops which satisfy one of the identities called "Central identity" as named by F. Fenyves [9], [10]. Closely related to the central identity are left central (LC) and right central (RC) identities. The expressions for the mentioned identities are as follows:

\[(yx \cdot x)z = y(x \cdot xz)\] central identity

\[(1.1)\]

i. \(xx \cdot yz = (x \cdot xy)z \equiv\)

ii. \((x \cdot xy)z = x(x \cdot yz) \equiv\)

iii. \((xx \cdot y)z = x(x \cdot yz)\)  

LC- identities

\[(1.2)\]

i. \(yz \cdot xx = y(zx \cdot x) \equiv\)

ii. \((yz \cdot x)x = y(zx \cdot x) \equiv\)

iii. \((yz \cdot x)x = y(z \cdot xx)\)  

RC- identities

\[(1.3)\]

Recently Phillips and Vojtechovsky [20], found out that in addition to the identities above, LC and RC identity can also be defined respectively by,

\[(y \cdot xx)z = y(x \cdot xz) \text{ and } (yx \cdot x)z = y(xx \cdot z)\]  

\[(1.4)\]

C-loops are one of the least studied loops. Few publications that have considered C-loops include Fenyves [9], [10], Phillips and Vojtechovsky [18] [20] [19], Chein [5]. The difficulty in studying them is as a result of the nature of their identities when compared with other Bol-Moufang identities (the element occurring twice on both sides has no other element separating it from itself).

2. Preliminaries

**Theorem 2.1.** ([10], [20]) Let \((L, \cdot)\) be an LC-loop (RC-loop). Then:

1. \((L, \cdot)\) is a left (right) alternative loop,
2. \((L, \cdot)\) is a left (right) inverse property loop,
3. \((L, \cdot)\) is a left (right) nuclear square loop,
4. \((L, \cdot)\) is a left (right) power alternative loop,
5. \((L, \cdot)\) is a middle square loop,
6. \((L, \cdot)\) is power associative loop.

**Definition 2.1.** A triple \((\alpha, \beta, \gamma)\) of bijections is called an isotopism of loop \((L, \cdot)\) onto a loop \((H, \circ)\) provided \(x\alpha \circ y\beta = (x \cdot y)\gamma\ \forall\ x, y \in L\). \((H, \circ)\) is called an isotope of \((L, \cdot)\). The loops \((L, \cdot)\) and \((H, \circ)\) are said to be isotopic to each other.

**Definition 2.2.** Let \(\alpha\) and \(\beta\) be a permutation of \(L\) and let \(\iota\) denotes the identity map on \(L\). Then \((\alpha, \beta, \iota)\) is a principal isotopism of \((L, \cdot)\) onto a loop \((L, \circ)\) which imply that \((\alpha, \beta, \iota)\) is an isotopism of \((L, \cdot)\) onto \((L, \circ)\).

**Definition 2.3.** An isotopism of \((L, \cdot)\) onto \((L, \circ)\) is called an autotopism of \((L, \cdot)\). The group of autotopisms of \(L\) is denoted by \(A(L)\).

**Remark 2.1.** The components of isotopism are usually denoted by lower case Greek letters. However, we shall denote the components of autotopism by capital letters, thus if \(T = (U, V, W)\) is an autotopism of a loop \((L, \cdot)\), then
\[xU \cdot yV = (xy)W, \forall\ x, y \in L.\]

The set of all autotopism of a loop is a group with the inverse of \(TT^{-1} = (U, V, W)^{-1} = (U^{-1}, V^{-1}, W^{-1})\). The identity element of the group being \((I, I, I)\) where \(I\) is the identity map of \(L\). If \(T = (U, U, U)\), then \(T\) is called the automorphism \((L, \cdot)\)

**Definition 2.4.** If \(\langle U, V, W \rangle\) is autotopism of an inverse property loop \((L, \cdot)\) then \(\langle W, JV, JU \rangle\) and \(\langle JUJ, W, V \rangle\) are autotopism of \(L\). Moreover if \(\langle U, V, W \rangle = \langle S, SR_c, SR_c \rangle\) the \(S\) is called a pseudoautomorphism of \(L\) with companion \(c\). The set of all pseudoautomorphisms of \(L\) is denoted by \(PS(L, \cdot)\).

**Definition 2.5.** Let \((L, \cdot)\) be an inverse property loop with the nucleus denoted by \(N\). Then an automorphism \(\alpha\) of \((L, \cdot)\) is left nuclear iff \(a \alpha \cdot a^{-1} \in N\) for all \(a \in L\).

