The signature in actions of semisimple Lie groups on pseudo-Riemannian manifolds

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Abstract

We study the relationship between the signature of a semisimple Lie group and a pseudo-Riemannian manifold on which the group acts topologically transitively and isometrically. We also provide a description of the bi-invariant pseudo-Riemannian metrics on a semisimple Lie group over $\mathbb{R}$ in terms of the complexification of the Lie algebra associated to the group, and then we utilize it to prove a remark of Gromov.

Keywords: semisimple Lie groups, bi-invariant metric, local freeness.

1. Introduction

On a semisimple Lie group $G$ the Killing–Cartan form is invariant by automorphisms, and it defines an $Ad(g)$-invariant scalar product on $Lie(G) = g$. Then the left action of $G$ on itself joint to the Killing–Cartan form of $g$ provide a pseudo-Riemannian structure on $G$ which is bi-invariant. This permits us to study semisimple Lie groups from the point of view of geometry, i.e. we choose an appropriate pseudo-Riemannian metric and compute the various geometrical objects, such as curvature, and geodesics.

It is known that there is a bijective correspondence between the $Ad(g)$-invariant nondegenerate symmetric bilinear forms on $g$ and the bi-invariant pseudo-Riemannian metrics on $G$. Under such correspondence, a bilinear form on $g$ which is not a multiple of the Killing–Cartan form defines a pseudo-Riemannian metric on $G$ that might be expected to provides a geometry that differs from the one given by the Killing–Cartan form. The first thing we want to prove is the fact that such situation does not occur, i.e. every bi-invariant pseudo-Riemannian metric on a semisimple Lie group is a finite sum of Killing–Cartan forms.

We inquire about the relationship of the pseudo-Riemannian invariants of $G$ and $M$, respectively, for some bi-invariant pseudo-Riemannian metric on $G$. In this work, we restrict our attention to the signature, which we will denote with $(m_1, m_2)$ and $(n_1, n_2)$ for $M$ and $G$, respectively.

The second goal of this work is to obtain an estimate between the signatures of $G$ and $M$, in the case of $G = G_1 \cdots G_l$ and each $G_i$ is a connected simple Lie group and carries a bi-invariant pseudoRiemannian metric. If we denote $n_{i0} = \min\{n_{i1}, n_{i2}\}$ and $m_0 = \min\{m_1, m_2\}$, then we are going to prove that $n_{i0} + \cdots + n_{i0} \leq m_0$.

The organization of this article is as follows. In section 2 we collect some basic results about complexification of a real Lie algebra and invariant bilinear forms on a simple Lie algebra that will needed in the proof of the main theorem on that section. Also we give the classification of the $Ad(g)$-invariant bilinear forms on a semisimple Lie algebra. This is mentioned in [2], but the generalization to semisimple Lie groups is new. As a consequence we give the classification of the bi-invariant pseudo-Riemannian metrics on $G$. In section 3 we use the results obtained previously to obtain an estimated between the signatures of the metrics on $M$ and $G$, respectively.

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2. Complexification of a real Lie algebra

Let $V$ be a vector space over $R$ of finite even dimension. A complex structure on $V$ is an $R$-linear endomorphism $J$ of $V$ such that $J^2 = -I$. When there exist a complex structure $J$ on $V$, we define $V_J$ as the complex vector space associated to $V$ by the rule: $(a + ib)X = aX + bJX$ for $X \in V$ and $a, b \in R$.

A Lie algebra $\mathfrak{g}$ over $R$ is said to have a compatible complex structure $J$ if $J$ is a complex structure on the real vector space $\mathfrak{g}$ and in addition $[X, JY] = J[X, Y]$ for $X, Y \in \mathfrak{g}$. It is easy to see that $\mathfrak{g}_C$ then becomes a complex Lie algebra.

If $V$ is an arbitrary finite dimensional vector space over $R$, the $R$-linear map $J : (X, Y) \mapsto (-Y, X)$ is a complex structure on $V \times V$. The complex vector space $(V \times V)_J$ is called the complexification of $V$ and will be denoted by $V^C$. We write $X + iy$ instead of $(X, Y)$ in $V^C$.

