Bounded linear operators for some new matrix transformations

M. AIYUB
University of Bahrain, Kingdom of Bahrain
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Abstract
In this paper, we define \((\sigma, \theta)\)-convergence and characterize \((\sigma, \theta)\)-
conservative, \((\sigma, \theta)\)-regular, \((\sigma, \theta)\)-coercive matrices and we also deter-
mine the associated bounded linear operators for these matrix classes.

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trix transformation; bounded linear operators.
1. Introduction and preliminaries

We shall write \( w \) for the set of all complex sequences \( x = (x_k)_{k=0}^\infty \). Let \( \phi, \ell_\infty, c \) and \( c_0 \) denote the sets of all finite, bounded, convergent and null sequences respectively; and \( cs \) be the set of all convergent series. We write \( \ell_p := \{ x \in w : \sum_{k=0}^\infty |x_k|^p < \infty \} \) for \( 1 \leq p < \infty \). By \( e \) and \( e^{(n)}(n \in \mathbb{N}) \), we denote the sequences such that \( e_k = 1 \) for \( k = 0, 1, \ldots \), and \( e^{(n)}_0 = 1 \) and \( e^{(n)}_k = 0(k \neq n) \). For any sequence \( x = (x_k)_{k=0}^\infty \), let \( x^{[n]} = \sum_{k=0}^n x_k e^{(k)} \) be its \( n \)-section.

Note that \( c_0, c \), and \( \ell_\infty \) are Banach spaces with the sup-norm \( \| x \|_\infty = \sup_k |x_k| \), and \( \ell_p(1 \leq p < \infty) \) are Banach spaces with the norm \( \| x \|_p = (\sum |x_k|^p)^{1/p} \); while \( \phi \) is not a Banach space with respect to any norm.

A sequence \( (b^{(n)}_{(n)})_{n=0}^\infty \) in a linear metric space \( X \) is called Schauder basis if for every \( x \in X \), there is a unique sequence \( (\beta_n)_{n=0}^\infty \) of scalars such that \( x = \sum_{n=0}^\infty \beta_n b^{(n)} \).

Let \( X \) and \( Y \) be two sequence spaces and \( A = (a_{nk})_{n,k=1}^\infty \) be an infinite matrix of real or complex numbers. We write \( Ax = (A_n(x)) \), \( A_n(x) = \sum_k a_{nk}x_k \) provided that the series on the right converges for each \( n \). If \( x = (x_k) \in X \) implies that \( Ax \in Y \), then we say that \( A \) defines a matrix transformation from \( X \) into \( Y \) and by \( (X, Y) \) we denote the class of such matrices.

Let \( \sigma \) be a one-to-one mapping from the set \( \mathbb{N} \) of natural numbers into itself. A continuous linear functional \( \varphi \) on the space \( \ell_\infty \) is said to be an invariant mean or a \( \sigma \)-mean if and only if (i) \( \varphi(x) \geq 0 \) if \( x \geq 0 \) (i.e. \( x_k \geq 0 \) for all \( k \)), (ii) \( \varphi(e) = 1 \), where \( e = (1, 1, 1, \cdots) \), (iii) \( \varphi(x) = \varphi((x_{\sigma(k)})) \) for all \( x \in \ell_\infty \).

Throughout this paper we consider the mapping \( \sigma \) which has no finite orbits, that is, \( \sigma^p(k) \neq k \) for all integer \( k \geq 0 \) and \( p \geq 1 \), where \( \sigma^p(k) \) denotes the \( p \)th iterate of \( \sigma \) at \( k \). Note that, a \( \sigma \)-mean extends the limit functional on the space \( c \) in the sense that \( \varphi(x) = \lim x \) for all \( x \in c \), (cf [10]). Consequently, \( c \subset V_\sigma \), the set of bounded sequences all of whose \( \sigma \)-means are equal. We say that a sequence \( x = (x_k) \) is \( \sigma \)-convergent if and only if \( x \in V_\sigma \).

