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Matrix representation of the q -Jacobsthal numbers

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Abstract

In this paper, we consider a q -Jacobsthal sequence $\{J_{q,n}\}$, with initial conditions $J_{q,0} = 0$ and $J_{q,1} = 1$. Then give a generating matrix for the terms of sequence $\{J_{q,kn}\}$ for a positive integer k . With the aid of this matrix, we derive some new identities for the sequence $\{J_{q,kn}\}$.

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1. Introduction

In [H1], Horadam introduce a sequence $\{W_n(a, b; p, q)\}$, or briefly $\{W_n\}$, defined by the recurrence relation

$$(1.1) \quad W_n = pW_{n-1} - qW_{n-2}, n \geq 2,$$

with $W_0 = a$, $W_1 = b$, where a, b, p and q are integers with $p > 0$, $q \neq 0$.

We are interested in the following two special cases of $\{W_n\}$, the q -Jacobsthal sequence $\{J_{q,n}\}$ defined by

$$(1.2) \quad J_{q,n} = J_{q,n-1} - qJ_{q,n-2}, \quad J_{q,0} = 0, \quad J_{q,1} = 1, \quad n \geq 2,$$

and the q -Jacobsthal-Lucas sequence $\{j_{q,n}\}$ defined by

$$(1.3) \quad j_{q,n} = j_{q,n-1} - qj_{q,n-2}, \quad j_{q,0} = 2, \quad j_{q,1} = 1, \quad n \geq 2.$$

The above recurrences involve the characteristic equation

$$(1.4) \quad x^2 - x + q = 0$$

with roots $\alpha_q = \frac{1+\sqrt{1-4q}}{2}$ and $\beta_q = \frac{1+\sqrt{1-4q}}{2}$. Explicit closed form expressions for $J_{q,n}$ and $j_{q,n}$ are ($n \geq 1$)

$$(1.5) \quad J_{q,n} = \frac{\alpha_q^n - \beta_q^n}{\alpha_q - \beta_q},$$

and

$$(1.6) \quad j_{q,n} = \alpha_q^n + \beta_q^n.$$

Particular cases of the previous definition are:

- If $q = -1$, the classic Fibonacci sequence appears by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$: $\{F_n\}_{n \in \mathbb{N}} = \{0, 1, 1, 2, 3, 5, 8, \dots\}$.
- If $q = -2$, the Jacobsthal sequence introduced in [H2], and defined by $J_{n+1} = J_n + 2J_{n-1}$ for $n \geq 1$, $J_0 = 0$, $J_1 = 1$: $\{J_n\}_{n \in \mathbb{N}} = \{0, 1, 1, 3, 5, 11, \dots\}$.

We define J_q be the 2×2 matrix

$$(1.7) \quad J_q = \begin{bmatrix} 1 & -q \\ 1 & 0 \end{bmatrix},$$

then for an integer n with $n \geq 1$, J_q^n has the form

$$(1.8) \quad J_q^n = \begin{bmatrix} J_{q,n+1} & -qJ_{q,n} \\ J_{q,n} & -qJ_{q,n-1} \end{bmatrix}.$$

The particular case $q = -2$, was introduced by Köken and Bozkurt in [KB2, KB3]. Moreover, they have obtained the Cassini formula for the Jacobsthal numbers. In this paper, we study new relations on q -Jacobsthal sequence, using the matrix J_q defined in (1.7).

Initially, the q -Jacobsthal numbers are defined for $n \geq 0$ but their existence for $n < 0$ is readily extended, yielding

$$J_{q,-n} = -q^{-n}J_{q,n} \text{ and } j_{q,-n} = q^{-n}j_{q,n}.$$

For $n \geq 2$ and a fixed positive integer k , in [KS] the authors study the sequence $\{W_{q,kn}(x)\}$ and prove the following relation:

$$(1.9) \quad J_{q,kn} = j_{q,k}J_{q,k(n-1)} - q^k J_{q,k(n-2)},$$

$$(1.10) \quad j_{q,kn} = j_{q,k}j_{q,k(n-1)} - q^k j_{q,k(n-2)},$$

where the initial conditions of the sequences $\{J_{q,n}\}$ and $\{j_{q,n}\}$ are 0 and $\{J_{q,k}\}$, and 2 and $\{j_{q,k}\}$, respectively.

If $\alpha_{q,k}$ and $\beta_{q,k}$ are the roots of equation $\lambda^2 - j_{q,k}\lambda + q^k = 0$, then the Binet formulas of the sequences $\{J_{q,kn}\}$ and $\{j_{q,kn}\}$ are given by

$$J_{q,kn} = J_{q,k} \left(\frac{\alpha_{q,k}^n - \beta_{q,k}^n}{\alpha_{q,k} - \beta_{q,k}} \right) \text{ and } j_{q,kn} = \alpha_{q,k}^n + \beta_{q,k}^n,$$

respectively. It is clear that $\alpha_{q,1} = \alpha_q$ and $\beta_{q,1} = \beta_q$.

