Generalized difference entire sequence spaces

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Abstract

In this paper we introduce difference entire sequence spaces and
difference analytic sequence spaces defined by a sequence of modulus
function \(F = (f_k)\) and study some topological properties and some
inclusion relations between these spaces. We also make an effort
to study some properties and inclusion relation between the spaces
\(\Gamma_F(\Delta^m, u, p, q, ||\cdot\cdot\cdot||)\) and \(\Lambda_F(\Delta^m, u, p, q, ||\cdot\cdot\cdot||)\).

Subjclass [2000] : 40A05, 40C05, 40D05.

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entire sequences, analytic sequences, paranorm space, \(n\)-normed space.
1. Introduction and Preliminaries

The notion of difference sequence spaces was introduced by Kizmaz [11], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [5] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let $w$ be the space of all complex or real sequences $x = (x_k)$ and let $m, s$ be non-negative integers, then for $Z = l_\infty, c, c_0$ we have sequence spaces

$$Z(\Delta^m_s) = \{x = (x_k) \in w : (\Delta^m_s x_k) \in Z\},$$

where $\Delta^m_s x_k = (\Delta^m_s x_k) = (\Delta^{m-1}_s x_k - \Delta^{m-1}_s x_{k+1})$ and $\Delta^0_s x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^m_s x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+sv}.$$

Taking $s = 1$, we get the spaces which were studied by Et and Çolak [5]. Taking $m = s = 1$, we get the spaces which were introduced and studied by Kizmaz [11].

A complex sequence, whose $k^{th}$ term is $x_k$, is denoted by $(x_k)$. Let $\varphi$ be the set of all finite sequences. A sequence $x = (x_k)$ is said to be analytic if $\sup |x_k|^\frac{1}{k} < \infty$. The vector space of all analytic sequences will be denoted by $\Lambda$. A sequence $x = (x_k)$ is called entire sequence if $\lim_{k \to \infty} |x_k|^\frac{1}{k} = 0$. The vector space of all entire sequences will be denoted by $\Gamma$.

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0, y \geq 0$,
3. $f$ is increasing
4. $f$ is continuous from right at 0.

It follows that $f$ must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x^p}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p, 0 < p < 1$, then the modulus $f(x)$ is unbounded. Subsequently, modulus function has been discussed
in ([1], [2], [3], [4], [12], [13], [17], [18]) and references therein. Let $F = (f_k)$
be a sequence of modulus function.

The space consisting of all those sequences $x$ in $w$ such that
$f_k \left( \frac{|x_k|^{1/k}}{\rho} \right) \to 0$ as $k \to \infty$ for some arbitrary fixed $\rho > 0$ is
denoted by $\Gamma_F$ and is known as a space of entire sequences defined by a
sequence of modulus function. The space $\Gamma_F$ is a metric space with the
metric $d(x, y) = \sup_k f_k \left( \frac{|x_k - y_k|^{1/k}}{\rho} \right)$ for all $x = (x_k)$ and $y = (y_k)$
in $\Gamma_F$. The space consisting of all those sequences $x$ in $w$ such that
$\left( \sup_k \left( f_k \left( \frac{|x_k|^{1/k}}{\rho} \right) \right) \right) < \infty$ for some arbitrarily fixed $\rho > 0$ is denoted
by $\Lambda_F$ and is known as a space of analytic sequences defined by a sequence
of modulus function.

A sequence space $E$ is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever
$(x_k) \in E$ and for all sequences of scalars $(\alpha_k)$ with $|\alpha_k| \leq 1$ (see [10]).

Let $X$ be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,
4. if $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and $(x_n)$ is a
   sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then
   $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm $p$ for which $p(x) = 0$ implies $x = 0$ is called total paranorm
and the pair $(X, p)$ is called a total paranormed space. It is well known
that the metric of any linear metric space is given by some total paranorm
(see [19], Theorem 10.4.2, P-183).

The following inequality will be used throughout the paper. Let $p = (p_k)$
be a sequence of positive real numbers with $0 \leq p_k \leq \sup p_k = G$, $K = \max(1, 2^{G-1})$ then

\[ |a_k + b_k|^{p_k} \leq K \{ |a_k|^{p_k} + |b_k|^{p_k} \} \]
for all $k$ and $a_k, b_k \in \mathbb{C}$. Also $|a|^p_k = \max(1, |a|^G)$ for all $a \in \mathbb{C}$.

Let $F = (f_k)$ be a sequence of modulus functions and $X$ be locally convex Hausdorff topological linear space whose topology is determined by a set of continuous seminorms $q$. The symbol $\Lambda(X)$ and $\Gamma(X)$ denotes the space of all analytic and entire sequences respectively defined over $X$. If $p = (p_k)$ be bounded sequences of strictly positive real numbers and $u = (u_k)$ be sequences of positive real numbers, then we define the following sequence spaces:

\[
\Lambda_F(\Delta^m_s, u, p, q) = \{ x \in \Lambda(X) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{|(u_k \Delta^m_s x_k)|^{1/k}}{\rho} \right) \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \}
\]

and

\[
\Gamma_F(\Delta^m_s, u, p, q) = \{ x \in \Gamma(X) : \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|(u_k \Delta^m_s x_k)|^{1/k}}{\rho} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty, \text{ for some } \rho > 0 \}.\]

If we take $p = (p_k) = 1$, we get

\[
\Lambda_F(\Delta^m_s, u, q) = \{ x \in \Lambda(X) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{|(u_k \Delta^m_s x_k)|^{1/k}}{\rho} \right) \right) \right] < \infty, \text{ for some } \rho > 0 \}
\]

and

\[
\Gamma_F(\Delta^m_s, u, q) = \{ x \in \Gamma(X) : \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|(u_k \Delta^m_s x_k)|^{1/k}}{\rho} \right) \right) \right] \to 0 \text{ as } n \to \infty, \text{ for some } \rho > 0 \}.\]
The purpose of this paper is to introduce and study a concept of difference entire sequence spaces and difference analytic sequence spaces using sequence of modulus functions. We examine some topological properties and inclusion relation between the spaces $\Lambda_F(\Delta^m_s, u, p, q)$ and $\Gamma_F(\Delta^m_s, u, p, q)$ in the second section and third section devoted to the study of some properties of $n$-normed spaces $\Lambda_F(\Delta^m_s, u, p, q, ||\cdot||, \cdots, ||\cdot||)$ and $\Gamma_F(\Delta^m_s, u, p, q, ||\cdot||, \cdots, ||\cdot||)$.