\[ V_\sigma = \{ x \in \ell_\infty : \lim_{p \to \infty} t_{pn}(x) = L, \text{ uniformly in } n \}. \]

where \( L = \sigma - \lim x \), where
\[ t_{pn}(x) = \frac{1}{p+1} \sum_{m=0}^p x_{\sigma^m(n)}. \]

Using the concept of Schaefer [17] defined and characterized the \( \sigma \)-conservative, \( \sigma \)-regular and \( \sigma \)-coercive matrices. If \( \sigma \) is translation then
the $\sigma$-mean often called Banach Limit [2] and the set $V_\sigma$ reduces to the set $f$ of almost convergent sequence studied by Lorenz [9]. By a lacunary sequence we mean an increasing sequence $\theta = (k_r)$ of integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by $\theta$ will be denoted by $I_r := (k_{r-1} - k_r]$, and the ratio $k_r/k_{r-1}$ will be abbreviated by $q_r$ (see Fredman et al[8]). Recently, Aydin[1] defined the concept of almost lacunary convergent as follow: A bounded sequence $x = (x_k)$ is said be almost lacunary convergent to the number $c$ if and only if
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{j \in I_r} x_{j+n} = \ell , \text{ uniformly in } n.
\]

The idea of $\sigma$-convergence for double sequences was introduced in [4] and further studied recently in [3] and [15]. In [11]-[14] we study various classes of four dimensional matrices, e.g. $\sigma$-regular, $\sigma$-conservative, regularly $\sigma$-conservative, boundedly $\sigma$-conservative and $\sigma$-coercive matrices.

In this paper, we define $(\sigma, \theta)$-convergence. We also generalize the above matrices by characterizing the $(\sigma, \theta)$-conservative, $(\sigma, \theta)$-regular and $(\sigma, \theta)$-coercive matrices. Further, we also determine the associated bounded linear operators for these matrix classes, which is the generalized result of Mur-saleen, M.A. Jarrah and S.Mouhiddin see ref [15]

2. $(\sigma, \theta)$ -Lacunary convergent sequences

We define the following:

**Definition 2.1.** [Sir paper, 2009] A bounded sequence $x = (x_k)$ of real numbers is said to be $(\sigma, \theta)$-lacunary convergent to a number $\ell$ if and only if
\[
\lim_{r \to \infty} \frac{1}{h_r} \sum_{j \in I_r} x_{\sigma_j(n)} = \ell , \text{ uniformly in } n, \text{ and let } V_\sigma(\theta), \text{ denote the set of all such sequences, i.e where}
\]
\[
V_\sigma(\theta) = \{ x \in \ell_\infty : \lim_{r \to \infty} \frac{1}{h_r} \sum_{j \in I_r} x_{\sigma_j(n)} = \ell , \text{ uniformly in } n \}
\]

Note that for $\sigma(n) = n + 1$, $\sigma$-lacunary convergence is reduced to almost lacunary convergence. Results similar to that Aydin[1] can easily be proved for the space $V_\sigma(\theta)$.

**Definition 2.2.** A bounded sequence $x = (x_k)$ of real numbers is said to be $\sigma$-lacunary bounded if and only if $\sup_{r,n} \left| \frac{1}{h_r} \sum_{j \in I_r} x_{\sigma_j(n)} \right| < \infty$, and we let $V_\sigma^\infty(\theta)$, denot the set of all such sequences
\[
V_\sigma^\infty(\theta) = \{ x \in \ell_\infty : \sup_{r,n} \left| \tau_{r,n}(x) \right| < \infty \}.
\]
Where
\[ \tau_{tn}(x) = \frac{1}{h_r} \sum_{j \in I_r} x_{\sigma_j(n)}, \]
Note that \( c \subset V_\sigma(\theta) \subset V_\sigma^\infty(\theta) \subset \ell_\infty. \)

**Definition 2.3.** An infinite matrix \( A = (a_{nk}) \) is said to be \((\sigma, \theta)\)-conservative if and only if \( Ax \in V_\sigma(\theta) \) for all \( x = (x_k) \in c \) and we denote this by \( A \in (c, V_\sigma(\theta)) \).

