Improving some sequences convergent to Euler-Mascheroni constant

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Abstract

We obtain the following very fast sequences convergent to Euler-Mascheroni constant:

$$\theta_n = H_n - \log \left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5} \right)$$

and

$$\phi_n = H_n - \log \left(n + \frac{1}{2} + \frac{1}{24(n + \frac{1}{2})} - \frac{37}{5760(n + \frac{1}{2})^3} + \frac{10313}{2903040(n + \frac{1}{2})^5} \right),$$

where H_n are the harmonic numbers defined by $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$.

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1. Introduction

Euler's constant (or Euler Mascheroni constant) γ was introduced by Leonhard Euler (1707-1783) in 1734 as the limit of the sequence

$$(1.1) \quad D_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n.$$

This constant is known to be the third most important mathematical constant, next to π and e . It appears in a lot of places in mathematics such as number theory, analysis, theory of probability, special functions, and differential equations. The convergence of D_n to γ is very slowly since

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \quad (\text{R.M. Young [12]}),$$

which shows that it converges to γ as n^{-1} . A faster convergent sequence to γ were introduced by DeTemle in [10, 11]. He proved that the sequence R_n defined by

$$(1.2) \quad R_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log \left(n + \frac{1}{2} \right).$$

converges to γ with the speed like n^{-2} , since

$$(1.3) \quad \frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}.$$

Recently Chen [3] obtained sharp form of (1.3) as follows: For all $n \in \mathbb{N}$

$$\frac{1}{24(n+a)^2} < R_n - \gamma < \frac{1}{24(n+b)^2}$$

with the best possible constants

$$a = \frac{1}{\sqrt{24(1 - \gamma - \log(3/2))}} = 0.55106\dots, \text{ and } b = \frac{1}{2}.$$

In 1997, Negoi [9] introduced the sequence

$$T_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log \left(n + \frac{1}{2} + \frac{1}{24n} \right)$$

and showed that

$$\frac{1}{48(n+1)^3} < T_n - \gamma < \frac{1}{48n^3},$$

which shows that the approximation $T_n \approx \gamma$ has a significant superiority over the approximation $R_n \approx \gamma$. For other faster convergences of Euler-Mascheroni constant we refer to [1, 3, 5, 6, 7, 8]. To accelerate the sequence (T_n) , Chen and Mortici [2] established the following approximation formula

$$(1.4) \quad \gamma \approx H_n - \log \left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \dots \right).$$

Our first aim here is to improve (1.4) and obtain bounds for H_n in this form. Our second aim is to establish similar formulas to improve (R_n) . For this purpose we shall consider the following sequences:

$$(1.5) \quad A_n = H_n - \log \left(n + \frac{1}{2} + \frac{1}{24n} + \frac{\alpha}{n^2} + \frac{\beta}{n^3} + \frac{\delta}{n^4} + \frac{\epsilon}{n^5} \right),$$

and

$$B_n = H_n - \log \left(n + \frac{1}{2} + \frac{a}{n + \frac{1}{2}} + \frac{b}{(n + \frac{1}{2})^2} + \frac{c}{(n + \frac{1}{2})^3} + \frac{c}{(n + \frac{1}{2})^3} + \frac{d}{(n + \frac{1}{2})^4} + \frac{e}{(n + \frac{1}{2})^5} \right),$$

where $\alpha, \beta, \delta, \epsilon$ and a, b, c, d, e are real parameters. Precisely, we introduce the sequences (θ_n) and (ϕ_n) by

$$\theta_n = H_n - \log \left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5} \right),$$

and

$$\phi_n = H_n - \log \left(n + \frac{1}{2} + \frac{1}{24(n + \frac{1}{2})} - \frac{37}{5760(n + \frac{1}{2})^3} + \frac{10313}{2903040(n + \frac{1}{2})^5} \right),$$

both of which converge to γ like n^{-7} , since

$$\lim_{n \rightarrow \infty} n^7(\theta_n - \gamma) = -\frac{2501}{1161216}$$

and

$$\lim_{n \rightarrow \infty} n^7\phi_n = -\frac{5509121}{1393459200}.$$

In order to prove our main results we need the following lemma, which is a strong tool to measure and improve the speed of convergence of some sequences having limit equal to zero.

Lemma 1.1. *If (s_n) is convergent to zero and there exists the limit*

$$(1.6) \quad \lim_{n \rightarrow \infty} n^k (s_n - s_{n+1}) = c \in \mathbf{R}$$

then there exists the limit

$$\lim_{n \rightarrow \infty} n^{k-1} s_n = \frac{c}{k-1},$$

see [4]. From this lemma it is clear that the speed of convergence of the sequence (s_n) is as higher as the value of k satisfying (1.6) is as greater as.