2. Some Topological properties of the spaces $\Lambda_F(\Delta^m_s, u, p, q)$ and $\Gamma_F(\Delta^m_s, u, p, q)$

In this section of the paper we study very interesting properties like linearity, paranorm and some attractive inclusion relations between the spaces $\Lambda_F(\Delta^m_s, u, p, q)$ and $\Gamma_F(\Delta^m_s, u, p, q)$.

**Theorem 2.1** Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be bounded sequence of strictly positive real numbers, then $\Gamma_F(\Delta^m_s, u, p, q)$ and $\Lambda_F(\Delta^m_s, u, p, q)$ are linear spaces over the set of complex numbers $\mathbb{C}$.

**Proof.** Let $x = (x_k), y = (y_k) \in \Gamma_F(\Delta^m_s, u, p, q)$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result, we need to find some $\rho_3 > 0$ such that

$$\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta^m_s (\alpha x_k + \beta y_k)|}{\rho_3} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty.$$

Since $x = (x_k), y = (y_k) \in \Gamma_F(\Delta^m_s, u, p, q)$, there exist some positive $\rho_1$ and $\rho_2$ such that

$$\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta^m_s x_k|}{\rho_1} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty$$

and

$$\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta^m_s y_k|}{\rho_2} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty.$$
Since \( F = (f_k) \) is a non-decreasing function, \( q \) is a seminorm and \( \Delta^m_{x_k} \) is linear, then

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{u_k^m(m(\alpha x_k + \beta y_k))}{\rho_3} \right)^{1 \over 2} \right) \right]^{p_k} \\
\leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{|\alpha u_k^m x_k|}{\rho_3} + \frac{|\beta u_k^m y_k|}{\rho_3} \right)^{1 \over 2} \right) \right]^{p_k}
\]

so that

\[
\sum_{k=1}^{n} \left[ f_k \left( \left( \frac{|u_k^m(m(\alpha x_k + \beta y_k))|}{\rho_3} \right)^{1 \over 2} \right) \right]^{p_k} \\
\leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{|\alpha u_k^m x_k|}{\rho_3} + \frac{|\beta u_k^m y_k|}{\rho_3} \right)^{1 \over 2} \right) \right]^{p_k} \cdot \rho_3
\]

Take \( \rho_3 > 0 \) such that \( \frac{1}{\rho_3} = \min \left\{ \frac{1}{|\alpha| \rho_1}, \frac{1}{|\beta| \rho_2} \right\} \)

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{|u_k^m(m(\alpha x_k + \beta y_k))|}{\rho_3} \right)^{1 \over 2} \right) \right]^{p_k} \\
\leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{|\alpha u_k^m x_k|}{\rho_1} + \frac{|\beta u_k^m y_k|}{\rho_2} \right)^{1 \over 2} \right) \right]^{p_k}
\]

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{|\alpha u_k^m x_k|}{\rho_1} \right)^{1 \over 2} \right) \right]^{p_k} + \left[ f_k \left( \left( \frac{|\beta u_k^m y_k|}{\rho_2} \right)^{1 \over 2} \right) \right]^{p_k}
\]

\[
\leq K \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{|\alpha u_k^m x_k|}{\rho_1} \right)^{1 \over 2} \right) \right]^{p_k} + K \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{|\beta u_k^m y_k|}{\rho_2} \right)^{1 \over 2} \right) \right]^{p_k}
\]

\[
\to 0 \quad \text{as} \quad n \to \infty.
\]

Hence

\[
\sum_{k=1}^{n} \left[ f_k \left( \left( \frac{|\alpha u_k^m x_k + \beta u_k^m y_k|}{\rho_3} \right)^{1 \over 2} \right) \right]^{p_k} \to 0 \quad \text{as} \quad n \to \infty.
\]

This proves that \( \Gamma_F(\Delta^m_{x_k}, u, p, q) \) is a linear space. Similarly, we can prove that \( \Lambda_F(\Delta^m_{x_k}, u, p, q) \) is a linear space.
Theorem 2.2 Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be bounded sequence of strictly positive real numbers. Then $\Gamma_F(\Delta^m u, p, q)$ is a paranormed space with paranorm defined by

$$g(x) = \inf \left\{ \rho^{\frac{mp}{p}} : \sup_{k \geq 1} \left[ f_k \left( q \left( \left( \frac{|u_k \Delta^m u_k x_k|}{\rho} \right)^\frac{1}{p} \right) \right) \right]^{p_k} \leq 1, \quad \rho > 0, \ m \in \mathbb{N} \right\},$$

where $H = \max(1, \sup_k p_k)$.

**Proof.** Clearly $g(x) \geq 0$, $g(x) = g(-x)$ and $g(\theta) = 0$, where $\theta$ is the zero sequence of $X$.

Let $(x_k), (y_k) \in \Gamma_F(\Delta^m u, p, q)$. Let $\rho_1, \rho_2 > 0$ be such that

$$\sup_{k \geq 1} \left[ f_k \left( q \left( \left( \frac{|u_k \Delta^m u_k x_k|}{\rho_1} \right)^\frac{1}{p} \right) \right) \right]^{p_k} \leq 1$$

and

$$\sup_{k \geq 1} \left[ f_k \left( q \left( \left( \frac{|u_k \Delta^m u_k y_k|}{\rho_2} \right)^\frac{1}{p} \right) \right) \right]^{p_k} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$.

Then by using Minkowski’s inequality, we have

$$\sup_{k \geq 1} f_k \left( q \left( \left( \frac{|u_k \Delta^m u_k (x_k + y_k)|}{\rho} \right)^\frac{1}{p} \right) \right)^{p_k} \leq \left( \frac{\rho}{\rho_1 + \rho_2} \right) \sup_{k \geq 1} f_k \left( q \left( \left( \frac{|u_k \Delta^m u_k x_k|}{\rho_1} \right)^\frac{1}{p} \right) \right)^{p_k}$$

$$+ \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_{k \geq 1} f_k \left( q \left( \left( \frac{|u_k \Delta^m u_k y_k|}{\rho_2} \right)^\frac{1}{p} \right) \right)^{p_k} \leq 1.$$

Hence

$$g(x + y) \leq \inf \left\{ (\rho_1 + \rho_2)^{\frac{mp}{p}} : \sup_{k \geq 1} f_k \left( q \left( \left( \frac{|u_k \Delta^m u_k x_k|}{\rho_1 + \rho_2} \right)^\frac{1}{p} \right) \right) \right\}^{p_k} \leq 1, \rho_1, \rho_2 > 0, \ m \in \mathbb{N} \}$$
Thus we have
\[ g(x + y) \leq g(x) + g(y). \] Hence \( g \) satisfies the triangle inequality.