**Definition 2.4.** We say that, infinite matrix \( A = (a_{nk}) \) is said to be \((\sigma, \theta)\)-regular if and only if it is \( V_\sigma(\theta) \)-conservative and \((\sigma, \theta)\)-lim \( Ax = \lim x \) for all \( x \in c \) and we denote this by \( A \in (c, V_\sigma(\theta))_{\text{reg}} \).

**Definition 2.5.** A matrix \( A = (a_{nk}) \) is said to be \((\sigma, \theta)\)-coercive if and only if \( Ax \in V_\sigma(\theta) \) for all \( x = (x_k) \in \ell_\infty \) and we denote this by \( A \in (\ell_\infty, V_\sigma(\theta)) \).

**Remark 2.6.** If we take \( h_r = r \) then \( V_\sigma(\theta) \) is reduced to the space \( V_\sigma \) and \((\sigma, \theta)\)-conservative, \((\sigma, \theta)\)-regular, \((\sigma, \theta)\)-coercive matrices are respectively reduced to \( \sigma \)-conservative, \( \sigma \)-regular, \( \sigma \)-coercive matrices (cf [15]); and in addition if \( \sigma(n) = n + 1 \) then the space \( V_\sigma(\theta) \) is reduced to the space \( f \) of almost convergent sequences (cf [9]) and these matrices are reduced to the almost conservative, almost regular (cf [7]) and almost coercive matrices respectively (cf [6]).

3. \((\sigma, \theta)\)-conservative matrices and bounded linear operators

In the following theorem we characterize \((\sigma, \theta)\)-conservative matrices and find the associated bounded linear operator.

**Theorem 3.1.** A matrix \( A = (a_{nk}) \) is \((\sigma, \theta)\)-conservative, i.e. \( A \in (c, V_\sigma(\theta)) \) if and only if it satisfies the condition

\begin{enumerate}
  \item \( \|A\| = \sup_n \sum_k |a_{nk}| < \infty; \)
  \item \( a_{(k)} = (a_{nk})_{n=1}^\infty \in V_\sigma(\theta) \), for each \( k \);
  \item \( a = \left( \sum_k a_{nk} \right)_{n=1}^\infty \in V_\sigma(\theta). \)
\end{enumerate}

In this case, the \((\sigma, \theta)\)-limit of \( Ax \) is \( \lim x \left[ u - \sum_k u_k \right] + \sum_k x_k u_k \), where \( u = (\sigma, \theta)\)-lim \( a \) and \( u_k = (\sigma, \theta)\)-lim \( a_k \), \( k = 1, 2, \cdots \).
Proof. Sufficiency. Let the conditions hold. Let $r$ be any non-negative integer and $x = (x_k) \in c$. For every positive integer $n$; write \[ \tau_{r_n}(x) = \frac{1}{n^r} \sum_{k=1}^{\infty} \sum_{j \in I_r} a_{\sigma^i(n),k} x_k \] Then we have \[ -\tau_{r_n}(x) \leq \frac{1}{n^r} \sum_{k=1}^{\infty} \sum_{j \in I_r} |a_{\sigma^i(n),k}| \|x_k\| \leq \frac{\|\|\|}{n^r} \sum_{k=1}^{\infty} \sum_{j \in I_r} |a_{\sigma^i(n),k}| \|x\| \]. Since $\tau_{r_n}$ is obviously linear on $c$, it follows that $\tau_{r_n} \in c'$ and $\|\tau_{r_n}\| \leq \|A\|$. 

