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On some refinements of companions of Fejér's inequality via superquadratic functions

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Abstract

In this paper some companions of Fejér's inequality for superquadratic functions are given, we also get refinements of some known results proved in [18].

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1. Introduction

Let $\emptyset \neq I \subseteq R$, $a, b \in I$ with $a < b$, let $f : I \rightarrow R$ be a convex function and $p : [a, b] \rightarrow R$ be a non-negative integrable and symmetric about $x = \frac{a+b}{2}$. The following two inequalities are of great significance in literature: the first known as Hermite-Hadamard inequality:

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

with the reversed inequality for the concave function f , and the second, known as Fejér's inequality:

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \int_a^b p(x)dx \leq \int_a^b f(x)p(x)dx \leq \frac{f(a)+f(b)}{2} \int_a^b p(x)dx.$$

These inequalities attracted the attention of many mathematicians over the decades and they generalize, improve and extend these inequalities in a number of ways, see [6, 7, 8, 9, 11, 19]. Let us now define some mappings and quote the results established by K.L. Tseng, S. R. Hwang and S.S. Dragomir in [18]:

$$G(t) = \frac{1}{2} \left[f\left(ta + (1-t)\frac{a+b}{2}\right) + f\left(tb + (1-t)\frac{a+b}{2}\right) \right],$$

$$H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx,$$

$$I(t) = \int_a^b \frac{1}{2} \left[f\left(t\frac{x+a}{2} + (1-t)\frac{a+b}{2}\right) + f\left(t\frac{x+b}{2} + (1-t)\frac{a+b}{2}\right) \right] p(x)dx,$$

$$L_p(t) = \frac{1}{2} \int_a^b [f(ta + (1-t)x) + f(tb + (1-t)x)] p(x)dx,$$

$$L_p(t) = \frac{1}{2(b-a)} \int_a^b [f(ta + (1-t)x) + f(tb + (1-t)x)] dx$$

and

$$S_p(t) = \frac{1}{4} \int_a^b \left[f\left(ta + (1-t)\frac{x+a}{2}\right) + f\left(ta + (1-t)\frac{x+b}{2}\right) \right. \\ \left. + f\left(tb + (1-t)\frac{x+a}{2}\right) + f\left(tb + (1-t)\frac{x+b}{2}\right) \right] p(x) dx,$$

where $f : [a, b] \rightarrow R$ is a convex function and $p : [a, b] \rightarrow R$ is non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $t \in [0, 1]$.

Now we quote some results from [18]:

Theorem 1. [18] Let f, p, I be defined as above. Then:

1. The following inequality holds:

$$(1.3) \quad f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \\ \leq 2 \left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)p(4x-2a-b)dx + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)p(4x-a-2b)dx \right] \\ \leq \int_0^1 I(t) dt \leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \right. \\ \left. + \int_a^b \frac{1}{2} \left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right) \right] p(x) dx \right].$$

2. If f is differentiable on $[a, b]$ and p is bounded on $[a, b]$, then for all $t \in [0, 1]$ we have the inequality

$$(1.4) \quad 0 \leq \int_a^b \frac{1}{2} \left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right) \right] p(x) dx - I(t) \\ \leq (1-t) \left[\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)dx \right] \|p\|_\infty,$$

where $\|p\|_\infty = \sup_{x \in [a, b]} |p(x)|$.

3. If f is differentiable on $[a, b]$, then for all $t \in [0, 1]$ we have the inequality

$$(1.5) \quad 0 \leq \frac{f(a) + f(b)}{2} \int_a^b p(x) dx - I(t) \leq \frac{(f'(b) - f'(a))(b - a)}{4} \int_a^b p(x) dx.$$

Theorem 2. [18] Let f, p, G, I be defined as above. Then:

1. The following inequality holds for all $t \in [0, 1]$:

$$(1.6) \quad I(t) \leq G(t) \int_a^b p(x) dx.$$

2. If f is differentiable on $[a, b]$ and p is bounded on $[a, b]$, then for all $t \in [0, 1]$ we have the inequality

$$(1.7) \quad 0 \leq I(t) - f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \leq (b-a) [G(t) - H(t)] \|p\|_{\infty},$$

where $\|p\|_{\infty} = \sup_{x \in [a, b]} |p(x)|$.

Theorem 3. [18] Let f, p, G, I, S_p be defined as above. Then we have the following results:

1. S_p is convex on $[0, 1]$.
2. The following inequalities hold for all $t \in [0, 1]$:

$$(1.8) \quad G(t) \int_a^b p(x) dx \leq S_p(t) \\ \leq (1-t) \int_a^b \frac{1}{2} \left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right) \right] p(x) dx$$

$$+t \cdot \frac{f(a)+f(b)}{2} \int_a^b p(x) dx \leq \frac{f(a)+f(b)}{2} \int_a^b p(x) dx,$$

$$(1.9) \quad I(1-t) \leq S_p(t)$$

and

$$(1.10) \quad \frac{I(t) + I(1-t)}{2} \leq S_p(t).$$

3. The following equality holds:

$$(1.11) \quad \sup_{x \in [0,1]} S_p(t) = S_p(1) = \frac{f(a)+f(b)}{2} \int_a^b p(x) dx.$$

They used the following Lemma to prove the above results:

Lemma 4. [17, p. 3] $f : [a, b] \longrightarrow R$ be convex function and let $a \leq A \leq C \leq D \leq B \leq b$ with $A + B = C + D$. Then

$$f(A) + f(B) \leq f(C) + f(D).$$

Let us now recall the definition, some of the properties and results related to superquadratic functions to be used in the sequel.