**Definition 2.6.** Let \((L, \cdot)\) be a loop and \(BS(L, .)\) be the set of all permutations \(\theta\) of \(Q\) such that
\[< \theta R_g^{-1}, \theta L_f^{-1}, \theta >\]
is an autotopism of \((L, \cdot)\) for some \(f, g \in L\), then \(BS(L, .)\) is called the Bryant-Schneider group of the loop.
**Definition 2.7.** Let \((L, \cdot)\) be a loop, \(A(L)\) a group of automorphisms of loop \((L, \cdot)\) and let \(H = A(L) \times L\) and define
\[
(\alpha, x) \circ (\beta, y) = (\alpha \beta, x \beta \cdot y)
\]
\(\forall (\alpha, x), (\beta, y) \in H\). Then the loop \((H, \circ)\) is called the \(A(L)\)-holomorph of \((L, \cdot)\) or simply holomorphy of \((L, \cdot)\).

3. Holomorphy

**Theorem 3.1.** Let \((L, \cdot)\) be an LC-loop and \(A(L)\) be a group of automorphism of \((L, \cdot)\). Then the \(A(L)\)-holomorph \((H, \circ)\) of \((L, \cdot)\) is an LC-loop if and only if
\[
(x \alpha \cdot xy)z = x(\alpha \cdot yz)
\]
\(\forall x, y, z \in L\) and \(\forall \alpha \in A(L)\).

**Proof.**

Suppose \(A(L)\)-holomorph \((H, \circ)\) of \((L, \cdot)\) is an LC-loop we have
\[
\{(\alpha, x)\circ[(\alpha, x)\circ(\beta, y)]\circ(\gamma, z) = (\alpha, x)\circ((\alpha, x)\circ(\beta, y)\circ(\gamma, z))\}
\]
\(\forall x, y, z \in L\) and \(\forall \alpha, \beta, \gamma \in A(L)\). Thus
\[
\{(\alpha, x)\circ(\alpha \beta, x \beta \cdot y)\circ(\beta, y)\circ(\gamma, z) = (\alpha, x)\circ((\alpha, x)\circ(\beta, \gamma, y \gamma \cdot z))\}
\]
\(\forall x, y, z \in L\) and \(\forall \alpha, \beta, \gamma \in A(L)\). Therefore
\[
\{x(\alpha \beta, x \beta \cdot y)\gamma \cdot z = x(\alpha \beta \cdot x \beta \gamma, y \gamma \cdot z)
\]
\(\forall x, y, z \in L\) and \(\forall \alpha, \beta, \gamma \in A(L)\).

Therefore,
\[
\{x(\alpha \beta \cdot x \beta \gamma, y \gamma \cdot z) = x(\alpha \beta \gamma, x \beta \gamma, y \gamma \cdot z)
\]
putting \(\phi = \beta \gamma\), gives
\[
\{x(\alpha \phi, x \phi \cdot y \gamma) = x(\alpha \phi \cdot y \gamma, z)
\]
hence
\[\{x\alpha \cdot (x \cdot y\gamma\phi^{-1})\} \cdot z\phi^{-1} = \{x\alpha \cdot x(y\gamma\phi^{-1} \cdot z\phi^{-1})\}\]
\[\forall x, y, z \in L \text{ and } \forall \alpha, \phi, \gamma \in A(L).\]
If we put \(\mathfrak{y} = y\gamma\phi^{-1}\) and \(\mathfrak{z} = z\phi^{-1}\), we obtain
\[(x\alpha \cdot x\mathfrak{y})\mathfrak{z} = x\alpha \cdot (x \cdot \mathfrak{y} \mathfrak{z})\]
And replacing \(\mathfrak{y}\) and \(\mathfrak{z}\) by \(y\) and \(z\) respectively we have
\[(x\alpha \cdot xy)z = x\alpha(x \cdot yz)\]
\[\forall x, y, z \in L \text{ and } \forall \alpha \in A(L), \text{ which is equation (3.1)}.\]

The converse is obtained by reversing the process.