Now, \[ \tau_{r_n}(e) = \frac{1}{n^r} \sum_{k=1}^{\infty} \sum_{j \in I_r} a_{\sigma^i(n),k} = \frac{1}{n^r} \sum_{k=1}^{\infty} \sum_{j \in I_r} a_{\sigma^i(n),k} \text{that is,} \lim_{r} \tau_{r_n}(e) \text{ exists uniformly in } n \text{ and } \lim_{r} \tau_{r_n}(e) = u \text{ uniformly in } n, \text{ the } (\sigma, \theta)\text{-limit of } a \text{, since } a \in V^\sigma_\sigma(\theta) \text{. Similarly, } \lim_{r} \tau_{r_n}e^k = u_k, \text{ the } (\sigma, \theta)\text{-limit of } a_{(k)} \text{ for each } k \text{, uniformly in } n \text{. Since } \{e,e^1,e^2,\cdots\} \text{ is a fundamental set in } c \text{, and } \sup_\tau |\tau_{r_n}(x)| \text{ is finite for each } x \in c \text{, it follows that } \lim_{r} \tau_{r_n}(x) = \tau_n(x) \text{, exists for all } x \in c \text{ (cf [5]). Furthermore, } \|\tau_n\| \leq \liminf_{r} [\|\tau_{r_n}\| \leq \|A\| \text{ for each } n \text{ and } \tau_n \in c'. Thus, each } x \in c \text{ has a unique representation } \quad x = (\lim x) \left[ e - \sum_k e_k \right] + \sum_k x_k e_k \tau_n(x) = (\lim x) \left[ t_n(e) - \sum_k t_n(e_k) \right] + \sum_k x_k t_n(e_k) \tau_n(x) = (\lim x) \left[ u - \sum_k u_k \right] + \sum_k x_k u_k. \text{By } L(x), \text{ we denote the right hand side of the above expression which is independent of } n. \text{ Now, we have to show that } \lim_{r} \tau_{r_n}(x) = L(x) \text{ uniformly in } n. \text{ Put } F_{r_n}(x) = \tau_{r_n}(x) - L(x). \text{Then } F_{r_n} \in c', \|F_{r_n}\| \leq 2\|A\| \text{ for all } r, n, \lim_{r} F_{r_n}(e) = 0 \text{ uniformly in } n, \text{ and } \lim_{r} F_{r_n}(e^k) = 0 \text{ uniformly in } n \text{ for each } k. \text{ Let } K \text{ be an arbitrary positive integer. Then } x = (\lim x) e + \sum_{k=1}^{K}(x_k - \lim x)e^k + \sum_{k=K+1}^{\infty}(x_k - \lim x)e^k. \text{Now applying } F_{r_n} \text{ on both sides of the above equality, we have } F_{r_n}(x) = (\lim x) F_{r_n}(e) + \sum_{k=1}^{K}(x_k - \lim x) F_{r_n}(e^k) + F_{r_n} \left( \sum_{k=K+1}^{\infty}(x_k - \lim x)e^k \right). (3.1.1) \text{Now, } \left| F_{r_n} \left( \sum_{k=K+1}^{\infty}(x_k - \lim x)e^k \right) \right| \leq 2\|A\| \sum_{k\geq K+1}|x_k - \lim x|, \text{ for all } r, n. \text{ After choosing fixed } K \text{ large enough, it is easy to see that the absolute value of each term on the right hand side of (3.1.1) can be made uniformly small for all sufficiently large } r. \text{ Therefore, } \lim_{r} F_{r_n}(x) = 0 \text{ uniformly in } n; \text{ so that } Ax \in V^\sigma_\sigma(\theta) \text{ and the matrix } A \text{ is } (\sigma, \theta)\text{-conservative.} \n
Necessity. Suppose that } A \text{ is } (\sigma, \theta)\text{-conservative. Then } Ax = (A_n(x))_{n=1}^{\infty} = \left( \sum_k a_{nk}x_k \right)_{n=1}^{\infty} \in V^\sigma_\sigma(\theta), \text{ for all } x \in c. \text{ Let } x = (x_k) = e^k. \text{ Therefore } (\sigma, \theta)\text{-lim}_n \sum_k a_{nk}e^k = (\sigma, \theta)\text{-lim}_n a_{nk} = a_{(k)}. \text{Hence } (ii) \text{ holds. Now, let } x = e. \text{ Then } (\sigma, \theta)\text{-lim}_n \sum_k a_{nk}e = (\sigma, \theta)\text{-lim}_n \sum_k a_{nk} = a, \text{ so that } (iii) \text{ must hold. Since } Ax = (A_n(x)) \in V^\sigma_\sigma(\theta) \subset c_\infty. \text{ It follows that } \sup_n |A_n(x)| < \infty.
$\infty, (A_n)$ is a sequence of bounded operators. Therefore, by Banach-Steinhaus theorem, $\sup_n |A_n| < \infty$, which implies $\sup_n \sum_k |a_{nk}| < \infty$ and hence $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$, i.e. (i).