Definition 5. [3, Defintion 2.1] Let $I = [0, a]$ or $[0, \infty)$ be an interval in R . A function $f : I \longrightarrow R$ is superquadratic if for each x in I there exists a real number $C(x)$ such that

$$(1.12) \quad f(y) - f(x) \geq C(x)(y - x) + f(|y - x|)$$

for all $y \in I$. If $-f$ is superquadratic then f is called subquadratic.

For examples of superquadratic functions see [2, p. 1049].

Theorem 6. [3, Theorem 2.3] The inequality

$$(1.13) \quad f\left(\int g d\mu\right) \leq \int \left(f(g(s)) - f\left(\left|g(s) - \int g d\mu\right|\right)\right) d\mu(s)$$

holds for all probability measure μ and all non-negative μ -integrable function g , if and only if f is superquadratic.

The following discrete version of the above theorem will be helpful in the sequel of the paper:

Lemma 7. [2, Lemma A, p.1049] Suppose that f is superquadratic. Let $x_r \geq 0$, $1 \leq r \leq n$, and let $\bar{x} = \sum_{r=1}^n \lambda_r x_r$ where $\lambda_r \geq 0$ and $\sum_{r=1}^n \lambda_r = 1$. Then

$$(1.14) \quad \sum_{r=1}^n \lambda_r f(x_r) \geq f(\bar{x}) + \sum_{r=1}^n \lambda_r f(|x_r - \bar{x}|).$$

The following Lemma shows that positive superquadratic functions are also convex:

Lemma 8. [3, Lemma 2.2] Let f be superquadratic function with $C(x)$ as in Definition 1. Then

1. $f(0) \leq 0$.
2. If $f(0) = f'(0) = 0$ then $C(x) = f'(x)$ whenever f is differentiable at $x > 0$.
3. If $f \geq 0$, then f convex and $f(0) = f'(0) = 0$.

In [4] a converse of Jensen's inequality for superquadratic functions was proved:

Theorem 9. [4, Theorem 1] Let (Ω, A, μ) be a measurable space with $0 < \mu(\Omega) < \infty$ and let $f : [0, \infty) \rightarrow R$ be a superquadratic function. If $g : \Omega \rightarrow [m, M] \subseteq [0, \infty)$ is such that $g, f \circ g \in L_1(\mu)$, then we have for $\bar{g} = \frac{1}{\mu(\Omega)} \int g d\mu$,

$$(1.15) \quad \frac{1}{\mu(\Omega)} \int f(g) d\mu \leq \frac{M - \bar{g}}{M - m} f(m) + \frac{\bar{g} - m}{M - m} f(M) \\ - \frac{1}{\mu(\Omega)} \frac{1}{M - m} \int ((M - g) f(g - m) + (g - m) f(M - g)) d\mu.$$

The discrete version of this theorem is:

Theorem 10. [4, Theorem 2] Let $f : [0, \infty) \rightarrow R$ be a superquadratic function. Let $(x_1, \dots, x_n)$ be an n -tuple in $[m, M]^n$ ($0 \leq m \leq M < \infty$), and $(p_1, \dots, p_n)$ be a non-negative n -tuple such that $P_n = \sum_{i=1}^n p_i > 0$. Denote $\bar{x} = \frac{1}{P_n} \sum_{i=1}^n p_i x_i$, then

$$(1.16) \quad \frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) \leq \frac{M - \bar{x}}{M - m} f(m) + \frac{\bar{x} - m}{M - m} f(M) \\ - \frac{1}{P_n (M - m)} \sum_{i=1}^n p_i [(M - x_i) f(x_i - m) + (x_i - m) f(M - x_i)]$$

For recent results on Fejér and Hermite-Hadamard type inequalities for superquadratic functions, we refer interested readers to [4], [5] and [2]. In this paper we deal with mappings $G(t)$, $I(t)$, $S_p(t)$ and $L(t)$ when f is superquadratic function. In case when superquadratic function f is also non-negative and hence convex we get refinements of some parts of Theorem 1, Theorem 2 and of Theorem 3.

2. Main Results

In this section we prove our main results by using the same techniques as used in [17] and [2]. Moreover, we assume that all the considered integrals in this section exist.