If $\mathfrak{g}$ is a Lie algebra over $R$, owing to the conventions above, the complex space $\mathfrak{g}^C$ consists of all symbols $X + iY$ with $X, Y \in \mathfrak{g}$, and it is a complex Lie algebra whose Lie bracket is given by

$$[X + iY, Z + iT] = [X, Y] - [Y, T] + i\left([Y, Z] + [X, T]\right).$$

**Definition 1.** A real Lie algebra $\mathfrak{g}_0$ is called a real form of a complex Lie algebra $\mathfrak{g}$ if its complexification $(\mathfrak{g}_0)^C$ is isomorphic to $\mathfrak{g}$ as a complex Lie algebra.

For semisimple Lie algebras over $C$ the existence of a real form is proved in [1, Thm. III.6.3]. This real form is also compact, which means that the Killing-Cartan form of $\mathfrak{g}$ is strictly negative definite.

If $\mathfrak{g}$ is a real Lie algebra and $T: \mathfrak{g} \rightarrow \mathfrak{g}$ is a $R$-linear map, then there is a $C$-linear map $T^C : \mathfrak{g}^C \rightarrow \mathfrak{g}^C$ defined by $T^C(X + iY) = TX + iTY$.

The complexification of the adjoint representation $ad: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ is the $C$-linear map $ad^C: \mathfrak{g}^C \rightarrow \mathfrak{gl}(\mathfrak{g}^C)$ given by

$$ad^C(X + iY)(Z + iW) = [X + iY, Z + iW]$$


A trivial calculation shows that if $T$ and $ad^C(X)$ commute for every $X \in \mathfrak{g}$, then $T^C$ and $ad^C(X + iY)$ commute for every $X, Y \in \mathfrak{g}$.

For an $R$-linear map $T: \mathfrak{g} \rightarrow \mathfrak{g}$ and a given complex structure $J$ on $\mathfrak{g}$, we can consider the $C$-linear map $T_J: \mathfrak{g}_J \rightarrow \mathfrak{g}_J$ defined by $T_J(X) = T(X) - JTJ(X)$. Consider also the adjoint representation $ad: \mathfrak{g}_J \rightarrow \mathfrak{gl}(\mathfrak{g}_J)$. It is easy to show that if $T$ and $ad(X)$ commute for each $X \in \mathfrak{g}$, then $T_J$ commutes with $ad(Z)$ for each $Z \in \mathfrak{g}_J$. 
The next proposition (for a proof see [1, Thm. X.1.5]) it will be useful later.

**Proposition 1.** For a simple Lie algebra \( g \) over \( R \) there are two possibilities:

1. \( g^C \) is simple;

2. \( g^C \) is nonsimple; in this case \( g \) admits a complex structure \( J \) whose associated complex Lie algebra \( g_J \) is simple.

For the above proposition, the classification of \( \text{Ad}(G) \)-invariant bilinear forms should be separated into two cases. The first case is the following.

**Theorem 1.** Let \( g \) be a simple Lie algebra over \( R \) which has simple complexification. Then the \( \text{Ad}(G) \)-invariant bilinear forms of \( g \) are multiples of the Killing–Cartan form of \( g \).

**Proof.** Let \( D \) be an \( \text{Ad}(G) \)-invariant bilinear form on \( g \).

Since the Killing–Cartan form \( B \) of \( g \) is nondegenerate, there is a linear map \( T: g \to g \) such that \( D(X,Y) = B(X,TY) \) for each \( X,Y \in g \). Using the \( \text{Ad}(G) \)-invariance of \( D \) we obtain

\[
B(X,TY) = D(X,Y) = D(\text{Ad}(g)X, \text{Ad}(g)Y) = B(\text{Ad}(g)X, T\text{Ad}(g)Y) = B(\text{Ad}(g^{-1})T\text{Ad}(g)Y),
\]

so \( TY - \text{Ad}(g^{-1})T\text{Ad}(g)Y = 0 \). Therefore \( \text{Ad}(g)T = T\text{Ad}(g) \) for each \( g \in G \).

We claim that \( \text{ad}(x)T = T\text{ad}(x) \) for each \( x \in g \). Taking \( g = \exp(tX) \) we obtain \( \text{Ad}(\exp(tX))T = T\text{Ad}(\exp(tX)) \), and from the identity \( \text{Ad}(\exp(X)) = e^{\text{ad}(X)} \) we conclude that \( Te^{\text{ad}(X)} = e^{\text{ad}(X)}T \).