**Corollary 3.1.** Let \((L, \cdot)\) be a loop, and \(A(L)\) be the group of all automorphism of \(L\), then \(L\) is an LC-loop if
\[B = \langle L_xL_{x\alpha}, I, L_xL_{x\alpha} \rangle\]
is an autotopism of \(L\), \(\forall x, y, z \in L\) and \(\forall \alpha \in A(L)\)

**Proof.** This is a consequence of (3.1)

**Theorem 3.2.** Let \((L, \cdot)\) be a loop and \(A(L)\) be a group of automorphism of \((L, \cdot)\). Then the \(A(L)\)-holomorph \((H, o)\) of \((L, \cdot)\) is an RC-loop if and only if
\[y((z \cdot x\alpha)x) = (yz \cdot x\alpha)x\]
\[\forall x, y, z \in L \text{ and } \forall \alpha \in A(L).\]

**Proof.**

The procedure for the proof is like that of Theorem 3.1 above hence it is omitted.

**Corollary 3.2.** Let \((L, \cdot)\) be any loop and \(A(L)\) be the group of all automorphisms of \(L\), then \(L\) is an RC-loop if and only if
\[B = \langle I, R_{x\alpha}R_x, R_{x\alpha}R_x \rangle\]
is an autotopism of \(L\), for all \(x, y, z \in L\) and all \(\alpha \in A(L)\).
Proof.
From (3.4)
\[ y((z \cdot x)x) = (yz \cdot x)x \]
\[ \Rightarrow y \cdot zR_{x\alpha}R_x = yzR_{x\alpha}R_x \]
\[ \forall \ x, y, z \in L \ and \ \forall \ \alpha \in A(L). \]
\[ \Rightarrow \langle I, R_{x\alpha}R_x, R_{x\alpha}R_x \rangle \]
is an autotopism of \((L, \cdot) \ \forall \ x \in L \ and \ \forall \ \alpha \in A(L). \]

Conversely, suppose (3.5) hold, then \( \forall \ y, z \in L \) we have
\[ yI \cdot zR_{x\alpha}R_x = yzR_{x\alpha}R_x \]
\[ y((z \cdot x)x) = yz(x\alpha \cdot x) \]
\[ \forall \ x, y, z \in L \ and \ \forall \ \alpha \in A(L). \]

**Theorem 3.3.** Let \((L, \cdot)\) be a loop and \(A(L)\) be a group of automorphism of \((L, \cdot)\). Then the \(A(L)\)-holomorph \((H, o)\) of \((L, \cdot)\) is a C-loop if and only if
\[ (y \cdot x)x \cdot z = y(x\alpha \cdot xz) \]
\[ \forall \ x, y, z \in L \ and \ \forall \ \alpha \in A(L). \]

**Proof.**
The procedure for the proof is like that of theorem 3.1 hence it is omitted.

**Corollary 3.3.** Let \((L, \cdot)\) be a loop and \(A(L)\) be the group of all automorphisms of \(L\), then \(L\) is a C-loop if and only if
\[ B = \langle R_{x\alpha}R_x, L^{-1}_{x\alpha}L_x, I \rangle \]
is an autotopism of \(L\), for all \( x, y, z \in L \) and all \( \alpha \in A(L) \)

**Proof.** From (3.6)
\[ (y \cdot x\alpha)x \cdot z = y(x\alpha \cdot xz) \]
\[ \Rightarrow yR_{x\alpha}R_x \cdot z = y \cdot z_{L_xL_{x\alpha}} \]
\[ \forall \ x, y, z \in L \ and \ \forall \ \alpha \in A(L). \]
substituting \( \tau = z_{L_xL_{x\alpha}} \) we have
\[ yR_{x\alpha}R_x \cdot \tau_{L(\alpha^{-1})}L_{x^{-1}} = y\tau \]
∀ x, y, z \in L and ∀ α ∈ A(L).

\[ \Rightarrow \langle R_{xα}R_x, L_{xα}^{-1}L_x^{-1}, I \rangle \]

is an autotopism of \((L, \cdot) \forall x \in L \text{ and } \forall \alpha \in A(L)\).

Conversely, suppose equation (3.7) is an autotopism of \((L, \cdot)\), therefore
∀ y, z ∈ L we have

\[ yR_{xα}R_x \cdot zL_{xα}^{-1}L_x^{-1} = yz \cdot I \]
\[ yR_{xα}R_x \cdot z = y \cdot zL_{xα}L_xI \]
\[ (y \cdot xα)z = y(xα \cdot xz) \]

∀ x, y, z ∈ L and ∀ α ∈ A(L) hence \((L, \cdot)\) is a C-loop.