**Theorem 2.3** Let \( F' = (f'_k) \) and \( F'' = (f''_k) \) be two sequences of modulus functions. Then

\[
\Gamma^{F'}(\Delta^m, u, p, q) \cap \Gamma^{F''}(\Delta^m, u, p, q) \subseteq \Gamma^{F'F''}(\Delta^m, u, p, q).
\]

**Proof.** Let \( x = (x_k) \in \Gamma^{F'}(\Delta^m, u, p, q) \cap \Gamma^{F''}(\Delta^m, u, p, q). \) Then there exist \( \rho_1 \) and \( \rho_2 \) such that

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f'_k \left( q \left( \frac{|u_k \Delta^m x_k|}{\rho_1} \right)^{\frac{1}{p}} \right) \right]^p \to 0 \text{ as } n \to \infty.
\]

and

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f''_k \left( q \left( \frac{|u_k \Delta^m x_k|}{\rho_2} \right)^{\frac{1}{p}} \right) \right]^p \to 0 \text{ as } n \to \infty.
\]
Since $\rho > 0$ such that $\frac{1}{\rho} = \min \left( \frac{1}{p_1}, \frac{1}{p_2} \right)$. Then we have $\frac{1}{n} \sum_{k=1}^{n} \left[ (f_k' + f_k'') \left( q \left( \frac{|u_k \Delta_{m} x_k|}{\rho} \right) \right) \right]^{p_k}$

\[
\leq K \left[ \frac{1}{n} \sum_{k=1}^{n} \left[ f_k' \left( q \left( \frac{|u_k \Delta_{m} x_k|}{\rho_1} \right) \right) \right]^{p_k} \\
+ K \left[ \frac{1}{n} \sum_{k=1}^{n} \left[ f_k'' \left( q \left( \frac{|u_k \Delta_{m} x_k|}{\rho_2} \right) \right) \right]^{p_k} \right]
\]

$\to 0$ as $n \to \infty$

Then

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ (f_k' + f_k'') \left( q \left( \frac{|u_k \Delta_{m} x_k|}{\rho} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty.
\]

Therefore $x = (x_k) \in \Gamma_{F^* + F''}(\Delta_{m}^{s}, u, p, q)$.

**Theorem 2.4** Let $m \geq 1$. Then we have the following inclusions:
(i) $\Gamma_{F}(\Delta_{m-1}^{s}, u, p, q) \subseteq \Gamma_{F}(\Delta_{m}^{s}, u, p, q)$,
(ii) $\Lambda_{F}(\Delta_{m-1}^{s}, u, p, q) \subseteq \Lambda_{F}(\Delta_{m}^{s}, u, p, q)$.

**Proof.** Let $x = (x_k) \in \Gamma_{F}(\Delta_{m-1}^{s}, u, p, q)$. Then we have

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_{m-1} x_k|}{\rho} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty, \text{ for some } \rho > 0.
\]

Since $F = (f_k)$ is non-decreasing and $q$ is a seminorm, we have

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_{m} x_k|}{\rho} \right) \right) \right]^{p_k} \\
\leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_{m} x_k - u_k \Delta_{m-1} x_k|}{\rho} \right) \right) \right]^{p_k} \\
\leq K \left\{ \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_{m} x_k|}{\rho} \right) \right) \right]^{p_k} \right\}
\]
\[ + \frac{1}{n} \sum_{k=1}^{n} \left\{ f_k \left( q \left( \frac{|u_k \Delta_{m-1} x_k|}{\rho} \right)^{\frac{r}{k}} \right) \right\}^{p_k} \]

\[ \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Therefore \( \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_{m} x_k|}{\rho} \right)^{\frac{1}{k}} \right) \right]^{p_k} \rightarrow 0 \text{ as } n \rightarrow \infty. \)

Hence \( x \in \Gamma_F(\Delta_{m}, u, p, q) \). This completes the proof of (i). Similarly, we can prove (ii).

**Theorem 2.5** Let \( 0 \leq p_k \leq r_k \) and let \( \{ \frac{p_k}{r_k} \} \) be bounded. Then \( \Gamma_F(\Delta_{m}, u, r, q) \subset \Gamma_F(\Delta_{m}, u, p, q) \).

**Proof.** Let \( x = (x_k) \in \Gamma_F(\Delta_{m}, u, r, q) \). Then

\[ \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_{m} x_k|}{\rho} \right)^{\frac{1}{r_k}} \right) \right]^{r_k} \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Let \( t_k = \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_{m} x_k|}{\rho} \right)^{\frac{1}{r_k}} \right) \right]^{q_k} \)

and \( \lambda_k = \frac{p_k}{r_k} \).

Since \( p_k \leq r_k \), we have \( 0 \leq \lambda_k \leq 1 \). Take \( 0 < \lambda < \lambda_k \). Define

\[ u_k = \begin{cases} t_k & \text{if } t_k \geq 1 \\ 0 & \text{if } t_k < 1 \end{cases} \]

and

\[ v_k = \begin{cases} 0 & \text{if } t_k \geq 1 \\ t_k & \text{if } t_k < 1 \end{cases} \]

\[ t_k = u_k + v_k, \quad t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}. \]

It follows that \( u_k^{\lambda_k} \leq u_k \leq t_k, \quad v_k^{\lambda_k} \leq v_k \).

Since \( t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k} \), then \( t_k^{\lambda_k} \leq t_k + v_k^{\lambda_k} \). Thus

\[ \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_{m} x_k|}{\rho} \right)^{\frac{1}{r_k}} \right) \right]^{r_k} \lambda_k \]

\[ \leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_{m} x_k|}{\rho} \right)^{\frac{1}{r_k}} \right) \right]^{r_k} \]
\[
\sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta^m u_k|}{\rho} \right)^{\frac{1}{p_k}} \right) \right]^{r_k} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

But
\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta^m u_k|}{\rho} \right)^{\frac{1}{p_k}} \right) \right]^{r_k} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Therefore
\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta^m u_k|}{\rho} \right)^{\frac{1}{p_k}} \right) \right]^{p_k} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Hence \( x = (x_k) \in \Gamma_F(\Delta^m u, r, q) \). Thus, we have
\[
\Gamma_F(\Delta^m u, r, q) \subset \Gamma_F(\Delta^m u, p, q).
\]

**Theorem 2.6**

(i) Let \( 0 < \inf p_k \leq p_k \leq 1 \). Then
\[
\Gamma_F(\Delta^m u, p, q) \subset \Gamma_F(\Delta^m u, q),
\]

(ii) Let \( 1 \leq p_k \leq \sup p_k < \infty \). Then \( \Gamma_F(\Delta^m u, q) \subset \Gamma_F(\Delta^m u, p, q) \).