Differentiating the last equation at \( t = 0 \), we obtain \( \text{ad}(X)T = T\text{ad}(X) \).

From the first remark above we obtain that \( T^C \) and \( \text{ad}^C(X+iY) \) commute for every \( X,Y \in g \). Let \( \lambda \in C \) be an eigenvalue of \( T^C \). Since \( g^C \) is simple and \( \text{ad}^C \) is an irreducible representation, by Schur’s Lemma we conclude that \( T^C = \lambda I \), then \( T^C(X+iY) = \lambda(X+iY) \). If we write \( \lambda = \lambda_1 + i\lambda_2 \), then it is very easy to conclude that \( T = \lambda_1 I \), where \( \lambda_1 \in R \), and then \( D = \lambda_1 B \). \( \square \)
Theorem 2. Let \( \mathfrak{g} \) be a simple Lie algebra over \( R \) which has nonsimple complexification and let \( J \) be a fixed complex structure on \( \mathfrak{g} \). Let \( D \) be an \( \text{Ad}(G) \)-invariant bilinear form on \( \mathfrak{g} \), then \( D(X,Y) = aB(X,Y) + bB(X,JY) \) for some \( a, b \in R \) and for each \( X, Y \in \mathfrak{g} \).

Proof. There is a linear map \( T: \mathfrak{g} \to \mathfrak{g} \) such that \( D(X,Y) = B(X,TY) \) for each \( X, Y \in \mathfrak{g} \), where \( B \) is the Killing–Cartan form of \( \mathfrak{g} \).

We shall prove that \( T \) satisfies \( TJ + JT = \lambda_1 I + \lambda_2 J \) for some \( \lambda_1, \lambda_2 \in R \).

By the \( \text{Ad}(G) \)-invariance of \( D \) we conclude that \( T \) and \( \text{Ad}(g) \) commute for each \( g \in G \). From the second remark above we obtain that \( T \) and \( \text{Ad}(X) \) commute for each \( X \in \mathfrak{g} \).

Using the irreducibility of the adjoint representation of \( \text{ad}J \), its follow by Schur’s Lemma that there is \( \lambda = \lambda_2 - i\lambda_1 \in \mathbb{C} \) such that \( T \) and \( \text{Ad}(g) \) commute for each \( g \in G \). From this we obtain \( TJ + JT = \lambda_1 I + \lambda_2 J \), where \( \lambda_1, \lambda_2 \in R \).

Define \( \tilde{T} = T - \frac{1}{2\lambda_2} \lambda_1 I + \frac{1}{2\lambda_1} J \). Then \( \tilde{T} \) and \( \text{Ad}(X) \) commute for each \( X \in \mathfrak{g} \) and it is easy to check that \( \tilde{T}J + JT = 0 \).

We can find an orthogonal basis \( \{X_1, \ldots, X_r\} \) of the complex Lie algebra \( \mathfrak{g} \) such that \( \{X_1, \ldots, X_r, J(X_1), \ldots, J(X_r)\} \) is a basis of \( \mathfrak{g} \) over \( R \). Let \( \mathfrak{g}_k \) be the \( R \)-linear subspace given by

\[
\mathfrak{g}_k := \sum_{x \in \Delta} R(iH_\alpha) + \sum_{x \in \Delta} R(X_\alpha - X_\alpha) + \sum_{x \in \Delta} R(i(X_\alpha + X_\alpha)),
\]

where \( \Delta \) is the corresponding set of nonzero roots of \( \mathfrak{g}_J \), and for each \( \alpha \in \Delta \) we select \( X_\alpha \in \mathfrak{g}^\alpha \) with the properties of [1, Thm. III.5.5]. It follows that \( B \), the Killing–Cartan form of \( \mathfrak{g}_J \), is strictly negative definite on \( \mathfrak{g}_k \). Moreover, \( \mathfrak{g}_J = \mathfrak{g}_k \oplus J\mathfrak{g}_k \).