### 3.1. Nuclear Automorphism

**Theorem 3.4.** Let \((L, \cdot)\) be a loop and \(A(L)\) be a group of automorphism of \((L, \cdot)\). Then the \(A(L)\)-holomorph \((H, o)\) of \((L, \cdot)\) is a C-loop iff \((L, \cdot)\) is a C-loop and each \(\alpha \in A(L)\) is a left nuclear automorphism of \((L, \cdot)\).

**Proof.** Suppose \((H, o)\) is a C-loop. Since \((L, \cdot)\) is isomorphic to a subloop of \((H, o)\), it follows that \((L, \cdot)\) must be a C-loop. From Theorem (3.1), equation (3.1) holds \(\forall x, y, z \in L \text{ and } \forall \alpha \in A(L)\). Furthermore, by Theorem (3.1) and Corollary (3.3),

\[ A(x) = \langle R_x^2, L_x^{-2}, I \rangle \text{ and } B(x) = \langle R_xR_{xα}, L_x^{-1}L_{xα}^{-1}, I \rangle \]

are autotopisms of \((L, \cdot)\), \(\forall x \in L \text{ and } \forall \alpha \in A(L)\). Therefore by Theorem (3.1) and we have

\[ A_λ(x) = \langle L_x^{-2}, I, L_x^{-2} \rangle, A_μ^{-1}(x) = \langle I, R_x^2, R_x^2 \rangle, \]
\[ B_λ^{-1}(x) = \langle L_{xα}L_x, I, L_{xα}L_x \rangle \text{ and } B_μ(x) = \langle I, R_xR_{xα}, R_xR_{xα} \rangle \]

are also autotopisms of \((L, \cdot)\), \(\forall x \in L \text{ and } \forall \alpha \in A(L)\). If these are combined we have

\[ A_λ(x)B_λ^{-1}(x) = \langle L_x^{-2}, I, L_x^{-2} \rangle \langle L_xL_{xα}, I, L_xL_{xα} \rangle \]

(3.8)

\[ A_λ(x)B_λ^{-1}(x) = \langle L_x^{-1}L_{xα}, I, L_x^{-1}L_{xα} \rangle \]

and

\[ B_μ(x)A_μ^{-1}(x) = \langle I, R_xR_{xα}, R_xR_x \rangle \langle I, R_x^{-2}, R_x^{-2} \rangle \]
as autotopisms of \((L, \cdot)\), \(\forall x \in L\) and \(\forall \alpha \in A(L)\). Now if we apply (3.8) and (3.9) to \(1 \cdot b\) and \(a \cdot 1\) respectively, we have
\[
L_x^{-1} L_{x\alpha} \cdot b = (1 \cdot b) L_x^{-1} L_{x\alpha}
\]
\[
(x\alpha \cdot x^{-1}) b = b L(x)^{-1} L_{x\alpha}
\]
\[
b L_{x\alpha} L_x^{-1} = b L_x^{-1} L_{x\alpha}
\]
and
\[
a \cdot 1 R_{x\alpha} R_x^{-1} = (a \cdot 1) R_{x\alpha} R_x^{-1}
\]
\[
a (x\alpha \cdot x^{-1}) = a R_{x\alpha} R_x^{-1}
\]
\[
a R_{x\alpha} R_x^{-1} = a R_{x\alpha} R_x^{-1}
\]
and respectively we have
\[
L_{x\alpha \cdot x^{-1}} = L_{x}^{-1} L_{x\alpha}
\]
\[
R_{x\alpha \cdot x^{-1}} = R_{x\alpha} R_x^{-1}
\]
\(\forall x \in L \text{ and } \forall \alpha \in A(L)\). If we put equations (3.10) and (3.11) into equations (3.8) and (3.9) respectively, we have
\[
A_{\lambda}(x) B_{\mu}^{-1}(x) = \langle L_{x\alpha \cdot x^{-1}}, I, L_{x\alpha \cdot x^{-1}} \rangle
\]
and
\[
B_{\mu}(x) A_{\mu}^{-1}(x) = \langle I, R_{x\alpha \cdot x^{-1}}, R_{x\alpha \cdot x^{-1}} \rangle
\]
\(\forall x \in L \text{ and } \forall \alpha \in A(L)\). These therefore imply that \(x\alpha \cdot x^{-1} \in N_{\lambda}(L)\) and \(x\alpha \cdot x^{-1} \in N_{\mu}(L)\). Consequently, \(x\alpha \cdot x^{-1} \in N(L)\) since \((L, \cdot)\) is an inverse property loop. Hence \(\alpha \in A(L)\), is left nuclear.