2. Main results

Let A_n be the sequence defined by (1.5). We are interested to find the values of α, β, δ and ϵ which provide the fastest sequence A_n . First, let us write

$$\begin{aligned} A_n - A_{n+1} &= -\frac{1}{n+1} - \log \left(n + \frac{1}{2} + \frac{1}{24n} + \frac{\alpha}{n^2} + \frac{\beta}{n^3} \frac{\delta}{n^4} + \frac{\epsilon}{n^5} \right) \\ &+ \log \left(n + 1 + \frac{1}{2} + \frac{1}{24(n+1)} + \frac{\alpha}{(n+1)^2} + \frac{\delta}{(n+1)^4} + \frac{\epsilon}{(n+1)^5} \right). \end{aligned}$$

We are concentrated to compute a limit of the form (1.6). In order to do this we use a computer software to obtain the following power series representation in $\frac{1}{n}$:

$$\begin{aligned} A_n - A_{n+1} &= \left(-\frac{1}{16} - 3\alpha \right) \frac{1}{n^4} + \left(\frac{263}{1440} + 8\alpha - 4\beta \right) \frac{1}{n^5} \\ &+ \left(-\frac{139}{384} - \frac{385\alpha}{24} + \frac{25\beta}{2} - 5\delta \right) \frac{1}{n^6} + \left(\frac{3685}{6048} + \frac{229\alpha}{8} + 3\alpha^2 - \frac{115\beta}{4} + 18\delta - 6\epsilon \right) \frac{1}{n^7} \\ &+ \left(-\frac{8663}{9216} - \frac{27517\alpha}{576} - 14\alpha^2 + \frac{1379\beta}{24} + 7\alpha\beta - \frac{1127\delta}{24} + \frac{49\epsilon}{2} \right) \frac{1}{n^8} + O(n^{-9}). \end{aligned}$$

By Lemma 1.1 faster convergences are obtained by imposing the conditions that the first four coefficients vanish. Now this results in

$$\left\{ \begin{array}{l} -\frac{1}{16} - 3\alpha = 0, \\ \frac{263}{1440} + 8\alpha - 4\beta = 0, \\ \frac{139}{384} - \frac{385\alpha}{24} + \frac{25\beta}{2} - 5\delta = 0, \\ \frac{3685}{6048} + \frac{229\alpha}{8} + 3\alpha^2 - \frac{115\beta}{4} + 18\delta - 6\epsilon = 0, \\ -\frac{8663}{9216} - \frac{27517\alpha}{576} - 14\alpha^2 + \frac{1379\beta}{24} + 7\alpha\beta - \frac{1127\delta}{24} + \frac{49\epsilon}{2} = 0, \end{array} \right.$$

namely,

$$(2.1) \quad \alpha = -\frac{1}{48}, \quad \beta = \frac{23}{5760}, \quad \delta = \frac{17}{3840}, \quad \text{and} \quad \epsilon = -\frac{10099}{2903040}.$$

These solutions correspond to the following sequence

$$(2.2) \quad \theta_n = H_n - \log \left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5} \right).$$

By replacing the solutions (2.1) above

$$\theta_n - \theta_{n+1} = -\frac{2501}{165888n^8} + O(n^{-9}).$$

Now we can state the following

Theorem 2.1. *Let (θ_n) be the sequence defined by (2.2). Then*

$$\lim_{n \rightarrow \infty} n^8(\theta_n - \theta_{n+1}) = -\frac{2501}{165888} \quad \text{and} \quad \lim_{n \rightarrow \infty} n^7(\theta_n - \gamma) = -\frac{2501}{1161206},$$

namely, the speed of convergences of the sequence (θ_n) is like n^{-7}

Let (B_n) be as defined by (1). Then we have

$$(2.3) \quad \begin{aligned} B_n - B_{n+1} = & -\frac{1}{n+1} - \log \left(n + \frac{1}{2} + \frac{a}{(n+\frac{1}{2})} + \frac{b}{(n+\frac{1}{2})^2} + \frac{c}{(n+\frac{1}{2})^3} + \frac{d}{(n+\frac{1}{2})^4} \right. \\ & \left. + \frac{e}{(n+\frac{1}{2})^5} \right) + \log \left((n+1) + \frac{1}{2} + \frac{a}{(n+\frac{3}{2})} + \frac{b}{(n+\frac{3}{2})^2} + \frac{c}{(n+\frac{3}{2})^3} + \frac{d}{(n+\frac{3}{2})^4} + \frac{e}{(n+\frac{3}{2})^5} \right). \end{aligned}$$