**Proof.** (i) Let \( x = (x_k) \in \Gamma_F(\Delta^m u, p, q) \). Then
\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta^m u_k|}{\rho} \right)^{\frac{1}{p_k}} \right) \right]^{p_k} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Since \( 0 < \inf p_k \leq p_k \leq 1 \),
\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta^m u_k|}{\rho} \right)^{\frac{1}{p_k}} \right) \right] \leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta^m u_k|}{\rho} \right)^{\frac{1}{p_k}} \right) \right]^{p_k} \rightarrow 0
\]
as $n \to \infty$.
Thus, it follows that, $x = (x_k) \in \Gamma_F(\Delta_m, u, q)$. Thus $\Gamma_F(\Delta_m, u, p, q) \subset \Gamma_F(\Delta_m, u, q)$.

$(ii)$ Let $p_k \geq 1$ for each $k$ and sup $p_k < \infty$ and let $x = (x_k) \in \Gamma_F(\Delta_m, u, q)$. Then

$$\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_m x_k|}{\rho} \right)^{\frac{1}{p_k}} \right) \right] \to 0 \text{ as } n \to \infty$$

Since $1 \leq p_k \leq \text{sup } p_k < \infty$, we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_m x_k|}{\rho} \right)^{\frac{1}{p_k}} \right) \right]^{p_k} \leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_m x_k|}{\rho} \right)^{\frac{1}{p_k}} \right) \right]^{p_k} \to 0 \text{ as } n \to \infty.$$ 

This implies that $x = (x_k) \in \Gamma_F(\Delta_m, u, p, q)$. Therefore

$$\Gamma_F(\Delta_m, u, q) \subset \Gamma_F(\Delta_m, u, p, q).$$

**Theorem 2.7** Suppose $\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_m x_k|}{\rho} \right)^{\frac{1}{p_k}} \right) \right]^{p_k} \leq |x_k|^{1/k}$, then $\Gamma \subset \Gamma_F(\Delta_m, u, p, q)$.

**Proof.** Let $x = (x_k) \in \Gamma$. Then we have,

$$|x_k|^{1/k} \to 0 \text{ as } k \to \infty.$$ 

But $\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_m x_k|}{\rho} \right)^{\frac{1}{p_k}} \right) \right]^{p_k} \leq |x_k|^{1/k}$, by our assumption, implies that

$$\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta_m x_k|}{\rho} \right)^{\frac{1}{p_k}} \right) \right]^{p_k} \to 0 \text{ as } n \to \infty$$

Then $x = (x_k) \in \Gamma_F(\Delta_m, u, p, q)$ and $\Gamma \subset \Gamma_F(\Delta_m, u, p, q)$.

**Theorem 2.8** $\Gamma_F(\Delta_m, u, p, q)$ is solid.
Proof. Let \(|x_k| \leq |y_k|\) and let \(y = (y_k) \in \Gamma_F(\Delta^m, u, p, q)\), because \(F = (f_k)\) is non-decreasing

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta^m x_k|}{\rho} \right)^{\frac{1}{p}} \right) \right]^{p_k} \leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta^m y_k|}{\rho} \right)^{\frac{1}{p}} \right) \right]^{p_k}
\]

Since \(y = (y_k) \in \Gamma_F(\Delta^m, u, p, q)\). Therefore,

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta^m y_k|}{\rho} \right)^{\frac{1}{p}} \right) \right]^{p_k} \to 0 \text{ as } n \to \infty
\]

and so that

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|u_k \Delta^m x_k|}{\rho} \right)^{\frac{1}{p}} \right) \right]^{p_k} \to 0 \text{ as } n \to \infty.
\]

Therefore \(x = (x_k) \in \Gamma_F(\Delta^m, u, p, q)\).

Theorem 2.9 \(\Gamma_F(\Delta^m, u, p, q)\) is monotone.

Proof. It is trivial so we omit it.

3. Difference Entire sequence spaces over \(n\)-normed spaces

The concept of 2-normed spaces was initially developed by Gähler[6] in the mid of 1960’s, while that of \(n\)-normed spaces one can see in Misiak[14]. Since then, many others have studied this concept and obtained various results, see Gunawan ([7],[8]) and Gunawan and Mashadi [9]. For more details about the sequence spaces over \(n\)-normed spaces see ([15],[16]).

Let \(n \in \mathbb{N}\) and \(X\) be a linear space over the field \(\mathbb{K}\), where \(\mathbb{K}\) is field of real or complex numbers of dimension \(d\), where \(d \geq n \geq 2\). A real valued function \(|\cdot, \cdots, \cdot|\) on \(X^n\) satisfying the following four conditions:

1. \(|x_1, x_2, \cdots, x_n| = 0\) if and only if \(x_1, x_2, \cdots, x_n\) are linearly dependent in \(X\);

2. \(|x_1, x_2, \cdots, x_n|\) is invariant under permutation;
3. \[ ||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n|| \] for any \( \alpha \in K \), and

4. \[ ||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n|| \]
is called an \( n \)-norm on \( X \), and the pair \((X, |||, \cdots, |||)\) is called a \( n \)-normed space over the field \( K \). For example, we may take \( X = \mathbb{R}^n \) being equipped with the \( n \)-norm \( ||x_1, x_2, \cdots, x_n||_E = \) the volume of the \( n \)-dimensional parallelepiped spanned by the vectors \( x_1, x_2, \cdots, x_n \) which may be given explicitly by the formula

\[ ||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|, \]

where \( x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n \) for each \( i = 1, 2, \cdots, n \).

Let \((X, |||, \cdots, |||)\) be an \( n \)-normed space of dimension \( d \geq n \geq 2 \) and \( \{a_1, a_2, \cdots, a_n\} \) be linearly independent set in \( X \). Then the following function \( |||, \cdots, |||_\infty \) on \( X^{n-1} \) defined by

\[ ||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\} \]
defines an \((n - 1)\)-norm on \( X \) with respect to \( \{a_1, a_2, \cdots, a_n\} \).

A sequence \((x_k)\) in a \( n \)-normed space \((X, |||, \cdots, |||)\) is said to converge to some \( L \in X \) if

\[ \lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \] for every \( z_1, \cdots, z_{n-1} \in X \).

A sequence \((x_k)\) in a \( n \)-normed space \((X, |||, \cdots, |||)\) is said to be Cauchy if

\[ \lim_{k,p \to \infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \] for every \( z_1, \cdots, z_{n-1} \in X \).