Conversely, suppose \((L, \cdot)\) is a C-loop and each \(\alpha \in A(L)\) is left nuclear. Then for each \(\alpha \in A(L)\) and each \(x \in L\) the element \(x\alpha \cdot x^{-1} \in N_{\mu}(L)\), thus
\[
x\alpha \cdot y = ((x\alpha \cdot x^{-1}) x) y
\]
\[
x\alpha \cdot y = (x\alpha \cdot x^{-1}) xy
\]
\(\forall y \in L\)
\[
y L_{x\alpha} = y L_x L_{x\alpha \cdot x^{-1}} \Rightarrow L_x^{-1} L_{x\alpha} = L_{x\alpha \cdot x^{-1}}
\]
∀ x ∈ L and ∀ α ∈ A(L). But for ∀ x ∈ L and ∀ α ∈ A(L), we know that 
\[ xα \cdot x^{-1} ∈ N_λ(L) \]. Hence,

\[ C = \langle L_{α\cdot x^{-1}}, I, L_{xα\cdot x^{-1}} \rangle = \langle L_x^{-1}L_{2α}, I, L_{x^{-1}}L_{xα} \rangle \]
is an autotopism of \((L, \cdot)\), ∀ x ∈ L and ∀ α ∈ A(L). But again , \( A = \langle L_x^2, I, L_x^2 \rangle \) is an autotopism of \((L, \cdot)\), ∀ x ∈ L. Therefore,

\[ AC = \langle L_xL_{xα}, I, L_xL_{xα} \rangle \]
is an autotopism of \((L, \cdot)\), ∀ x ∈ L and ∀ α ∈ A(L). Therefore, if we put \( z = zL_{xα}^{-1}L_x^{-1} \), in this we have

\[ yR_{xα}Rx \cdot zL_{xα}^{-1}L_x^{-1} = yz \]

∀ x, y, z ∈ L and ∀ α ∈ A(L). Replacing \( z \) by \( z \), ∀ x, y, z ∈ L and ∀ α ∈ A(L) and we have a central identity. Hence, \((H, o)\) is a C-loop.

**Theorem 3.5.** The set \( S(L) \) of all left nuclear automorphism of an C-loop \((L, \cdot)\), is a normal subgroup of the automorphism group of \((L, \cdot)\).

**Proof.** \( S(L) \neq \emptyset \), from the Theorem 3.4 it was shown that

\[ L_{uα\cdot u^{-1}} = L_u^{-1}L_{uα} \]

∀ u ∈ L and ∀ α ∈ S(L) (since for an inverse property loop \( L, L_{u^{-1}} = L_u^{-1} \)
∀u ∈ L). Then \( uα \cdot u^{-1} ∈ N_λ(L, \cdot), ∀ u ∈ L \) and ∀ α ∈ S(L). It follows then that

\[ A(α, u) = \langle L_{uα\cdot u^{-1}}, I, L_{uα\cdot u^{-1}} \rangle = \langle L_u^{-1}L_{uα}, I, L_u^{-1}L_{uα} \rangle \]

∀ u ∈ L and for all α ∈ L. Hence if α, β ∈ S(L), we have

\[ A(α, u)A(β, uα) = \langle L_u^{-1}L_{uα}, I, L_u^{-1}L_{uα} \rangle \langle L_{uα\cdot u^{-1}}^{-1}L_{uα}, I, L_u^{-1}L_{uα} \rangle \]

(3.12) \[ A(α, u)A(β, uα) = \langle L_u^{-1}L_{uα}, I, L_u^{-1}L_{uα} \rangle \]
is an autotopism of \((L, \cdot), \forall u \in L\). Therefore \(\forall y \in L\) we have

\[
1L_u^{-1}L_{ua\beta} \cdot y = (1 \cdot y)L_u^{-1}L_{ua\beta}
\]

\[
(u\alpha\beta \cdot u^{-1}) \cdot y = yL_u^{-1}L_{ua\beta}
\]

\[
yL_{ua\beta \cdot u^{-1}} = yL_u^{-1}L_{ua\beta}
\]

\[
\Rightarrow L_{ua\beta \cdot u^{-1}} = L_u^{-1}L_{ua\beta}
\]

(3.13)

Thus, (3.13) into (3.12) gives

(3.14) \[A(\alpha, u)A(\beta, u\alpha) = \langle L_{ua\beta \cdot u^{-1}}, I, L_{ua\beta \cdot u^{-1}} \rangle\]

From equation (3.14), \(u\alpha\beta \cdot u^{-1} \in N_\lambda(L, \cdot), \forall u \in L\), hence \(u\alpha\beta \cdot u^{-1} \in N\), for all \(u \in L\) and so \(\alpha\beta \in S(L)\), since \((L, \cdot)\) is an inverse property loop.