In order to prove our main results we go through some calculations. From Lemma 2 and Theorem 6 for $n = 2$, we get that

$$f(z) \leq \frac{M - z}{M - m} f(m) + \frac{z - m}{M - m} f(M) - \frac{M - z}{M - m} f(z - m) - \frac{z - m}{M - m} f(M - z) \quad (2.1)$$

and

$$f(M + m - z) \leq \frac{z - m}{M - m} f(m) + \frac{M - z}{M - m} f(M) - \frac{z - m}{M - m} f(M - z) - \frac{M - z}{M - m} f(z - m) \quad (2.2)$$

hold for superquadratic function f , $0 \leq m \leq z \leq M$, $m < M$.

Therefore from (2.1) and (2.2), we have

$$f(z) + f(M + m - z) \leq f(m) + f(M) - 2 \frac{z - m}{M - m} f(M - z) - 2 \frac{M - z}{M - m} f(z - m). \quad (2.3)$$

Now for $0 \leq t \leq \frac{1}{2}$ and $0 \leq a \leq x \leq \frac{a+b}{2}$, we obtain from (2.3) the following inequalities:

By setting $z = \frac{a+b}{2}$, $M = \frac{3(a+b)}{4} - \frac{x}{2}$, $m = \frac{x}{2} + \frac{a+b}{4}$ in (2.3), we have that

$$2f\left(\frac{a+b}{2}\right) \leq f\left(\frac{x}{2} + \frac{a+b}{4}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}\right) - 2f\left(\frac{1}{2}\left(\frac{a+b}{2} - x\right)\right) \quad (2.4)$$

holds.

Also, by replacing $z = \frac{x}{2} + \frac{a+b}{4}$, $M = tx + (1-t)\frac{a+b}{2}$, $m = t\frac{a+b}{2} + (1-t)x$ in (2.3), we get that

$$2f\left(\frac{x}{2} + \frac{a+b}{4}\right) \leq f\left(t\frac{a+b}{2} + (1-t)x\right) + f\left(tx + (1-t)\frac{a+b}{2}\right) - 2f\left(\left(\frac{1}{2} - t\right)\left(\frac{a+b}{2} - x\right)\right) \quad (2.5)$$

holds.

Further, for $z = \frac{3(a+b)}{4} - \frac{x}{2}$, $M = t\frac{a+b}{2} + (1-t)(a+b-x)$, $m = t(a+b-x) + (1-t)\frac{a+b}{2}$ in (2.3), we observe that

$$2f\left(\frac{3(a+b)}{4} - \frac{x}{2}\right) \leq f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) + f\left(t\frac{a+b}{2} + (1-t)(a+b-x)\right) - 2f\left(\left(\frac{1}{2} - t\right)\left(\frac{a+b}{2} - x\right)\right) \quad (2.6)$$

holds.

Again, for $z = t\frac{a+b}{2} + (1-t)x$, $M = \frac{a+b}{2}$, $m = x$ in (2.3), we observe that

$$f\left(t\frac{a+b}{2} + (1-t)x\right) + f\left(tx + (1-t)\frac{a+b}{2}\right) \leq f(x) + f\left(\frac{a+b}{2}\right) - 2tf\left((1-t)\left(\frac{a+b}{2} - x\right)\right) - 2(1-t)f\left(t\left(\frac{a+b}{2} - x\right)\right) \quad (2.7)$$

holds.

Finally, by setting $z = t(a+b-x) + (1-t)\frac{a+b}{2}$, $M = a+b-x$, $m = \frac{a+b}{2}$ in (2.3), we get that

$$\begin{aligned}
& f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) + f\left(t\frac{a+b}{2} + (1-t)(a+b-x)\right) \\
(2.8) \quad & \leq f\left(\frac{a+b}{2}\right) + f(a+b-x) - 2tf\left((1-t)\left(\frac{a+b}{2} - x\right)\right) \\
& \quad - 2(1-t)f\left(t\left(\frac{a+b}{2} - x\right)\right)
\end{aligned}$$

holds.

Now we are ready to state and prove our main results based on the calculations done above.

Theorem 1. *Let f be superquadratic integrable function on $[0, b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Let I be defined as above, then we have the following inequalities:*

$$\begin{aligned}
(2.9) \quad & f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \leq 2 \left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)p(4x-2a-b)dx \right. \\
& \left. + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)p(4x-a-2b)dx \right] - \int_a^b f\left(\frac{1}{4}(b-x)\right)p(x)dx,
\end{aligned}$$

$$\begin{aligned}
(2.10) \quad & 2 \left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)p(4x-2a-b)dx + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)p(4x-a-2b)dx \right] \\
& \leq \int_0^1 I(t)dt - \int_a^b \int_0^1 f\left(\left|\frac{1}{2} - t\right|\left(\frac{b-x}{2}\right)\right)p(x) dt dx
\end{aligned}$$

and

$$\begin{aligned}
(2.11) \quad & \int_0^1 I(t)dt \leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx + \int_a^b \frac{1}{2} \left[f\left(\frac{x+a}{2}\right) \right. \right. \\
& \left. \left. + f\left(\frac{x+b}{2}\right) \right] p(x) dx \right] - 2 \int_a^b \int_0^1 (1-t)f\left(t\frac{b-x}{2}\right)p(x) dt dx.
\end{aligned}$$