Using again a computer software we get

$$\begin{aligned}
B_n - B_{n+1} = & \left(\frac{1}{12} - 2a \right) \frac{1}{n^3} + \left(-\frac{1}{4} + 6a - 3b \right) \frac{1}{n^4} \\
& + \left(\frac{41}{80} - 13a + 2a^2 + 12b - 4c \right) \frac{1}{n^5} \\
& + \left(-\frac{43}{48} + 25a - 10a^2 - \frac{65b}{2} + 5ab + 20c - 5d \right) \frac{1}{n^6} \\
& + \left(\frac{645}{448} - \frac{363a}{8} + \frac{65a^2}{2} - 2a^3 + 75b - 30ab + 3b^2 - 65c + 6ac + 30d - 6e \right) \frac{1}{n^7} \\
& \left(-\frac{141}{64} + \frac{637a}{8} - \frac{175a^2}{2} + 14a^3 - \frac{2541b}{16} + \frac{455ab}{4} - 7a^2b - 21b^2 + 175c \right. \\
& \left. - 42ac + 7bc - \frac{455d}{4} + 7ad + 42e \right) \frac{1}{n^8} + O(n^{-9}).
\end{aligned}$$

(2.4)

According to Lemma 1.1 we can see that the fastest sequence ϕ_n is obtained in the case when as many of the first coefficients of (2.3) is cancelled. As we have five parameters a, b, c, d, e , they produce the best result if and only if

$$\begin{aligned}
\frac{1}{12} - 2a &= 0, \\
-\frac{1}{4} + 6a - 3b &= 0, \\
\frac{41}{80} - 13a + 2a^2 + 12b - 4c &= 0, \\
-\frac{43}{48} + 25a - 10a^2 - \frac{65b}{2} + 5ab + 20c - 5d &= 0, \\
\frac{645}{448} - \frac{363a}{8} + \frac{65a^2}{2} - 2a^3 + 75b - 30ab + 3b^2 - 65c \\
+ 6ac + 30d - 6e &= 0, \\
-\frac{141}{64} + \frac{637a}{8} - \frac{175a^2}{2} + 14a^3 - \frac{2541b}{16} + \frac{455ab}{4} - 7a^2b \\
- 21b^2 + 175c - 42ac + 7bc - \frac{455d}{4} + 7ad + 42e &= 0.
\end{aligned}$$

From these we obtain the following solutions:

$$(2.5) \quad a = \frac{1}{24}, \quad b = 0, \quad c = -\frac{37}{5760}, \quad d = 0, \quad e = \frac{10313}{2903040},$$

and these solutions correspond to the following sequence

$$(2.6) \quad \phi_n = H_n - \log \left(n + \frac{1}{2} + \frac{1}{24(n + \frac{1}{2})} - \frac{37}{5760(n + \frac{1}{2})^3} + \frac{10313}{2903040(n + \frac{1}{2})^5} \right).$$

By replacing the solutions given in (2.5) in (2.4) we get

$$\phi_n - \phi_{n+1} = -\frac{5509121}{174182400n^8} + O(n^{-9}).$$

These can be summarized as follow.

Theorem 2.2. *Let (ϕ_n) be the sequence given by (2.6). Then it holds that*

$$\lim_{n \rightarrow \infty} n^8(\phi_n - \phi_{n+1}) = -\frac{5509121}{174182400} \quad \text{and} \quad \lim_{n \rightarrow \infty} n^7\phi_n = -\frac{5509121}{1393459200},$$

that is, the speed of convergence of (ϕ_n) is like n^{-7} .

Theorem 2.3. *Let the sequences (θ_n) and (ϕ_n) be as defined (2.2) and (2.6). Then, both (θ_n) and (ϕ_n) are strictly decreasing for $n \geq 2$ and all natural numbers n , respectively.*

Proof. We set $\theta_n - \theta_{n+1} = f(n)$, where

$$f(x) = -\frac{1}{x+1} - \log \left(x + \frac{1}{2} + \frac{1}{24x} - \frac{1}{48x^2} + \frac{23}{5760x^3} + \frac{17}{3840x^4} - \frac{10099}{2903040x^5} \right) \\ + \log \left(x + \frac{3}{2} + \frac{1}{24(x+1)} - \frac{1}{48(x+1)^2} + \frac{23}{5760(x+1)^3} + \frac{17}{3840(x+1)^4} - \frac{10099}{2903040(x+1)^5} \right).$$

Differentiation gives

$$(2.7) \quad f'(x) = \frac{p(x)}{q(x)},$$

where

$$p(x) = -223661795575 - 1556403370554x - 4175585115408x^2 \\ - 4951284518880x^3 - 1613300443776x^4 + 1495234411776x^5 \\ + 1016470425600x^6$$

and

$$\begin{aligned}
q(x) = & -44732359115x - 285157435534x^2 - 655259062139x^3 \\
& -595777525560x^4 - 16441205760x^5 + 7510856117184x^6 \\
& +67212592098624x^7 + 276219358571520x^8 + 661238937354240x^9 \\
& +1014009112166400x^{10} + 1032643563356160x^{11} \\
& +698089616179200x^{12} + 301990477824000x^{13} \\
& +75848771174400x^{14} + 8427641241600x^{15},
\end{aligned}$$