If every cauchy sequence in \( X \) converges to some \( L \in X \), then \( X \) is said to be complete with respect to the \( n \)-norm. Any complete \( n \)-normed space is said to be \( n \)-Banach space.

Let \( F = (f_k) \) be a sequence of modulus functions and let \( X \) be locally convex Hausdorff topological linear space whose topology is determined by
a set of continuous seminorms \( q \). The symbol \( \Lambda(X) \), \( \Gamma(X) \) denotes the space of all analytic and entire sequences respectively defined over \( X \). In this section we define the following sequences spaces:

\[
\Lambda_F(\Delta^m_s, u, p, q, ||, \cdots, ||) = \left\{ \left( x \in \Lambda(X) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[ f_k\left( q\left( \left( \frac{u_k \Delta^m_s x_k}{\rho} \right)^{1/k}, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \right) < \infty, \text{ for some } \rho > 0 \right\},
\]

\[
\Gamma_F(\Delta^m_s, u, p, q, ||, \cdots, ||) = \left\{ \left( x \in \Gamma(X) : \frac{1}{n} \sum_{k=1}^{n} \left[ f_k\left( q\left( \left( \frac{u_k \Delta^m_s x_k}{\rho} \right)^{1/k}, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for some } \rho > 0 \right\}.
\]

If we take \( p = (p_k) = 1 \), we get

\[
\Lambda_F(\Delta^m_s, u, q, ||, \cdots, ||) = \left\{ \left( x \in \Lambda(X) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[ f_k\left( q\left( \left( \frac{u_k \Delta^m_s x_k}{\rho} \right)^{1/k}, z_1, \cdots, z_{n-1} \right) \right) \right] \right) < \infty, \text{ for some } \rho > 0 \right\},
\]

\[
\Gamma_F(\Delta^m_s, u, q, ||, \cdots, ||) = \left\{ \left( x \in \Gamma(X) : \frac{1}{n} \sum_{k=1}^{n} \left[ f_k\left( q\left( \left( \frac{u_k \Delta^m_s x_k}{\rho} \right)^{1/k}, z_1, \cdots, z_{n-1} \right) \right) \right] \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for some } \rho > 0 \right\}.
\]

In this section of the paper we study some topological properties of the spaces \( \Lambda_F(\Delta^m_s, u, p, q, ||, \cdots, ||) \) and \( \Gamma_F(\Delta^m_s, u, p, q, ||, \cdots, ||) \). We also examine some inclusion relation between these spaces.

**Theorem 3.1** Let \( F = (f_k) \) be a sequence of modulus functions and \( p = (p_k) \) be bounded sequence of strictly positive real numbers, then

\( \Gamma_F(\Delta^m_s, u, p, q, ||, \cdots, ||) \) and \( \Lambda_F(\Delta^m_s, u, p, q, ||, \cdots, ||) \) are linear spaces
over the set of complex numbers \( \mathbb{C} \).

**Proof.** \( x = (x_k), y = (y_k) \in \Gamma_F(\Delta^m_s, u, p, q, ||\cdot||, \cdots, ||\cdot||) \) and \( \alpha, \beta \in \mathbb{C} \). In order to prove the result, we need to find some \( \rho_3 > 0 \) such that

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{u_k \Delta^m_s (\alpha x_k + \beta y_k) z_{1, \ldots, z_{n-1}}}{\rho_3} \right)^{\frac{1}{k}}, z_{1, \ldots, z_{n-1}} \right) \right]^{p_k} \to 0 \quad \text{as} \quad n \to \infty.
\]

Since \( x = (x_k), y = (y_k) \in \Gamma_F(\Delta^m_s, u, p, q, ||\cdot||, \cdots, ||\cdot||) \), there exist some positive \( \rho_1 \) and \( \rho_2 \) such that

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{u_k \Delta^m_s x_k}{\rho_1} \right)^{\frac{1}{k}}, z_{1, \ldots, z_{n-1}} \right) \right]^{p_k} \to 0 \quad \text{as} \quad n \to \infty
\]

and

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{u_k \Delta^m_s y_k}{\rho_2} \right)^{\frac{1}{k}}, z_{1, \ldots, z_{n-1}} \right) \right]^{p_k} \to 0 \quad \text{as} \quad n \to \infty.
\]

Since \( F = (f_k) \) is a non-decreasing function, \( q \) is a seminorm and \( \Delta^m_s \) is linear, then

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{u_k \Delta^m_s (\alpha x_k + \beta y_k) z_{1, \ldots, z_{n-1}}}{\rho_3} \right)^{\frac{1}{k}}, z_{1, \ldots, z_{n-1}} \right) \right]^{p_k}
\]

\[
\leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{u_k \Delta^m_s x_k}{\rho_1} \right)^{\frac{1}{k}}, z_{1, \ldots, z_{n-1}} \right) \right]^{p_k} + \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{u_k \Delta^m_s y_k}{\rho_2} \right)^{\frac{1}{k}}, z_{1, \ldots, z_{n-1}} \right) \right]^{p_k}
\]

so that

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{u_k \Delta^m_s (\alpha x_k + \beta y_k) z_{1, \ldots, z_{n-1}}}{\rho_3} \right)^{\frac{1}{k}}, z_{1, \ldots, z_{n-1}} \right) \right]^{p_k}
\]

\[
\leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( \left( \frac{\alpha u_k \Delta^m_s x_k}{\rho_3} \right)^{\frac{1}{k}}, z_{1, \ldots, z_{n-1}} \right) \right]^{p_k}
\]

\[
+ \left| \frac{\beta u_k \Delta^m_s y_k}{\rho_3} \right|^{p_k}, z_{1, \ldots, z_{n-1}} \right) \right]^{p_k}.
\]
Since $\rho_3 > 0$ such that $\frac{1}{\rho} = \min \left\{ \frac{1}{|\alpha| \rho_1}, \frac{1}{|\beta| \rho_2} \right\}$

$$\frac{1}{n} n \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{\| \frac{(u_k \Delta^m \alpha x_k + \beta y_k)}{\rho_1} \|^p}{\rho_1}, z_1, \cdots, z_{n-1} \| \right) \right) \right]^{p_k}$$

$$\leq \frac{1}{n} n \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{\| \frac{(u_k \Delta^m y_k)}{\rho_2} \|^p}{\rho_2}, z_1, \cdots, z_{n-1} \| \right) \right) \right]^{p_k}$$

$$+ \left[ f_k \left( q \left( \frac{\| \frac{(u_k \Delta^m \alpha x_k)}{\rho_1} \|^p}{\rho_1}, z_1, \cdots, z_{n-1} \| \right) \right) \right]^{p_k}$$

$$\leq K^{\frac{1}{n}} n \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{\| \frac{(u_k \Delta^m y_k)}{\rho_2} \|^p}{\rho_2}, z_1, \cdots, z_{n-1} \| \right) \right) \right]^{p_k}$$

$$+ K^{\frac{1}{n}} n \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{\| \frac{(u_k \Delta^m \alpha x_k)}{\rho_1} \|^p}{\rho_1}, z_1, \cdots, z_{n-1} \| \right) \right) \right]^{p_k}$$

$\rightarrow 0$ as $n \rightarrow \infty$.