Normalizing the basis given by \( \mathfrak{g}_k \) we obtain a new basis of \( \mathfrak{g}_J \) given by \( \{X_1, \ldots, X_r\} \) such that \( \{X_1, \ldots, X_r, J(X_1), \ldots, J(X_r)\} \) is a basis of \( \mathfrak{g} \) over \( R \). It is clear that \( B_\mathfrak{g}(Y_i, Y_j) = \delta_{ij} \epsilon_i \), for \( Y_i, Y_j \in \mathfrak{g} \), where \( \epsilon_i = 1 \) if \( i = 1, \ldots, r \) and \( \epsilon_i = -1 \) if \( i = r + 1, \ldots, 2r \). Using [1, Lm. III.6.1]) we conclude that \( B_\mathfrak{g}(X,Y) = (B(X,Y))(B(X,Y)) \) for \( X, Y \in \mathfrak{g} \).

In this basis the matrix representation of the complex structure \( J \) and the Killing–Cartan form of \( \mathfrak{g} \) are given by

\[
J = \begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}, \quad B_\mathfrak{g} = \begin{pmatrix} I_r & 0 \\ 0 & I_r \end{pmatrix}.
\]

If the matrix representation of \( T \) in that basis has the form
\[
T = \begin{pmatrix}
A_0 & B_0 \\
C_0 & D_0
\end{pmatrix},
\]
then we deduce from the identity \( TJ + JT = \lambda_1 I + \lambda_2 J \) that
\[
T = \begin{pmatrix}
A_0 & C_0 + \lambda_1 I \\
C_0 & -A_0 + \lambda_2 I
\end{pmatrix}.
\]

If we rewrite \( T \) in the following form
\[
2T = \begin{pmatrix}
2A_0 - \lambda_2 I_r & 2C_0 + \lambda_1 I_r \\
2C_0 + \lambda_1 I_r & -2A_0 + \lambda_2 I_2
\end{pmatrix} + \lambda_1 \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} I_r & 0 \\ 0 & I_r \end{pmatrix},
\]
and we denote \( 2E_0 = 2A_0 - \lambda_2 I_r \) and \( 2L_0 = 2C_0 + \lambda_1 I_r \), then we have
\[
2T = \begin{pmatrix}
2E_0 & 2L_0 \\
2L_0 & -2E_0
\end{pmatrix} - \lambda_1 J + \lambda_2 I.
\]

Define \( \hat{T} = T - \frac{1}{2} \lambda_2 I + \frac{1}{2} \lambda_1 J \). Then \( \hat{T} \) and \( ad(X) \) commute for each \( X \in \mathfrak{g} \) and it is easy to check that \( \hat{T}J + J\hat{T} = 0 \).

We prove now that \( \hat{T} = 0 \), and from this the theorem follows.

Let \( u \) be a real form of \( \mathfrak{g} \) (for a proof see [1, Thm. III.6.3]). Then for the map \( \hat{T}: \mathfrak{g} \to \mathfrak{g} \) there are \( R \)-linear maps \( E_0, L_0: u \to u \) such that \( \hat{T}(X) = E_0(X) + JL_0(X) \) for each \( X \in u \).

Let \( X, Y \) be in \( u \), then \( [X, Y] \in u \) and \( \hat{T}([X, Y]) = E_0[X, Y] + JL_0[X, Y] \). On the other hand, \( \hat{T}[X, Y] = [E_0X, Y] + J[L_0X, Y] \), because \( \hat{T} \) and \( ad(X) \) commute for each \( X \in \mathfrak{g} \). It follows that
\[
\hat{T}([X, Y]) - [\hat{T}X, Y] = E_0[X, Y] - [E_0X, Y] + JL_0[X, Y] - J([L_0X, Y]).
\]

The relationship \( \hat{T} \circ ad(X) = ad(X) \circ \hat{T} \), for all \( X \in \mathfrak{g} \), shows that \( \hat{T}([X, Y]) = [TX, Y] \), therefore we conclude that
\[
(2.1) \quad E_0[X, Y] = [E_0X, Y], \quad L_0[X, Y] = [L_0X, Y].
\]

It follows from the above relationship that
\[
\hat{T}([JX, JY]) = -\hat{T}([JY, JX])
= -\hat{T}ad(JY)(JX)
= -ad(JY)\hat{T}(JX)
= [\hat{T}(JX), JY]
\]
From this and the direct sum, \( g = u \oplus Ju \), we can conclude that

\[
E_0[X,Y] = -[E_0X,Y], \quad L_0[X,Y] = -[L_0X,Y].
\]