Proof. Using simple techniques of integration and by the assumptions on p , we have

$$f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx = 4 \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} f\left(\frac{a+b}{2}\right) p(2x-a) dt dx.$$

Therefore from (2.4), we get that

$$\begin{aligned} (2.12) \quad & f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \\ & \leq 2 \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f\left(\frac{x}{2} + \frac{a+b}{4}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}\right) \right] p(2x-a) dt dx \\ & \quad - 4 \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} f\left(\frac{1}{2}\left(\frac{a+b}{2} - x\right)\right) p(2x-a) dt dx. \end{aligned}$$

But

$$\begin{aligned} & 2 \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f\left(\frac{x}{2} + \frac{a+b}{4}\right) + f\left(\frac{3(a+b)}{4} - \frac{x}{2}\right) \right] p(2x-a) dt dx \\ (2.13) \quad & \end{aligned}$$

$$\begin{aligned} & = 2 \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} [f(x) + f(a+b-x)] p(4x-2a-b) dx \\ & = 2 \left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x) p(4x-2a-b) dx + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x) p(4x-a-2b) dx \right]. \end{aligned}$$

From (2.12), (2.13) and by the change of variable $x \rightarrow \frac{x+a}{2}$, we get (2.9).

From (2.5), (2.6) and (2.13), we have

$$\begin{aligned} & 2 \left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x) p(4x-2a-b) dx + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x) p(4x-a-2b) dx \right] \\ (2.14) \quad & \end{aligned}$$

$$\begin{aligned}
 &\leq \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f\left(t\frac{a+b}{2} + (1-t)x\right) + f\left(tx + (1-t)\frac{a+b}{2}\right) \right. \\
 &\quad \left. + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) \right. \\
 &\quad \left. + f\left(t\frac{a+b}{2} + (1-t)(a+b-x)\right) \right] p(2x-a) dt dx \\
 &\quad - 4 \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} f\left(\left(\frac{1}{2}-t\right)\left(\frac{a+b}{2}-x\right)\right) p(2x-a) dt dx.
 \end{aligned}$$

But

$$\int_0^1 I(t) dt = \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f\left(t\frac{a+b}{2} + (1-t)x\right) + f\left(tx + (1-t)\frac{a+b}{2}\right) \right.$$

(2.15)

$$\begin{aligned}
 &\quad \left. + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) \right. \\
 &\quad \left. + f\left(t\frac{a+b}{2} + (1-t)(a+b-x)\right) \right] p(2x-a) dt dx.
 \end{aligned}$$

From (2.14) and (2.15), we get that

$$2 \left[\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x) p(4x-2a-b) dx + \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x) p(4x-a-2b) dx \right]$$

(2.16)

$$\leq \int_0^1 I(t) dt - 4 \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} f\left(\left(\frac{1}{2}-t\right)\left(\frac{a+b}{2}-x\right)\right) p(2x-a) dt dx.$$

By the change of variables $t \rightarrow 1-t$ and $x \rightarrow \frac{x+a}{2}$ in (2.16), we get (2.10).

From (2.7), (2.8) and (2.15), we have

$$\int_0^1 I(t) dt \leq \int_0^{\frac{1}{2}} \int_a^{\frac{a+b}{2}} \left[f(x) + 2f\left(\frac{a+b}{2}\right) - 4tf\left((1-t)\left(\frac{a+b}{2}-x\right)\right) \right]$$

(2.17)

$$- 4(1-t) f\left(t\left(\frac{a+b}{2} - x\right)\right) + f(a+b-x) \Big] p(2x-a) dt dx.$$

But

$$\frac{1}{2} \left[f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx + \int_a^b \frac{1}{2} \left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right) \right] p(x) dx \right] \quad (2.18)$$

$$= \int_a^{\frac{a+b}{2}} \int_0^{\frac{1}{2}} \left[f(x) + 2f\left(\frac{a+b}{2}\right) + f(a+b-x) \right] p(2x-a) dt dx.$$

From (2.17), (2.18) and by the change of variables $x \rightarrow \frac{x+a}{2}$ and $t \rightarrow 1-t$, we get (2.11).

This completes the proof of the theorem as well. $\square$

Remark 2. If the superquadratic function f is non-negative and hence convex, then from (2.9) we get refinement of the first inequality of (1.3) in Theorem 1; from (2.10) we get refinement of the middle inequality of (1.3) in Theorem 1 and from (2.11) we get refinement of the last inequality of (1.3) in Theorem 1.