Hence

$$\sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{\| \frac{(u_k \Delta^m \alpha x_k + \beta y_k)}{\rho_3} \|^p}{\rho_3}, z_1, \cdots, z_{n-1} \| \right) \right) \right]^{p_k} \rightarrow 0$ as $n \rightarrow \infty$.

This proves that $\Gamma_F(\Delta^m, u, p, q, \|., .\|)$ is a linear space. Similarly, we can prove that $\Lambda_F(\Delta^m, u, p, q, \|., .\|)$ is a linear space.

**Theorem 3.2** Let $F = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be bounded sequence of strictly positive real numbers, $\Gamma_F(\Delta^m, u, p, q, \|., .\|)$ is paranormed space with paranorm defined by

$$g(x) = \inf \left\{ \rho^{\frac{m}{p}} : \sup_{k \geq 1} \left[ f_k \left( q \left( \frac{\| \frac{(u_k \Delta^m x_k)}{\rho} \|^p}{\rho}, z_1, \cdots, z_{n-1} \| \right) \right) \right]^{p_k} \leq 1, \rho > 0, m \in \mathbb{N} \right\},$$

where $H = \max(1, \sup_k p_k)$. 
Proof. Clearly \( g(x) \geq 0 \), \( g(x) = g(-x) \) and \( g(\theta) = 0 \), where \( \theta \) is the zero sequence of \( X \).

Let \( (x_k), (y_k) \in \Gamma_F(\Delta_m^u, u, p, q, ||., \cdot., .||) \). Let \( \rho_1, \rho_2 > 0 \) be such that

\[
\sup_{k \geq 1} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta_m^u x_k)}{\rho_1} \right\|_{z_1, \cdots, z_{n-1}} \right) \right) \right]^{p_k} \leq 1
\]

and

\[
\sup_{k \geq 1} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta_m^u y_k)}{\rho_2} \right\|_{z_1, \cdots, z_{n-1}} \right) \right) \right]^{p_k} \leq 1.
\]

Let \( \rho = \rho_1 + \rho_2 \). Then by using Minkowski’s inequality, we have

\[
\sup_{k \geq 1} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta_m^u (x_k + y_k))}{\rho} \right\|_{z_1, \cdots, z_{n-1}} \right) \right) \right]^{p_k}
\]

\[
\leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_{k \geq 1} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta_m^u x_k)}{\rho_1} \right\|_{z_1, \cdots, z_{n-1}} \right) \right) \right]^{p_k}
\]

\[
+ \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_{k \geq 1} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta_m^u y_k)}{\rho_2} \right\|_{z_1, \cdots, z_{n-1}} \right) \right) \right]^{p_k}
\]

\[
\leq 1.
\]

Hence

\[
g(x + y)
\]

\[
\leq \inf \left\{ (\rho_1 + \rho_2)^{\frac{m}{\rho_1 + \rho_2}} : \sup_{k \geq 1} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta_m^u x_k)}{\rho_1 + \rho_2} \right\|_{z_1, \cdots, z_{n-1}} \right) \right) \right]^{p_k} \leq 1, \rho_1, \rho_2 > 0, m \in N \right\}
\]

\[
\leq \inf \left\{ (\rho_1)^{\frac{m}{\rho_1}} : \sup_{k \geq 1} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta_m^u x_k)}{\rho_1} \right\|_{z_1, \cdots, z_{n-1}} \right) \right) \right]^{p_k} \leq 1, \rho_1 > 0, m \in N \right\}
\]

\[
+ \inf \left\{ (\rho_2)^{\frac{m}{\rho_2}} : \sup_{k \geq 1} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta_m^u y_k)}{\rho_2} \right\|_{z_1, \cdots, z_{n-1}} \right) \right) \right]^{p_k} \leq 1, \rho_2 > 0, m \in N \right\}
\]
Thus we have $g(x + y) \leq g(x) + g(y)$. Hence $g$ satisfies the triangle inequality.

$$g(\lambda x) = \inf \left\{ (\rho)^{\frac{m}{k}} : \sup_{k \geq 1} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta^m \Delta x_k)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right) \right]^{\frac{1}{pk}} \leq 1, \right\}$$

$$\rho > 0, \ m \in N$$

$$= \inf \left\{ (r|\lambda|)^{\frac{m}{k}} : \sup_{k \geq 1} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta^m \Delta x_k)}{r}, z_1, \cdots, z_{n-1} \right\| \right) \right) \right]^{\frac{1}{pk}} \leq 1, \right\}$$

$$r > 0, \ m \in N,$$

where $r = |\rho|$

Hence $\Gamma_F(\Delta^m u, p, q, ||, \cdots, ||)$ is a paranormed space.

**Theorem 3.3** Let $F' = (f'_k)$ and $F'' = (f''_k)$ be two sequences of modulus functions.

Then $\Gamma_{F'}(\Delta^m u, p, q, ||, \cdots, ||) \cap \Gamma_{F''}(\Delta^m u, p, q, ||, \cdots, ||)$

$$\subseteq \Gamma_{F' + F''}(\Delta^m u, p, q, ||, \cdots, ||).$$

**Proof.** Let $x = (x_k) \in \Gamma_{F'}(\Delta^m u, p, q, ||, \cdots, ||) \cap \Gamma_{F''}(\Delta^m u, p, q, ||, \cdots, ||)$.