By expanding $p(x)$ and $q(x)$ as a power series of $x - 2$ we get

$$\begin{aligned}
p(x) = & 27439716165461 + 185481302397702(x - 2) \\
& +290969152206768(x - 2)^2 + 204586956497952(x - 2)^3 \\
& +74327269209984(x - 2)^4 + 13692879518976(x - 2)^5 \\
& +1016470425600(x - 2)^6,
\end{aligned}$$

and

$$\begin{aligned}
q(x) = & 10423677493515991770 + 62668051134141291321(x - 2) \\
& + \dots + 328678008422400(x - 2)^{14} + 8427641241600(x - 2)^{15},
\end{aligned}$$

which is a polynomial with all positive coefficients. It follows $f'(x) > 0$ for $x \geq 2$, so that f is strictly increasing in $(2, \infty)$ with $\lim_{x \rightarrow \infty} f(x) = 0$. it results that $f(x) < 0$ for $x \geq 2$, namely θ_n is strictly increasing for $n \geq 2$. This completes the first part of Theorem 2.3. To prove the second part of the theorem we denote $\phi_n - \phi_{n+1} = g(n)$, where

$$\begin{aligned}
g(x) = & -\frac{1}{x+1} - \log \left(x + \frac{1}{2} + \frac{1}{24(x+\frac{1}{2})} - \frac{37}{5760(x+\frac{1}{2})^3} \right. \\
& \left. + \frac{10313}{2903040(x+\frac{1}{2})^5} \right) + \log \left(x + \frac{3}{2} + \frac{1}{24(x+\frac{3}{2})} - \frac{37}{5760(x+\frac{3}{2})^3} + \frac{10313}{2903040(x+\frac{3}{2})^5} \right).
\end{aligned}$$

By differentiation we get

$$(2.8) \quad g'(x) = \frac{t(x)}{s(x)},$$

where

$$\begin{aligned}
t(x) = & 9678358492223 + 57880272188784x + 144357200961720x^2 \\
& +192184418280960x^3 + 144005296337280x^4 + 57575515060224x^5 \\
& +9595919176704x^6,
\end{aligned}$$

and

$$\begin{aligned}
s(x) = & 5912418259515 + 110278703811038x + 996749749920191x^2 \\
& + \dots + 539369039462400x^{15} + 33710564966400x^{16},
\end{aligned}$$

which is a polynomial with all positive coefficients. Since both $t(x)$ and $s(x)$ are positive for $x \geq 1$, g is strictly increasing with $\lim_{x \rightarrow \infty} g(x) = 0$, consequently, the sequence (ϕ_n) is strictly increasing for $n = 1, 2, 3, \dots$. This completes the proof of Theorem 2.3. $\square$

As a direct consequence of the fact that θ is strictly increasing for $n = 2, 3, 4, \dots$ we have $\theta_2 \leq \theta_n < \lim_{n \rightarrow \infty} \theta_n = \gamma$ for all $n \geq 2$. As $\theta_2 = \frac{3}{2} - \log\left(\frac{58804553}{23224320}\right)$, we have

Corollary 2.4. *Let $n \geq 2$ be an integer. Then we have*

$$\begin{aligned} & \alpha + \log\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5}\right) \\ & \leq H_n < \beta + \log\left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} \right. \\ & \left. + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5}\right), \end{aligned}$$

where $\alpha = \frac{3}{2} - \log\left(\frac{58804553}{23224320}\right) = 0.5709807216\dots$ and $\beta = \gamma = 0.5772156\dots$ are the best possible.

Similarly from monotonic increase of the sequence (ϕ_n) with $\lim_{n \rightarrow \infty} \phi_n = \gamma$ and $\phi_1 = 1 - \log\left(\frac{6729631}{4408992}\right) = 0.57712577887\dots$, we get

$$\begin{aligned} & \alpha^* + \log\left(n + \frac{1}{2} + \frac{1}{24(n+\frac{1}{2})} - \frac{37}{5760(n+\frac{1}{2})^3} + \frac{10313}{2903040(n+\frac{1}{2})^5}\right) \leq H_n \\ & \beta^* + \log\left(n + \frac{1}{2} + \frac{1}{24(n+\frac{1}{2})} - \frac{37}{5760(n+\frac{1}{2})^3} + \frac{10313}{2903040(n+\frac{1}{2})^5}\right), \end{aligned}$$

where $\alpha^* = 1 - \log\left(\frac{6729631}{4408992}\right) = 0.57712577887\dots$ and $\beta^* = \gamma = 0.5772156\dots$ are the best possible constants.

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