This completes the proof of the theorem.

Now, we deduce the following.

**Corollary 3.2.** $A = (a_{nk})$ is $(\sigma, \theta)$-regular if and only if the conditions (i), (ii) with $(\sigma, \theta)$-limit zero for each $k$, and (iii) with $(\sigma, \theta)$-limit 1 of Theorem 3.1 hold.

**Proof.** For $x \in c$, $(\sigma, \theta)$- $Ax = L(x)$, which reduces to $\lim x$, since $u = 1$ and $u_k = 0$ for each $k$. Hence $A$ is $(\sigma, \theta)$-regular.

Conversely, let $A$ be $(\sigma, \theta)$-regular. Then $(\sigma, \theta)$- $Ae = 1 = (\sigma, \theta)$- $Aa$, $(\sigma, \theta)$- $Ae_k = 0 = (\sigma, \theta)$- $A(a_k)$ and $\|A\|$ is finite as condition (i) of Theorem 3.1.

This completes the proof of the Corollary 3.2.

**4. $(\sigma, \theta)$-coercive matrices**

We use the following lemma in our next theorem.

**Lemma 4.1.** Let $B(n) = (b_{mk}(n))$, $n = 0, 1, 2, \cdots$ be a sequence of infinite matrices such that

(i) $\|B(n)\| < H < +\infty$ for all $n$; and

(ii) $\lim b_{mk}(n) = 0$ for each $k$, uniformly in $n$.

Then $\lim \sum b_{mk}(n)x_k = 0$ uniformly in $n$ for each $x \in \ell_\infty(4.1.1)$ if and only if $\lim \sum |b_{mk}(n)| = 0$ uniformly in $n.(4.1.2)$

**Theorem 4.2.** A matrix $A = (a_{nk})$ is $(\sigma, \theta)$-coercive, i.e. $A \in (\ell_\infty, V_\sigma(\theta))$ if and only if (i) and (ii) of Theorem 3.1 hold, and

(iii) $\lim_r \sum_{k=1}^\infty \sum_{j \in I_r} a_{\sigma j(n),k} - u_k$ uniformly in $n$.

In this case, the $(\sigma, \theta)$-limit of $Ax$ is $\sum_k x_k u_k \forall x \in \ell_\infty$, where $u_k = (\sigma, \theta)$- $a_k$.

**Proof.** Sufficiency. Let the conditions hold. For any positive integer $K$ $\sum_{k=1}^K |u_k| = \sum_{k=1}^K \sum_{j \in I_r} |a_{\sigma j(n),k} - h_r| \le \sum_{j \in I_r} \sum_{k=1}^K |a_{\sigma j(n),k}|$
\[
\limsup_r \sum_{j \in I_r} \sum_{k=1}^\infty |a_{\sigma j(n),k}| h_r \leq \|A\|. \text{ This shows that } \sum_{k=1}^\infty |u_k| \text{ converges, and that } \sum_{k=1}^\infty u_kx_k \text{ is defined for every } x = (x_k) \in \ell_\infty.
\]