It is easy to conclude from (2.1) and (2.2) that \( E_0 = 0 = L_0 \) on \([u,u]\). Since \( u \) is a semisimple Lie algebra we have \([u,u] = u \). Hence \( T = \frac{1}{2}\lambda_2 I + \frac{1}{2}\lambda_1 J \), and

\[
D(X,Y) = B(X,TY) = aB(X,Y) + bB(X,JY).
\]

\[\square\]

The classification of the \( Ad(G) \)-invariant bilinear forms on a simple Lie algebra was done in the previous theorems. The next result gives the classification of the \( Ad(G) \)-invariant bilinear forms on a semisimple Lie algebra, but before that we will need the following easily proved result.

**Lemma 1.** Let \( F: g \times g \to R \) be a symmetric bilinear form that is \( Ad(G) \)-invariant, then \( F([X,W], Y) = F(X,[W,Y]) \) for \( W, X, Y \in g \). The converse holds if \( G \) is connected.

**Proof.** It is sufficient to show that \( F([W,X], X) = 0 \) for all \( W, X \in g \).

We consider the following function

\[
f(s) = F(Ad(\alpha(s))X, Ad(\alpha(s))X),
\]

We affirm that \( f \) is constant on each one-parameter subgroup \( \alpha \) of \( G \).

From this the direct assertion follows. \( \square \)

**Theorem 3.** Let \( g = g_1 \oplus g_2 \oplus \cdots \oplus g_l \) be a Lie algebra, where each \( g_i \) is a simple ideal of \( g \). We shall suppose the following:

- The complexification of each \( g_i \), for \( i = 1, \ldots, k \) is simple; and
- The complexification of each \( g_i \), for \( i = k+1, \ldots, l \) is not simple and so there exists a complex structure \( J_i \) for each \( g_i \).

Then every \( Ad(G) \)-invariant bilinear form \( D \) on \( g \) is given by the following

\[
D = \lambda_1 B_{g_1} \oplus \cdots \oplus \lambda_k B_{g_k} \oplus (\mu_{k+1} B_{g_{k+1}} \oplus \cdots \oplus (\mu_{l} B_{g_l} + \mu_{l+1} B_{J_{g_{l+1}}})), \text{ where each } B_{g_i} \text{ is the Killing–Cartan form on } g_i, \text{ for } i = 1, \ldots, l; \text{ all } \lambda_i \text{ and } \mu_i \text{ are real numbers, and } B_{g_i}^J(X,Y) = B_{g_i}(X,J_iY).
\]
Proof. It follows from Lemma 1 that
\[ D(X, [Y, Z]) = D([X, Y], Z), \quad \text{for all } X, Y, Z \in \mathfrak{g}. \]

On the other hand, it is easy to show the following properties:
1. \([\mathfrak{g}_i, \mathfrak{g}_j] = \{0\}\) for all \(i \neq j\); and
2. \([\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i\), for all \(i\).

Using the above information we have for \(Y \in \mathfrak{g}_j\) that there exist \(Z, W \in \mathfrak{g}_j\) such that \(Y = [Z, W]\), and for \(X \in \mathfrak{g}_i\) we conclude that
\[ D(X, Y) = D([X, Z], W) = 0. \]

Therefore \(\mathfrak{g}_i \perp \mathfrak{g}_j\) for all \(i \neq j\) with respect to \(D\). From this it follows that
\[ D = B\Big|_{\mathfrak{g}_i} \oplus \cdots \oplus B\Big|_{\mathfrak{g}_i}. \]

Now we use the classification of \(\text{Ad}(G)\)-invariant bilinear forms on a simple Lie algebra given in theorems 1 and 2. From this the result follows. □

From [3, Ch.11]) we obtain a relation between the geometry of \(G\) and its Lie algebra \(\mathfrak{g}\). Also we can deduce that the geometry of all bi-invariant metrics on \(G\) share most of the pseudo-Riemannian invariant.

We are going to give a classification of the bi-invariant pseudo-Riemannian metrics on a semisimple Lie group based on the classification of the bilinear \(\text{Ad}(G)\)-invariant forms on a semisimple Lie algebra.