If \(\alpha \in S(L)\), then \(A(\alpha, u)\) is an autotopism of \((L, \cdot)\), \(\forall u \in L\), so also is \(A(\alpha, u\alpha^{-1})^{-1} \forall \ u \in L\), i.e

\[
A(\alpha, u\alpha^{-1})^{-1} = \langle L_{u\alpha^{-1} \cdot u^{-1}}, I, L_{u\alpha^{-1} \cdot u^{-1}} \rangle^{-1}
\]

\[
= \langle L_{u\alpha^{-1} \cdot u^{-1}}, I, L_{u\alpha^{-1} \cdot u^{-1}} \rangle^{-1}
\]

\[
= \langle L_{u\alpha^{-1} \cdot u^{-1}}, I, L_{u\alpha^{-1} \cdot u^{-1}} \rangle^{-1}
\]

\[
= \langle L(u\alpha^{-1} \cdot u^{-1}), I, L(u\alpha^{-1} \cdot u^{-1}) \rangle
\]

Hence it follows that \(\alpha^{-1} \in S(L)\). Thus \(S(L)\) is a subgroup of the automorphism group of \((L, \cdot)\).

Let \(\alpha \in S(L)\), then \(u\alpha \cdot \alpha^{-1} \in N_\lambda(L, \cdot), \forall u \in L\) and

\[
(u\alpha \cdot \alpha^{-1})xy = (u\alpha \cdot \alpha^{-1})x'y
\]

\(\forall u, x, y \in L\), if \(\gamma\) is an automorphism of \((L, \cdot)\), then we have

\[
\{u\alpha \gamma \cdot (u\gamma)^{-1}\}(x\gamma \cdot y\gamma) = \{u\alpha \gamma \cdot (u\gamma)^{-1}\}x\gamma \cdot y\gamma
\]

\(\forall u, x, y \in L\), and if we replace \(u\) by \(u\gamma^{-1}\) in the last expression, we have

\[
(u\gamma^{-1} \alpha \gamma \cdot u^{-1})(x\gamma \cdot y\gamma) = (u\gamma^{-1} \alpha \gamma \cdot u^{-1})x\gamma \cdot y\gamma
\]

Thus, \(u\gamma^{-1} \alpha \gamma \cdot u^{-1} \in N_\lambda(L, \cdot)\) and since \(L\) is an inverse property loop, the three nuclei coincide, then \(u\gamma^{-1} \alpha \gamma \cdot u^{-1} \in N(L, \cdot)\) for all \(u \in L\) and all automorphism \(\gamma\) of \((L, \cdot)\). Hence \(\gamma^{-1} \alpha \gamma \in S(L)\) for all \(\alpha \in S(L)\) and all automorphism \(\gamma\) of \((L, \cdot)\). So \(S(L)\) is indeed normal in the automorphism group of \(A(L)\) of \((L, \cdot)\).
4. Bryant-Schneider group

**Theorem 4.1.** Let $\langle L, \cdot \rangle$ be a C-loop, an element $\theta$ of the Bryant-Schneider group of $L$ is an automorphism of $L$ provided

$$\langle \theta R_{y^{-1}}, \theta L_{f^{-1}}, \theta \rangle$$

is an autotopism of $\langle L, \cdot \rangle$ if $f$ and $g$ are elements of the nucleus of $\langle L, \cdot \rangle$.

**Proof:** Let $\langle L, \cdot \rangle$ is a C-loop then

$$\langle R_{y^{-1}}, R_{y^{-1}}, L_y L_y, I \rangle$$

is an autotopism for all $x \in L$. $\theta \in BS(L, \cdot)$ imply that $\langle \theta R_{y^{-1}}, \theta L_{f^{-1}}, \theta \rangle$ is also an autotopism for some $g, f \in \langle L, \cdot \rangle$.