Corollary 3. Let f be superquadratic integrable function on $[0, b]$. If $p(x) = \frac{1}{b-a}$, $x \in [a, b]$ and $0 \leq a < b$, then we have

$$(2.19) \quad f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x) dx - \frac{1}{b-a} \int_a^b f\left(\frac{1}{4}(b-x)\right) dx,$$

$$\frac{2}{b-a} \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x) dx \leq \int_0^1 H(t) dt - \frac{1}{b-a} \int_a^b \int_0^1 f\left(\left|\frac{1}{2} - t\right| \left(\frac{b-x}{2}\right)\right) dt dx \quad (2.20)$$

and

$$(2.21) \quad \int_0^1 H(t) dt \leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{1}{b-a} \int_a^b f(x) dx \right] - \frac{2}{b-a} \int_a^b \int_0^1 (1-t) f\left(t\frac{b-x}{2}\right) dt dx,$$

where

$$H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx, \quad t \in [0, 1].$$

Proof. If $p(x) = \frac{1}{b-a}$, $x \in [a, b]$, then $I(t) = H(t)$, $t \in [0, 1]$, and therefore the proof of the corollary follows directly from the above theorem. $\square$

Remark 4. If the superquadratic function f is non-negative and therefore convex, then the inequalities in Corollary 1 refine the inequalities in (1.3) of Theorem B from [18, p.2].

To proceed to our next result, we go again through the similar calculations as given before Theorem 7.

For $0 \leq a \leq x \leq \frac{a+b}{2}$, $t \in [0, 1]$, we have that

$$\begin{aligned} a \leq ta + (1-t) \frac{a+b}{2} &\leq tx + (1-t) \frac{a+b}{2} \leq t(a+b-x) + (1-t) \frac{a+b}{2} \\ &\leq tb + (1-t) \frac{a+b}{2} \leq b. \end{aligned}$$

Therefore, by replacing $z = tx + (1-t) \frac{a+b}{2}$, $M = tb + (1-t) \frac{a+b}{2}$, $m = ta + (1-t) \frac{a+b}{2}$ in (2.3), we get that

$$\begin{aligned} (2.22) \quad &f\left(t(a+b-x) + (1-t) \frac{a+b}{2}\right) + f\left(tx + (1-t) \frac{a+b}{2}\right) \\ &\leq f\left(tb + (1-t) \frac{a+b}{2}\right) + f\left(ta + (1-t) \frac{a+b}{2}\right) \\ &\quad - \frac{2(x-a)}{b-a} f(t(b-x)) - \frac{2(b-x)}{b-a} f(t(x-a)) \end{aligned}$$

holds.

Now we are ready to state and prove our next result based on the above calculations.

Theorem 5. Let f be superquadratic integrable function on $[0, b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Let I and G be defined as above, then the following inequality holds for all $t \in [0, 1]$:

$$\begin{aligned} (2.23) \quad I(t) &\leq G(t) \int_a^b p(x) dx - \int_a^b \frac{1}{2} \left[\frac{x-a}{b-a} f\left(t \left(\frac{2b-x-a}{2} \right) \right) \right. \\ &\quad \left. + \frac{2b-x-a}{b-a} f\left(t \left(\frac{x-a}{2} \right) \right) \right] p(x) dx. \end{aligned}$$

Proof. Using simple techniques of integration and by the assumptions on p , we have that the following identity holds for all $t \in [0, 1]$:

$$(2.24) \quad G(t) \int_a^b p(x) dx = \int_a^{\frac{a+b}{2}} \left[f \left(ta + (1-t) \frac{a+b}{2} \right) + f \left(tb + (1-t) \frac{a+b}{2} \right) \right] p(2x-a) dx.$$

Arguing similarly as in obtaining (2.15), by using (2.22) and (2.24), we get that

$$(2.25) \quad I(t) \leq G(t) \int_a^b p(x) dx - \int_a^{\frac{a+b}{2}} \left[\frac{2(x-a)}{b-a} f(t(b-x)) + \frac{2(b-x)}{b-a} f(t(x-a)) \right] p(2x-a) dx,$$

for all $t \in [0, 1]$.

By the change of variable $x \rightarrow \frac{x+a}{2}$ in (2.25), we get (2.23).

This completes the proof of the theorem. $\square$

Remark 6. If the superquadratic function f is non-negative and hence convex, then the inequality (2.23) represents a refinement of the inequality (1.6) in Theorem 2.

Corollary 7. Let f be superquadratic integrable function on $[0, b]$, let $p(x) = \frac{1}{b-a}$, $0 \leq a < b$ and G, H be defined as above. Then for all $t \in [0, 1]$, we have the following inequality

$$(2.26) \quad H(t) \leq G(t) - \int_a^b \frac{1}{2(b-a)} \left[\frac{x-a}{b-a} f \left(t \left(\frac{2b-x-a}{2} \right) \right) + \frac{2b-x-a}{b-a} f \left(t \left(\frac{x-a}{2} \right) \right) \right] dx.$$

Proof. This is a direct consequence of the above theorem, since for $p(x) = \frac{1}{b-a}$, $x \in [a, b]$, $I(t) = H(t)$, for all $t \in [0, 1]$. $\square$

Remark 8. If the superquadratic function f is non-negative and hence convex, then the inequality (2.26) represents refinement of the inequality (1.6) in [18, Theorem C, p. 2].