**Theorem 4.** Let \(G\) be a semisimple Lie group such that \(\text{Lie}(G) = \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l\), where each \(\mathfrak{g}_i\) is a simple ideal of the Lie algebra \(\mathfrak{g}\). We shall suppose the following:

- The complexification of each \(\mathfrak{g}_i\), for \(i = 1, \ldots, k\) is simple; and
- The complexification of each \(\mathfrak{g}_i\), for \(i = k + 1, \ldots, l\) is not simple and so there exists a complex structure \(J\) for each \(\mathfrak{g}_i\).

Then every bi-invariant pseudo-Riemannian metric \(\phi\) on \(\mathfrak{g}\) is given by \(\phi = \lambda_1 B_{\mathfrak{g}_1} \oplus \cdots \oplus \lambda_k B_{\mathfrak{g}_k} \oplus (\mu_{1,1} B_{\mathfrak{g}_{k+1}} + \mu_{2,1} B_{\mathfrak{g}_{k+1}}) \oplus \cdots \oplus (\mu_{1,l} B_{\mathfrak{g}_l} + \mu_{2,l} B_{\mathfrak{g}_l})\), where each \(B_{\mathfrak{g}_i}\) is the Killing–Cartan form on \(\mathfrak{g}_i\), for \(i = 1, \ldots, l\), all \(\lambda_i\) and \(\mu_i\) are real numbers, and
\[ B_{\mathfrak{g}_i}(X, Y) = B_{\mathfrak{g}_i}(X, J_i Y). \]
3. Group action

From now on $G = G_1 \cdots G_l$ will be a connected noncompact semisimple Lie group with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l$. We know that $G$ admits bi-invariant pseudoRiemannian metrics and all of them can be described in terms of the Killing-Cartan form on each $\mathfrak{g}_i$.

Let $M$ be a connected compact smooth manifold. We always assume that the action of $G$ on $M$ is smooth, faithful, and preserve a finite measure on $M$.

Definition 2. The dimension of maximal lightlike tangent subspaces for $M$ will be denoted with $m_0 = \min\{m_1, m_2\}$, where $(m_1, m_2)$ represent the signature of $M$, i.e., that $m_1$ correspond to the dimension of maximal timelike tangent subspaces and $m_2$ correspond to the dimension of the maximal spacelike tangent subspaces.

Definition 3. The dimension of maximal lightlike tangent subspaces for $G_i, i = 1, \ldots, l$, will be denoted with $n_0^i = \min\{n_1^i, n_2^i\}$, where $(n_1^i, n_2^i)$ represent the signature of $G_i$.

Gromov remarked in [2] that if $(n_1, n_2)$ is the signature of the metric given by the Killing–Cartan form on $\mathfrak{g}$, then any other bi-invariant pseudoRiemannian metric on $G$ has signature given by either $(n_1, n_2)$ or $(n_2, n_1)$.

We are interested in comparing the numbers $m_0$ and $n_0^1 + \cdots + n_0^l$. To better understand this result we are going to prove the following very useful lemma.

Lemma 2. Let $(V, g)$ be a scalar product space, i.e, $V$ is a finite dimensional vector space and $g$ a nondegenerate symmetric bilinear form. Suppose that $V = V_1 \oplus \cdots \oplus V_l$, where each $V_i$ is a subspace of $V$ and $g = g_1 \oplus \cdots \oplus g_l$, where each $g_i$ is a scalar product in $V_i$, for $i = 1, \ldots, l$. Let $n_0$ be the dimension of the maximal subspace of null vectors with respect to $g$ in $V$, and $n_0^i$, for $i = 1, \ldots, l$, is defined in a similar way for each $V_i$. Then the following inequality holds: $n_0 \geq n_0^1 + \cdots + n_0^l$.

Proof. The idea for this is to realize that for each $i$ we have $n_0^i = \min\{n_{-}^i, n_{+}^i\}$, where $n_{-}^i$ is the number of $-1$ and $n_{+}^i$ the number of $+1$ when $g$ is diagonalized. Without loss of generality we can suppose that for $i = 1, \ldots, k$ we have that $n_0^i = n_{-}^i$, and for $j = k + 1, \ldots, l$ also $n_0^j = n_{+}^j$. 
It follows that $n_{-} = n_{-}^{1} + \cdots + n_{-}^{l}$
$\geq n_{0}^{1} + \cdots + n_{0}^{l}$, and in a similar way we have $n_{+} = n_{+}^{1} + \cdots + n_{+}^{l}$
$\geq n_{0}^{0} + \cdots + n_{0}^{l}$.