Hence $\langle \theta R_{y^{-1}}, \theta L_{f^{-1}}, \theta \rangle = \langle R_{y^{-1}}, R_{y^{-1}}, L_y L_y, I \rangle$.

$\langle \theta R_{y^{-1}}, R_{y^{-1}}, L_y L_y, \theta \rangle$ is an autotopism of for all $y \in L$ and some $g, f \in L$. Since $\langle L, \cdot \rangle$ is an alternative property loop, then

$$R_{y^{-1}} R_{y^{-1}} = R_{(y^{-1})^2} = R_{(y^2)^{-1}}$$

and $L_y L_y = L_{y^2}$ therefore $\langle \theta R_{y^{-1}} R_{y^{-1}}, \theta L_{f^{-1}} L_y L_y, \theta \rangle = \langle \theta R_{y^{-1}} R_{y^{-1}}, \theta L_{f^{-1}} L_y L_y, \theta \rangle$. If $g = (y^2)^{-1}$ and $f = y^2$ we obtain $\langle \theta, \theta, \theta \rangle$. Hence $\theta$ is an automorphism of $\langle L, \cdot \rangle$. $g = (y^2)^{-1}$ and $f = y^2$ implies that $f = g^{-1} = y^2$. Then it follows that $f$ and $g$ are elements of $N(L, \cdot)$ the nucleus of $\langle L, \cdot \rangle$ since the square of every element $y \in L$ belongs to $N(L, \cdot)$.

**Theorem 4.2.** Let $\langle L, \cdot \rangle$ be a C-loop and let $\theta \in S(L, \cdot)$ (the symmetric group of $L$). Then $\theta \in BS(L, \cdot)$ if there is a unique $\alpha \in P(L, \cdot)$ (the set pseudo-automorphisms of $\langle L, \cdot \rangle$) and a unique $f \in N(L, \cdot)$ such that $\theta = \alpha R_f (\alpha = \theta R_f^{-1})$.

**Proof:**

Let $\langle L, \cdot \rangle$ be a C-loop then

$$A = \langle R_{x^{-1}} R_{x^{-1}}, L_x L_x, I \rangle$$

an autotopism of $\langle L, \cdot \rangle$ for all $x \in L$.

$B = \langle I, R_{x^2}, R_{x^2} \rangle = \langle R_{x^2}, R_{x^2}, \rho R_{x^2} \rho, I \rangle$ is also an autotopism for all $x \in L$.

Therefore by Bruck[4]

$$BA = \langle R_{x^2}, \rho R_{x^2} \rho, I \rangle < \langle R_{x^{-1}} R_{x^{-1}}, L_x L_x, I \rangle = \langle I, \rho R_{x^2} \rho L_x L_x, I \rangle$$
is an autotopism for all \( x \in L \). \( \theta \in BS(L,.) \) implies that 
\[
C = < \theta R_{f^{-1}}, \theta L_{g^{-1}}, \theta >
\]
is an autotopism for some \( f, g \in L \)
\[
CBA = \theta R_{f^{-1}}, \theta L_{g^{-1}}, \theta > < I, \rho R_{x^2} \rho L_x L_x, I > = \< \theta R_{f^{-1}}, \theta L_{g^{-1}} \rho R_{x^2} \rho L_x L_x, \theta >
\]
which implies that \( < \alpha, \theta L_{g^{-1}} \rho R_{x^2} \rho L_x L_x, \alpha R_f > \) is autotopism of for some \( f, g \in Q \) and all \( x \in L \). Now if
\[
< \alpha, \theta L_{g^{-1}} \rho R_{x^2} \rho L_x L_x, \alpha R_f >
\]
is an autotopism we have \( s \alpha t \beta = (s, t) \alpha R_f \) for all \( s, t \in L \) where \( \beta = \theta L_{g^{-1}} \rho R_{x^2} \rho L_x L_x \). If \( s \) is set to be \( e \) in the last autotopism and noting that \( e \alpha = e \theta e \theta = e \) we get \( \beta = \alpha R_f \) therefore \( < \alpha, \alpha R_f, \alpha R_f > \) is an autotopism of \( \alpha R_f \) for some \( f \in L \) hence \( \alpha \) is a pseudo-automorphism with companion \( f \). \( \theta = \alpha R_f \) implies that the elements of the Bryant-Schneider group of a C-loop \((L,.)\) can be expresses in terms of pseudo-automorphisms \( P(L,.) \) and right translations of elements of the nucleus of \((L,.)\). To show uniqueness, let \( \alpha_1 R_{x_1} = \alpha_2 R_{x_2} \) where \( \alpha_1, \alpha_2 \in P(L,.) \) and \( x_1, x_2 \in N(L,.) \). Then \( \alpha_1^{-1} \alpha_2 = R_{x_2} R_{x_1}^{-1} \) which implies that \( e \alpha_2^{-1} \alpha_1 = e R_{x_2} R_{x_1}^{-1} \). Then we observe that \( e = x_2 x_1^{-1} \) and therefore \( x_1 = x_2 \). It the follows that \( \alpha_1 = \alpha_2 \).