Now again we give some calculations for our next result.

For $0 \leq t \leq \frac{1}{2}$ and $0 \leq a \leq x \leq \frac{a+b}{2}$, we obtain from (2.3) the following inequalities:

By setting $m = ta + (1-t)x$, $M = ta + (1-t)(a+b-x)$ and $z = ta + (1-t)\frac{a+b}{2}$ in (2.3), we observe that

$$(2.27) \quad \begin{aligned} 2f\left(ta + (1-t)\frac{a+b}{2}\right) &\leq f(ta + (1-t)x) + f(ta + (1-t)(a+b-x)) \\ &\quad - 2f\left((1-t)\left(\frac{a+b}{2} - x\right)\right) \end{aligned}$$

holds.

Also, by replacing $m = tb + (1-t)x$, $M = tb + (1-t)(a+b-x)$ and $z = tb + (1-t)\frac{a+b}{2}$ in (2.3), we get that

$$(2.28) \quad \begin{aligned} 2f\left(tb + (1-t)\frac{a+b}{2}\right) &\leq f(tb + (1-t)x) + f(tb + (1-t)(a+b-x)) \\ &\quad - 2f\left((1-t)\left(\frac{a+b}{2} - x\right)\right) \end{aligned}$$

holds.

Theorem 9. Let f be superquadratic integrable function on $[0, b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Let S_p and G be defined as above, then the following inequality holds for all $t \in [0, 1]$:

$$(2.29) \quad G(t) \int_a^b p(x)dx \leq S_p(t) - \int_a^b f\left((1-t)\left(\frac{b-x}{2}\right)\right) p(x)dx.$$

Proof. By the simple techniques of integration and by the assumptions on p , we have the following identity for all $t \in [0, 1]$:

$$(2.30) \quad \begin{aligned} S_p(t) &= \frac{1}{2} \int_a^{\frac{a+b}{2}} [f(ta + (1-t)x) + f(ta + (1-t)(a+b-x)) \\ &\quad + f(tb + (1-t)x) + f(tb + (1-t)(a+b-x))] p(2x-a)dx. \end{aligned}$$

From (2.27), (2.28) and (2.30), we have that

$$\int_a^{\frac{a+b}{2}} \left[f \left(ta + (1-t) \frac{a+b}{2} \right) + f \left(tb + (1-t) \frac{a+b}{2} \right) \right] p(2x-a) dx \quad (2.31)$$

$$\leq S_p(t) - 2 \int_a^{\frac{a+b}{2}} f \left((1-t) \left(\frac{a+b}{2} - x \right) \right) p(2x-a) dx,$$

holds for all $t \in [0, 1]$.

From (2.24) and by the change of variable $x \rightarrow \frac{a+x}{2}$, we get from (2.31) that

$$G(t) \int_a^b p(x) dx \leq S_p(t) - \int_a^b f \left((1-t) \left(\frac{b-x}{2} \right) \right) p(x) dx,$$

for all $t \in [0, 1]$. Which is (2.29) and this completes the proof of the theorem as well. $\square$

Remark 10. The result of the above theorem refines the first inequality of Theorem 3, when superquadratic function f is non-negative and hence convex.

Corollary 11. Let f be superquadratic integrable function on $[0, b]$ and let $p(x) = \frac{1}{b-a}$, $x \in [a, b]$, $0 \leq a < b$. Let G be defined as above, then the following inequality holds for all $t \in [0, 1]$:

$$(2.32) \quad G(t) \leq L(t) - \frac{1}{b-a} \int_a^b f \left((1-t) \left(\frac{b-x}{2} \right) \right) dx.$$

Proof. Since for $p(x) = \frac{1}{b-a}$, $x \in [a, b]$, $S_p(t) = L_p(t) = L(t)$, for all $t \in [a, b]$. Therefore the proof of the corollary follows directly from the above theorem. $\square$

3. Inequalities for differentiable superquadratic functions

In this section we give results when f is a differentiable superquadratic function. Those results give refinements of (1.4) and (1.5) in Theorem 1 and refine (1.7) of Theorem 2 when superquadratic function f is non-negative and hence convex. Here we quote very important result which will be helpful in the sequel of the paper.

Theorem 1. [14, Theorem 10, p. 5] Let f be superquadratic integrable function on $[0, b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Let I be defined as above and let fp be integrable on $[a, b]$, then for $0 \leq s \leq t \leq 1$, $t > 0$, we have the following inequality:

$$(3.1) \quad I(s) \leq I(t) - \int_a^b \frac{t+s}{2t} f\left(\left(\frac{t-s}{2}\right)(b-x)\right) p(x) dx \\ - \int_a^b \frac{t-s}{2t} f\left(\left(\frac{t+s}{2}\right)(b-x)\right) p(x) dx.$$

Now we state and prove the first result of this section.