From this it follows that $n_{0} = \min\{n_{-}, n_{+}\} \geq n_{0}^{1} + \cdots + n_{0}^{l}$.

We will denote with $TO$ the tangent bundle to the orbits of the $G$-action on $M$. If $X \in g$, we define the infinitesimal generator $X^{*}$ as the vector field on $M$ induced by $X$. This new vector field is given by

$$X^{*}_{p} = \frac{d}{dt}|t=0} \exp (tX) \cdot p.$$  

It is clear that $X^{*}$ is a Killing vector field, and $X^{*}_{p} \in T_{p}(G \cdot p)$, for $p \in M$.

We will use the following two maps: $\varphi : M \times g \to TO$, given by $\varphi(p,X) = X^{*}_{p}$, and $\psi : M \to g^{*} \otimes g^{*}$, given by $\psi(p) = B_{p}$, where $B_{p}(X,Y) = h_{p}(X^{*}_{p}, Y^{*}_{p})$, and $h$ is the metric on $M$.

We can conclude from the next lemma that $\psi(p)$ is an $Ad(G)$-invariant bilinear form on $g$, for every $p \in M$.

**Lemma 3.** For every $p \in M$, $\psi(p)$ is $Ad(G)$-invariant.

**Proof.** This is a consequence of lemma 1 if we prove the following:

$$\psi(p)(ad(W)X, X) = 0,$$  

for all $X, W \in g$.

$$\psi(p)(ad(W)X, X) = h_{p}((ad(W)X)_{p}^{*}, X^{*}_{p})$$  

$$= h_{p}(-ad(W^{*})X^{*}_{p}, X^{*}_{p})$$  

$$= -h_{p}(X^{*}_{p}, (ad(W)X)_{p}^{*})$$  

$$= -\psi(p)(ad(W)X, X).$$  

We obtain a foliation of $M$ by orbits from the action of $G$ on $M$. If we restrict the given metric on $M$ to each orbit of $M$ we obtain a nondegenerate metric, therefore we can apply the classification given in the past section.

**Theorem 1.** For $G$ and $M$ as before suppose $G$ acts topologically transitively on $M$, i.e., there is a dense $G$-orbit, preserving its pseudo-Riemannian metric and satisfying $n_{0}^{1} + \cdots + n_{0}^{l} = m_{0}$. Then $G$ acts everywhere locally free with nondegenerate orbits.

**Proof.** Since the action is topologically transitively on $M$, it follows from a result in [5] that the action is everywhere locally free. open subset $U \subset M$, so that the the $G$-orbit of every point in $U$ is nondegenerate.

We are going to prove that there exist a $G$-invariant open subset $U$ on $M$ so that the $G$-orbit of every point in $U$ is nondegenerate.
Every basis of $\mathfrak{g}$ induces at every point $p \in M$ a family of vectors that also defines a base for the tangent space to the orbit at $M$. In particular, $\varphi$ trivializes $TO$.

We consider the $G$-action on $M \times \mathfrak{g}$ given by $g(p, X) = (gp, Ad(g)(X))$. The map $\varphi$ is $G$-equivariant,

$$g \varphi(p, X) = gX_p^*$$

$$= g \frac{d}{dt} \big|_{t=0} (exp (tX)p)$$

$$= \frac{d}{dt} \big|_{t=0} (g \exp (tX)gp)$$

$$= \frac{d}{dt} \big|_{t=0} (g \exp (tAd(g)(X)gp)$$

$$= Ad(g)(X)_g^* = \varphi(g(p, X)).$$

The map $\psi$, defined above, is $G$-equivariant. For $g \in G$, $X,Y \in \mathfrak{g}$ and $p \in M$ we have:

$$\psi(gp)(X,Y) = h_{gp}(X^*_g, Y^*_g)$$

$$= h_p(Ad(g^{-1})(X^*_g), Ad(g^{-1})(Y^*_g))$$

$$= \psi(p)(Ad(g^{-1})(X), Ad(g^{-1})(Y)).$$

Hence, since the $G$-action is tame on $\mathfrak{g}^* \otimes \mathfrak{g}^*$, such map is essentially constant on the support of almost every ergodic component of $M$, see [6, Ch.2]).