Then there exist $\rho_1$ and $\rho_2$ such that

$$\frac{1}{n} \sum_{k=1}^{n} \left[ f'_k \left( q \left( \left\| \frac{(u_k \Delta^m \Delta x_k)}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right) \right) \right]^{\frac{1}{pk}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

and

$$\frac{1}{n} \sum_{k=1}^{n} \left[ f''_k \left( q \left( \left\| \frac{(u_k \Delta^m \Delta x_k)}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right) \right) \right]^{\frac{1}{pk}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let $\frac{1}{\rho} = \min \left( \frac{1}{\rho_1}, \frac{1}{\rho_2} \right)$. Then we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[ (f'_k + f''_k) \left( q \left( \left\| \frac{(u_k \Delta^m \Delta x_k)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right) \right]^{\frac{1}{pk}}$$
\[
\leq K \left[ \frac{1}{n} \sum_{k=1}^{n} \left[ f_k' \left( q \left( \left\| \frac{(u_k \Delta^m x_k)_{\frac{1}{p}}}{\rho} \right\|, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \right] \\
+ K \left[ \frac{1}{n} \sum_{k=1}^{n} \left[ f_k'' \left( q \left( \left\| \frac{(u_k \Delta^m x_k)_{\frac{1}{p}}}{\rho} \right\|, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \right]
\]

\[\to 0 \text{ as } n \to \infty\]

Then

\[\frac{1}{n} \sum_{k=1}^{n} \left[ (f_k' + f_k'') \left( q \left( \left\| \frac{(u_k \Delta^m x_k)_{\frac{1}{p}}}{\rho} \right\|, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty.\]

Therefore \(x = (x_k) \in \Gamma_{F'+F''}(\Delta^m, u, p, q, ||\cdot||, \cdots, ||\cdot||).\)

**Theorem 3.4** Let \(m \geq 1.\) Then we have the following inclusions:

(i) \(\Gamma_{F}(\Delta^{m-1}, u, p, q, ||\cdot||, \cdots, ||\cdot||) \subseteq \Gamma_{F}(\Delta^m, u, p, q, ||\cdot||, \cdots, ||\cdot||),\)

(ii) \(\Lambda_{F}(\Delta^{m-1}, u, p, q, ||\cdot||, \cdots, ||\cdot||) \subseteq \Lambda_{F}(\Delta^m, u, p, q, ||\cdot||, \cdots, ||\cdot||).\)

**Proof.** Let \(x = (x_k) \in \Gamma_{F}(\Delta^m, u, p, q, ||\cdot||, \cdots, ||\cdot||).\) Then we have

\[\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta^m x_k)_{\frac{1}{p}}}{\rho} \right\|, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty, \text{ for some }\]

\(\rho > 0.\)

Since \(F = (f_k)\) is non-decreasing and \(q\) is a seminorm, we have

\[\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta^m x_k)_{\frac{1}{p}}}{\rho} \right\|, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k}
\]

\[\leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta^m x_k - u_k \Delta^{m-1} x_{k+1})_{\frac{1}{p}}}{\rho} \right\|, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k}
\]

\[\leq K \left\{ \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta^{m-1} x_{k+1})_{\frac{1}{p}}}{\rho} \right\|, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \right\}
\]

\[\to 0 \text{ as } n \to \infty.\]

Therefore \(\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \left\| \frac{(u_k \Delta^m x_k)_{\frac{1}{p}}}{\rho} \right\|, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty.\)
Hence \( x = (x_k) \in \Gamma_F(\Delta^m_u, u, p, q, ||, \cdots, ||) \). This completes the proof of (i). Similarly, we can prove (ii).

**Theorem 3.5** Let \( 0 \leq p_k \leq r_k \) and let \( \{ \frac{p_k}{r_k} \} \) be bounded. Then

\[
\Gamma_F(\Delta^m_u, u, r, q, ||, \cdots, ||) \subset \Gamma_F(\Delta^m_u, u, p, q, ||, \cdots, ||)
\]

**Proof.** Let \( x \in \Gamma_F(\Delta^m_u, u, r, q, ||, \cdots, ||) \). Then

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|| (u_k \Delta^m x_k) ^\frac{1}{r} \|, z_1, \cdots, z_{n-1} ||}{\rho} \right) \right) \right]^{r_k} \to 0 \quad \text{as} \; n \to \infty.
\]

Let \( t_k = \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|| (u_k \Delta^m x_k) ^\frac{1}{r} \|, z_1, \cdots, z_{n-1} ||}{\rho} \right) \right) \right]^{q_k} \) and \( \lambda_k = \frac{p_k}{r_k} \).

Since \( p_k \leq r_k \), we have \( 0 \leq \lambda_k \leq 1 \). Take \( 0 < \lambda < \lambda_k \). Define

\[
u_k = \begin{cases} \lambda_k & \text{if } t_k \geq 1 \\ 0 & \text{if } t_k < 1 \end{cases}
\]

and

\[
u_k = \begin{cases} 0 & \text{if } t_k \geq 1 \\ t_k & \text{if } t_k < 1 \end{cases}
\]

\( t_k = u_k + v_k \), \( t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k} \). It follows that \( u_k^{\lambda_k} \leq u_k \leq t_k \), \( v_k^{\lambda_k} \leq v_k^{\lambda_k} \).

Since \( t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k} \), then \( t_k^{\lambda_k} \leq t_k + v_k^{\lambda_k} \). So that

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|| (u_k \Delta^m x_k) ^\frac{1}{r} \|, z_1, \cdots, z_{n-1} ||}{\rho} \right) \right) \right]^{r_k} \lambda_k
\]

\leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|| (u_k \Delta^m x_k) ^\frac{1}{r} \|, z_1, \cdots, z_{n-1} ||}{\rho} \right) \right) \right]^{r_k}

This implies that

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|| (u_k \Delta^m x_k) ^\frac{1}{r} \|, z_1, \cdots, z_{n-1} ||}{\rho} \right) \right) \right]^{r_k} \frac{p_k}{r_k}
\]

\leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{|| (u_k \Delta^m x_k) ^\frac{1}{r} \|, z_1, \cdots, z_{n-1} ||}{\rho} \right) \right) \right]^{r_k}
\[ \Rightarrow \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \left\| \frac{u_k \Delta^m x_k}{\rho} \right\|^{\frac{1}{r}}, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \]

\[ \leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \left\| \frac{u_k \Delta^m x_k}{\rho} \right\|^{\frac{1}{r}}, z_1, \cdots, z_{n-1} \right) \right) \right]^{r_k}. \]

But

\[ \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \left\| \frac{u_k \Delta^m x_k}{\rho} \right\|^{\frac{1}{r}}, z_1, \cdots, z_{n-1} \right) \right) \right]^{r_k} \to 0 \text{ as } n \to \infty. \]

Therefore

\[ \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \left\| \frac{u_k \Delta^m x_k}{\rho} \right\|^{\frac{1}{r}}, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty. \]