Theorem 2. Let f be superquadratic function on $[0, b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Let f be differentiable on $[a, b]$ such that $f(0) = f'(0) = 0$ and p is bounded on $[a, b]$, then the following inequalities hold for all $t \in [0, 1]$:

$$(3.2) \quad \int_a^b \frac{1}{2} \left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right) \right] p(x) dx - I(t) \\ \leq (1-t) \left[\frac{f(a)+f(b)}{2} (b-a) - \int_a^b f(x) dx \right] \|p\|_\infty \\ - \int_a^b f\left((1-t)\left(\frac{b-x}{2}\right)\right) p(x) dx,$$

where $\|p\|_\infty = \sup_{x \in [a, b]} |p(x)|$ and

$$(3.3) \quad \frac{f(a)+f(b)}{2} \int_a^b p(x) dx - I(t) \leq \left[\frac{(f'(b) - f'(a))(b-a)}{4} \right. \\ \left. - f\left(\left|\frac{a-b}{2}\right|\right) \right] \int_a^b p(x) dx - \int_a^b f\left(t\left(\frac{b-x}{2}\right)\right) p(x) dx.$$

Proof. By integration by parts, we have that

$$(3.4) \quad \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right) [f'(a+b-x) - f'(x)] dx = \int_a^b \left(x - \frac{a+b}{2}\right) f'(x) dx$$

$$= \frac{f(a) + f(b)}{2}(b - a) - \int_a^b f(x)dx.$$

Using the substitution rules for integration, under the assumptions on p , we have

$$\begin{aligned} (3.5) \quad & \int_a^b \frac{1}{2} \left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right) \right] p(x) dx = \int_a^b \frac{1}{2} \left[f\left(\frac{x+a}{2}\right) \right. \\ & \quad \left. + f\left(\frac{a+2b-x}{2}\right) \right] p(x) dx \\ & = \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x)] p(2x-a) dx \end{aligned}$$

and

$$\begin{aligned} (3.6) \quad & I(t) = \int_a^b \frac{1}{2} \left[f\left(t\frac{x+a}{2} + (1-t)\frac{a+b}{2}\right) \right. \\ & \quad \left. + f\left(t\frac{a+2b-x}{2} + (1-t)\frac{a+b}{2}\right) \right] p(x) dx \\ & = \int_a^{\frac{a+b}{2}} \left[f\left(tx + (1-t)\frac{a+b}{2}\right) \right. \\ & \quad \left. + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right) \right] p(2x-a) dx, \end{aligned}$$

for all $t \in [0, 1]$.

Now by the assumptions on f , we have that

$$\begin{aligned} (3.7) \quad & \left[f(x) - f\left(tx + (1-t)\frac{a+b}{2}\right) \right] p(2x-a) + [f(a+b-x) \\ & \quad - f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right)] p(2x-a) \\ & \leq (1-t) \left(\frac{a+b}{2} - x \right) [f'(a+b-x) - f'(x)] p(2x-a) \\ & \quad - 2f\left((1-t)\left(\frac{a+b}{2} - x\right)\right) p(2x-a) \end{aligned}$$

$$\leq (1-t) \left(\frac{a+b}{2} - x \right) [f'(a+b-x) - f'(x)] \|p\|_{\infty} \\ - 2f \left((1-t) \left(\frac{a+b}{2} - x \right) \right) p(2x-a),$$

for all $t \in [0, 1]$ and $x \in \left[a, \frac{a+b}{2} \right]$.

From (3.4), (3.5), (3.6) and (3.7) and by the change of variable $x \rightarrow \frac{a+x}{2}$, we get (3.2).

By the assumptions on f and from Lemma 3, we get that

$$\frac{f(a) - f\left(\frac{a+b}{2}\right)}{2} \leq \frac{a-b}{4} f'(a) - \frac{1}{2} f\left(\left|\frac{a-b}{2}\right|\right)$$

and

$$\frac{f(b) - f\left(\frac{a+b}{2}\right)}{2} \leq \frac{b-a}{4} f'(b) - \frac{1}{2} f\left(\left|\frac{a-b}{2}\right|\right).$$

Adding these inequalities we get that

$$\frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) \leq \frac{(f'(b) - f'(a))(b-a)}{4} - f\left(\left|\frac{a-b}{2}\right|\right).$$

Thus

$$(3.8) \quad \frac{f(a) + f(b)}{2} \int_a^b p(x) dx - f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \\ \leq \left[\frac{(f'(a) - f'(b))(b-a)}{4} - f\left(\left|\frac{a-b}{2}\right|\right) \right] \int_a^b p(x) dx.$$

From (3.1), for $s = 0$, we have

$$(3.9) \quad f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \leq I(t) - \int_a^b f\left(t\left(\frac{b-x}{2}\right)\right) p(x) dx,$$

for all $t \in [0, 1]$.

From (3.8) and (3.9), we get (3.3). This completes the proof of the theorem. $\square$

Remark 3. The inequalities (3.2) and (3.3) refine the inequalities (1.4) and (1.5) in Theorem 1, when the superquadratic function f is non-negative and therefore convex.