Let \( x = (x_k) \) be any arbitrary bounded sequence. For every positive integer \( r \)
\[
\| \sum_{k=1}^\infty \left( \frac{1}{\hbar_r} \sum_{j \in I_r} a_{\sigma j(n),k} - u_k \right) x_k \| = \left\| \sum_{k=1}^\infty \sum_{j \in I_r} \frac{|a_{\sigma j(n),k} - u_k|}{\hbar_r} x_k \right\|
\leq \sup_{n} \left[ \sum_{k=1}^\infty \sum_{j \in I_r} \frac{|a_{\sigma j(n),k} - u_k|}{\hbar_r} \right] \leq \|x\| \sup_{n} \sum_{k=1}^\infty \sum_{j \in I_r} |a_{\sigma j(n),k} - u_k| / \hbar_r
\]

Letting \( r \to \infty \) and using condition (iii), we get
\[
\frac{1}{\hbar_r} \sum_{k=1}^\infty \sum_{j \in I_r} a_{\sigma j(n),k} x_k \to \sum_{k=1}^\infty u_kx_k.
\]

Hence \( Ax \in V_{\sigma}(\theta) \) with \((\sigma, \theta)\)-limit \( \sum_{k=1}^\infty u_kx_k \).

**Necessity.** Let \( A \) be \((\sigma, \theta)\)-coercive matrix. This implies that \( A \) is \((\sigma, \theta)\)-

conservative, then we have condition (i) and (ii) from Theorem 3.1. Now we have to show that (iii) holds.

Suppose that for some \( n \), we have \( \limsup_r \sum_{k=1}^\infty \sum_{j \in I_r} |a_{\sigma j(n),k} - u_k| / \hbar_r = N > 0 \). Since \( \|A\| \) is finite, therefore \( N \) is also finite. We observe that since \( \sum_{k=1}^\infty |u_k| < +\infty \) and \( A \) is \((\sigma, \theta)\)-coercive, the matrix \( B = (b_{nk}) \), where \( b_{nk} = a_{nk} - u_k \), is also \((\sigma, \theta)\)-coercive matrix. By an argument similar to

that of Theorem 2.1 in [6], one can find \( x \in \ell_\infty \) for which \( Bx \notin V_{\sigma}(\theta) \). This contradiction implies the necessity of (iii).

Now, we use Lemma 4.1 to show that this convergence is uniform in \( n \). Let \( t_{rk}(n) = \sum_{j \in I_r} |a_{\sigma j(n),k} - u_k| / \hbar_r \) and let \( T(n) \) be the matrix \( (t_{rk}(n)) \).

It is easy to see that \( \|H(n)\| \leq 2\|A\| \) for every \( n \); and from condition (ii) \( \lim t_{rk}(n) = 0 \) for each \( k \), uniformly in \( n \). For any \( x \in \ell_\infty \),

\[
\lim_{r} \sum_{j \in I_r} t_{rk}(n)x_k = (\sigma, \theta)\text{-lim} Ax - \sum_{k=1}^\infty u_kx_k
\]

and the limit exists uniformly in \( n \), since \( Ax \in V_{\sigma}(\theta) \). Moreover, this limit is zero since \( \left| \sum_{k=1}^\infty t_{rk}(n)x_k \right| \leq
\[ \|x\| \sum_{k=1}^{\infty} \left| \sum_{j \in J_k} [a_{\sigma'(n), k} - u_k] \right| = h_r. \] Hence \( \lim_{r \to \infty} \sum_{k=1}^{\infty} |t_{rk}(n)| = 0 \) uniformly in \( n \); i.e. the condition (iii) holds.

This completes the proof of the theorem.

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References


M. Aiyub
Department of Mathematics,
University of Bahrain,
P.O. Box-32038,
Kingdom of Bahrain
e-mail : maiyub2002@yahoo.com