**Remark 4.1.** Robinson[21] considered the Bryant-Schneider group of a Bol loop and found out that they can be expressed as a product of pseudo-automorphisms and right translations. Theorem 2.2 above shows that the Bryant-Schneider group of a C-loop can also be expressed in the same way. This further emphasis the fact that C-loops are analogous to Moufang loops since Moufang loops satisfies the Bol identities(right and left).

**Theorem 4.3.** Let \((L,.)\) be a C-loop . If \( x, y \in Q \), let \( \circ \) be a binary operation defined on the pseudo-automorphism \( PS(L,.) \) by
\[
\alpha \circ \beta = \alpha R_x \beta R_y R_{(x \beta, y)}^{-1}
\]
for all \( \alpha \beta \in PS(L,.) \). Let \( H = PS(L,.) \times Q \) and for
\[
(\alpha, x) \circ (\beta, y) = (\alpha \circ \beta, x \beta, y),
\]
Then \((H, \circ)\) a group which is isomorphic to \( BS(L,.) \).
Proof:
Let $\alpha, \beta \in PS(L,\cdot)$ and let $x, y \in N(L,\cdot)$ the nucleus of $(L,\cdot)$. Then we know from the immediate preceding theorem that there exist unique $\delta \in PS(L,\cdot)$ and unique $z \in N(L,\cdot)$ such that $\alpha R_x \beta R_y = \delta R_z$. Thus we observe that

$$(u\alpha.x)\beta y = u\delta.z$$

for all $u \in L$. If we set $u = e$ we obtain $x\beta.y = z$. Therefore $\alpha R_x \beta R_y = \delta R_{(x\beta.y)^{-1}}$ and so

$$\delta = \alpha R_x \beta R_y R_{(x\beta.y)^{-1}} = \alpha \circ \beta$$

Hence $\circ$ is a closed binary operation of $PS(L,\cdot)$. It is also obvious now that $(\alpha, x) \mapsto \alpha R_x$ provided $x \in N(L,\cdot)$ gives an isomorphism of $(H, \circ)$ onto the $BS(L,\cdot)$ of a C-loop. Hence the Bryant-Schneider group of a C-loop is a form generalized holomorph of the loop.

**Theorem 4.4.** A finite C-loop is isomorphic to all its loop isotopes if

$$[(L,\cdot) : N(L,\cdot)]^2 = [PS(L,\cdot) : A(L)]$$

where $A(L)$ is the automorphism group of $(L,\cdot)$

**Proof:**
By Theorem 4.2 it is clear that $|BS(L,\cdot)| = |L| |PS(L,\cdot)|$. By Bryant & Schneider[2] $(L,\cdot)$ is isomorphic to all its loop isotopes if

$$|L|^2 |A(L,\cdot)| = |BS(L,\cdot)||N_\mu(L,\cdot)|$$

But in a C-loop the nuclei coincide hence $|N_\mu(L,\cdot)| = |N(L,\cdot)|$. Now by Theorem 4.2 $|BS(L,\cdot)| = |PS(L,\cdot)||N(L,\cdot)|$ and therefore we have

$$|L|^2 |A(L,\cdot)| = |PS(L,\cdot)||N(L,\cdot)|^2$$

which implies that

$$\left[ \frac{|L|}{|N(L,\cdot)|} \right]^2 = \frac{|PS(L,\cdot)|}{|A(L,\cdot)|}$$

which is the same as

$$[L : N(L,\cdot)]^2 = [PS(L,\cdot) : A(L,\cdot)]$$

as required.

**Corollary 4.1.** Let $(L,\cdot)$ be a C-loop then

$$[PS(L,\cdot) : A(L,\cdot)] \neq 4$$

**Proof:**
The proof follows directly from Lemma 2.9 of [20] and Theorem 4.4
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