Corollary 4. Let f be superquadratic function on $[0, b]$ and differentiable on $[a, b]$ such that $f(0) = f'(0) = 0$. If $p(x) = \frac{1}{b-a}$, then we have the following inequalities:

$$(3.10) \quad \int_a^b \frac{1}{2(b-a)} \left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right) \right] dx - H(t) \\ \leq \frac{1-t}{b-a} \left[\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)dx \right] \\ - \frac{1}{b-a} \int_a^b f\left((1-t)\left(\frac{b-x}{2}\right)\right) dx$$

and

$$(3.11) \quad \frac{f(a)+f(b)}{2} - H(t) \leq \frac{(f'(b)-f'(a))(b-a)}{4} - f\left(\left|\frac{a-b}{2}\right|\right) dx \\ - \frac{1}{b-a} \int_a^b f\left(t\left(\frac{b-x}{2}\right)\right) dx,$$

for all $t \in [0, 1]$.

Now we give our last result and summarize the results related to it in the remark followed by Theorem 12.

Theorem 5. Let f be superquadratic function on $[0, b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Let f be differentiable on $[a, b]$ such that $f(0) = f'(0) = 0$ and p is bounded on $[a, b]$, then for all $t \in [0, 1]$ we have the inequality:

$$(3.12) \quad I(t) - f\left(\frac{a+b}{2}\right) \int_a^b p(x) dx \leq (b-a) [G(t) - H(t)] \|p\|_\infty \\ - \int_a^b f\left(\frac{t}{2}(b-x)\right) p(x) dx,$$

where $\|p\|_\infty = \sup_{x \in [a, b]} |p(x)|$.

Proof. By integration by parts, we observe that

$$\begin{aligned}
 (3.13) \quad & t \int_a^{\frac{a+b}{2}} \left[\left(x - \frac{a+b}{2} \right) f' \left(tx + (1-t) \frac{a+b}{2} \right) \right. \\
 & \left. + \left(\frac{a+b}{2} - x \right) f' \left(t(a+b-x) + (1-t) \frac{a+b}{2} \right) \right] dx \\
 & = t \int_a^b \left(x - \frac{a+b}{2} \right) f' \left(tx + (1-t) \frac{a+b}{2} \right) dx \\
 & = (b-a) G(t) - H(t),
 \end{aligned}$$

hold for all $t \in [0, 1]$.

Under the assumptions on f , we have that

$$\begin{aligned}
 (3.14) \quad & \left[f \left(tx + (1-t) \frac{a+b}{2} \right) - f \left(\frac{a+b}{2} \right) \right] p(2x-a) \\
 & + \left[f \left(t(a+b-x) + (1-t) \frac{a+b}{2} \right) - f \left(\frac{a+b}{2} \right) \right] p(2x-a) \\
 & \leq t \left(x - \frac{a+b}{2} \right) f' \left(tx + (1-t) \frac{a+b}{2} \right) p(2x-a) \\
 & + t \left(\frac{a+b}{2} - x \right) f' \left(t(a+b-x) + (1-t) \frac{a+b}{2} \right) p(2x-a) \\
 & \quad - 2f \left(t \left| \frac{a+b}{2} - x \right| \right) p(2x-a) \\
 & = t \left(\frac{a+b}{2} - x \right) \left[f' \left(t(a+b-x) + (1-t) \frac{a+b}{2} \right) \right. \\
 & \quad \left. - f' \left(tx + (1-t) \frac{a+b}{2} \right) \right] p(2x-a) - 2f \left(t \left| \frac{a+b}{2} - x \right| \right) p(2x-a) \\
 & \leq t \left(\frac{a+b}{2} - x \right) \left[f' \left(t(a+b-x) + (1-t) \frac{a+b}{2} \right) \right.
 \end{aligned}$$

$$-f' \left(tx + (1-t)\frac{a+b}{2} \right) \Big] \|p\|_{\infty} - 2f \left(t \left| \frac{a+b}{2} - x \right| \right) p(2x-a)$$

hold for all $t \in [0, 1]$ and $x \in \left[a, \frac{a+b}{2} \right]$.

Integrating (3.14) over x on $\left[a, \frac{a+b}{2} \right]$, using (3.13), by the change of variable $x \rightarrow \frac{x+a}{2}$, under the assumptions on p , we get (3.12). This completes the proof of the theorem. $\square$

Remark 6. The result of Theorem 12 refines (1.7) of Theorem 2, when superquadratic function f is non-negative and therefore convex.

Corollary 7. Let f be superquadratic function on $[0, b]$. Let f be differentiable on $[a, b]$ such that $f(0) = f'(0) = 0$. If $p(x) = \frac{1}{b-a}$, $x \in [a, b]$, then

$$H(t) - f \left(\frac{a+b}{2} \right) \leq [G(t) - H(t)] - \frac{1}{b-a} \int_a^b f \left(t \left(\frac{b-x}{2} \right) \right) dx, \quad (3.15)$$

for all $t \in [0, 1]$.

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