By the lemma 3, there is an $Ad(G)$-invariant bilinear form $B_U$ on $\mathfrak{g}$ so that the metric on $TO|_U \cong U \times \mathfrak{g}$ induced by $B_U$ on each fiber, where $U$ is the support of one ergodic component of $M$. It is very easy to prove that the kernel of $B_U$ is an ideal of $\mathfrak{g}$. If such kernel is $\mathfrak{g}$, then $TO|_U$ is lightlike which implies $\dim \mathfrak{g} \leq m_0$. This contradicts the condition $n_1 + \cdots + n_l = m_0$ since $n_0 < \dim \mathfrak{g} = \dim \mathfrak{g}_1 + \cdots + \dim \mathfrak{g}_s$. Hence, being $\mathfrak{g}$ semisimple, it follows that $B_U$ is nondegenerate, and so almost every $G$-orbit contained in $S$ is nondegenerate. We can conclude that almost every $G$-orbit in $M$ is nondegenerate. As a consequence, the set $U$ as defined before is conull and so nonempty.

The previous argument shows that the image under $\psi$ of a conull and hence dense, subset of $M$ lies in the set of $Ad(G)$-invariant elements of $\mathfrak{g}^* \otimes \mathfrak{g}^*$. It follows that $\psi(M)$ lies on it, since such set is closed. In particular, on every $G$-orbit the metric induced from that of $M$ is given by $Ad(G)$-invariant symmetric bilinear form on $\mathfrak{g}$.

Using topological transitivity, we obtain an $G$-orbit $O_\alpha$. This $G$-orbit is dense and so it must intersect $U$. It is clear that $O_\alpha \subset U$ since $U$ is $G$-invariant.

The metric restricted to $O_\alpha$, under the map $\varphi$, is given by the nondegenerate bilinear form $B_\alpha$ on $\mathfrak{g}$. It follows that $\psi(O_\alpha) = B_\alpha$ and so the
the continuity of $\psi$ together with the density of $O_\alpha$ imply that $\psi$ is the constant map given by $B_\alpha$.

We now prove the principal result in this work.

**Theorem 2.** Let $G = G_1 \cdots G_l$ be a connected semisimple Lie group without compact factors acting by isometries on a finite volume pseudoRiemannian manifold $M$ and no factor of $G$ acts trivially. Then $n_0^1 + \cdots + n_0^l \leq m_0$.

**Proof.** By results in [6] we have local freeness on an open subset $U \subset X$. Then the map $\psi: U \to g^* \otimes g^*$, defined above, is constant on the ergodic components in $U$ for the $G$-action. On any such ergodic component, the metric along the $G$-orbits comes from an $Ad(G)$-invariant bilinear form $B_U$ on $g$.

Let $\eta$ be the kernel of $B_U$. It is known that $\eta$ is an ideal of $g$. Since $g$ is semisimple we have to consider the following cases.

1. $\eta = g$. In this case it follows easily that $B_U = 0$, then $\dim g \leq m_0$.
   Since $\dim g_i \geq n_0^i$ and $\dim g = \dim g_1 + \cdots + \dim g_l$, we conclude that
   $$n_0^1 + \cdots + n_0^l < m_0.$$ 

2. $\eta = \{0\}$. In this case $B_U$ is nondegenerate and the $G$-orbits are nondegenerate submanifolds of $X$. Then we have that $n_0 \leq m_0$ and by Lemma 2 the claim follows.

3. $\eta = \bigoplus_{j \in J} g_j$ where $J \subset \{1, \ldots, l\}$. From this it follows that there is a subspace of null vectors in the tangent space to the $G$-orbits which has a dimension $\dim \bigoplus_{j \in J} g_j + n_0^{j_1} + \cdots + n_0^{j_s}$. Therefore,
   $$\dim \bigoplus_{j \in J} g_j + n_0^{j_1} + \cdots + n_0^{j_s} \leq m_0,$$
   where $j_1, \ldots, j_s \in \{1, \ldots, l\} \setminus J$. On the other hand,
   $$\dim \bigoplus_{j \in J} g_j + n_0^{j_1} + \cdots + n_0^{j_s} > n_0^1 + \cdots + n_0^l.$$ 

$\square$
References


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