Hence \( x = (x_k) \in \Gamma_F(\Delta^m, u, p, q, \| \cdot \|, \cdots, \| \cdot \|). \) Thus, we get

\[ \Gamma_F(\Delta^m, u, r, q, \| \cdot \|, \cdots, \| \cdot \|) \subset \Gamma_F(\Delta^m, u, p, q, \| \cdot \|, \cdots, \| \cdot \|). \]

**Theorem 3.6** (i) Let \( 0 < \inf p_k \leq p_k \leq 1. \) Then

\[ \Gamma_F(\Delta^m, u, p, q, \| \cdot \|, \cdots, \| \cdot \|) \subset \Gamma_F(\Delta^m, u, q, \| \cdot \|, \cdots, \| \cdot \|), \]

(ii) Let \( 1 \leq p_k \leq \sup p_k < \infty. \) Then

\[ \Gamma_F(\Delta^m, u, q, \| \cdot \|, \cdots, \| \cdot \|) \subset \Gamma_F(\Delta^m, u, p, q, \| \cdot \|, \cdots, \| \cdot \|). \]

**Proof.** (i) Let \( x = (x_k) \in \Gamma_F(\Delta^m, u, p, q, \| \cdot \|, \cdots, \| \cdot \|). \) Then

\[ \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \left\| \frac{u_k \Delta^m x_k}{\rho} \right\|^{\frac{1}{r}}, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty. \]

Since \( 0 < \inf p_k \leq p_k \leq 1, \)

\[ \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \left\| \frac{u_k \Delta^m x_k}{\rho} \right\|^{\frac{1}{r}}, z_1, \cdots, z_{n-1} \right) \right) \right] \]

\[ \leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \left\| \frac{u_k \Delta^m x_k}{\rho} \right\|^{\frac{1}{r}}, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty. \]
Thus, it follows that, $x = (x_k) \in \Gamma_F(\Delta_s^m, u, q, ||\cdot||, \ldots, ||\cdot||)$. Therefore $\Gamma_F(\Delta_s^m, u, p, q, ||\cdot||, \ldots, ||\cdot||) \subset \Gamma_F(\Delta_s^m, u, q, ||\cdot||, \ldots, ||\cdot||)$. 

(ii) Let $p_k \geq 1$ for each $k$ and sup $p_k < \infty$ and let

$$x = (x_k) \in \Gamma_F(\Delta_s^m, u, q, ||\cdot||, \ldots, ||\cdot||).$$

Then

$$\frac{1}{n} \sum_{k=1}^{n} \left[f_k \left( q \left( \frac{\|u_k \Delta_s^m x_k\|^{\frac{1}{k}}}{\rho}, z_1, \ldots, z_{n-1} \right) \right) \right] \to 0 \text{ as } n \to \infty.$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[f_k \left( q \left( \frac{\|u_k \Delta_s^m x_k\|^{\frac{1}{k}}}{\rho}, z_1, \ldots, z_{n-1} \right) \right) \right] \leq \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left( q \left( \frac{\|u_k \Delta_s^m x_k\|^{\frac{1}{k}}}{\rho}, z_1, \ldots, z_{n-1} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty.$$

Hence

$$\frac{1}{n} \sum_{k=1}^{n} \left[f_k \left( q \left( \frac{\|u_k \Delta_s^m x_k\|^{\frac{1}{k}}}{\rho}, z_1, \ldots, z_{n-1} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty.$$

This implies that $x = (x_k) \in \Gamma_F(\Delta_s^m, u, p, q, ||\cdot||, \ldots, ||\cdot||)$. Therefore $\Gamma_F(\Delta_s^m, u, q, ||\cdot||, \ldots, ||\cdot||) \subset \Gamma_F(\Delta_s^m, u, p, q, ||\cdot||, \ldots, ||\cdot||)$.

**Theorem 3.7** Suppose

$$\frac{1}{n} \sum_{k=1}^{n} \left[f_k \left( q \left( \frac{\|u_k \Delta_s^m x_k\|^{\frac{1}{k}}}{\rho}, z_1, \ldots, z_{n-1} \right) \right) \right]^{p_k} \leq |x_k|^{1/k},$$

then $\Gamma \subset \Gamma_F(\Delta_s^m, u, p, q, ||\cdot||, \ldots, ||\cdot||)$.

**Proof.** Let $x = (x_k) \in \Gamma$. Then we have,

$$|x_k|^{1/k} \to 0 \text{ as } k \to \infty.$$

But

$$\frac{1}{n} \sum_{k=1}^{n} \left[f_k \left( q \left( \frac{\|u_k \Delta_s^m x_k\|^{\frac{1}{k}}}{\rho}, z_1, \ldots, z_{n-1} \right) \right) \right]^{p_k} \leq |x_k|^{1/k},$$

by our assumption, implies that

$$\frac{1}{n} \sum_{k=1}^{n} \left[f_k \left( q \left( \frac{\|u_k \Delta_s^m x_k\|^{\frac{1}{k}}}{\rho}, z_1, \ldots, z_{n-1} \right) \right) \right]^{p_k} \to 0 \text{ as } n \to \infty \text{ by (10)}.$$
Then \( x = (x_k) \in \Gamma_F(\Delta^m, u, p, q, ||., .||) \) and \( \Gamma \subset \Gamma_F(\Delta^m, u, p, q, ||., .||) \).

**Theorem 3.8** \( \Gamma_F(\Delta^m, u, p, q, ||., .||) \) is solid.

**Proof.** Let \( |x_k| \leq |y_k| \) and let \( y = (y_k) \in \Gamma_F(\Delta^m, u, p, q, ||., .||) \), because \( F = (f_k) \) is non-decreasing, so that

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{\|u_k \Delta^m x_k\|^\frac{1}{r}}{\rho}, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \\
\leq \frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{\|u_k \Delta^m y_k\|^\frac{1}{r}}{\rho}, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k}
\]

Since \( y \in \Gamma_F(\Delta^m, u, p, q, ||., .||) \). Therefore,

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{\|u_k \Delta^m y_k\|^\frac{1}{r}}{\rho}, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \rightarrow 0 \text{ as } n \rightarrow \infty
\]

and

\[
\frac{1}{n} \sum_{k=1}^{n} \left[ f_k \left( q \left( \frac{\|u_k \Delta^m x_k\|^\frac{1}{r}}{\rho}, z_1, \cdots, z_{n-1} \right) \right) \right]^{p_k} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Therefore \( x = (x_k) \in \Gamma_F(\Delta^m, u, p, q) \).

**Theorem 3.9** \( \Gamma_F(\Delta^m, u, p, q, ||., .||) \) is monotone.

**Proof.** It is trivial so we omit it.

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