From the Binet formulas, one can see that $J_{q,2kn} = j_{q,kn}J_{q,kn}$.

2. Companion matrix for the sequence $\{J_{q,kn}\}$

In this section, we define a 2×2 matrix A_q and then we give some new results for the q -Jacobsthal numbers $J_{q,kn}$ by matrix methods.

Define the 2×2 matrix A_q as follows:

$$(2.1) \quad A_q = \begin{bmatrix} j_{q,k} & -q^k \\ 1 & 0 \end{bmatrix}.$$

By an inductive argument and using (1.9), we get

Proposition 2.1. For any integer $n \geq 1$ holds:

$$(2.2) \quad A_q^n = \frac{1}{J_{q,k}} \begin{bmatrix} J_{q,k(n+1)} & -q^k J_{q,kn} \\ J_{q,kn} & -q^k J_{q,k(n-1)} \end{bmatrix}.$$

Proof. (By induction). For $n = 1$:

$$(2.3) \quad A_q^1 = \begin{bmatrix} j_{q,k} & -q^k \\ 1 & 0 \end{bmatrix} = \frac{1}{J_{q,k}} \begin{bmatrix} J_{q,2k} & -q^k J_{q,k} \\ J_{q,k} & -q^k J_{q,0} \end{bmatrix}$$

since $J_{q,0} = 0$ and $J_{q,2k} = j_{q,k} J_{q,k}$. Let us suppose that the formula is true for $n - 1$:

$$(2.4) \quad A_q^{n-1} = \frac{1}{J_{q,k}} \begin{bmatrix} J_{q,kn} & -q^k J_{q,k(n-1)} \\ J_{q,k(n-1)} & -q^k J_{q,k(n-2)} \end{bmatrix}.$$

$$\begin{aligned} \text{Then, } A_q^n &= A_q^{n-1} A_q^1 = \frac{1}{J_{q,k}} \begin{bmatrix} J_{q,kn} & -q^k J_{q,k(n-1)} \\ J_{q,k(n-1)} & -q^k J_{q,k(n-2)} \end{bmatrix} \begin{bmatrix} j_{q,k} & -q^k \\ 1 & 0 \end{bmatrix} \\ &= \frac{1}{J_{q,k}} \begin{bmatrix} j_{q,k} J_{q,kn} - q^k J_{q,k(n-1)} & -q^k J_{q,kn} \\ j_{q,k} J_{q,k(n-1)} - q^k J_{q,k(n-2)} & -q^k J_{q,k(n-1)} \end{bmatrix} \\ &= \frac{1}{J_{q,k}} \begin{bmatrix} J_{q,k(n+1)} & -q^k J_{q,kn} \\ J_{q,kn} & -q^k J_{q,k(n-1)} \end{bmatrix} \quad \square \end{aligned}$$

Clearly the matrix A_q^n satisfies the recurrence relation, for $n \geq 1$

$$(2.5) \quad A_q^{n+1} = j_{q,k} A_q^n - q^k A_q^{n-1},$$

where $A_q^0 = I_2$, $A_q^1 = A_q$ and I_2 is the 2×2 unit matrix.

In this study, we define the q -Jacobsthal-Lucas j_q -matrix by

$$(2.6) \quad j_q = \begin{bmatrix} j_{q,k}^2 - 2q^k & -j_{q,k} q^k \\ j_{q,k} & -2q^k \end{bmatrix}.$$

It is easy to see that

$$\begin{bmatrix} j_{q,k(n+1)} \\ j_{q,kn} \end{bmatrix} = j_q \begin{bmatrix} J_{q,kn} \\ J_{q,k(n-1)} \end{bmatrix} \text{ and } \Delta \begin{bmatrix} J_{q,k(n+1)} \\ J_{q,kn} \end{bmatrix} = j_q \begin{bmatrix} j_{q,kn} \\ j_{q,k(n-1)} \end{bmatrix}$$

where $J_{q,kn}$, $j_{q,kn}$ are as above, and $\Delta = 1 - 4q$.

We obtain Cassini's formula and properties of these numbers by a similar matrix method to the Lucas numbers [KB1].

Proposition 2.2. Let j_q be a matrix as in (2.6). Then, for all integers $n \geq 1$, the following matrix power is held below

$$(2.7) \quad j_q^n = \begin{cases} \Delta^{\frac{n}{2}} \begin{bmatrix} J_{q,k(n+1)} & -q^k J_{q,kn} \\ J_{q,kn} & -q^k J_{q,k(n-1)} \end{bmatrix} & \text{if } n \text{ even} \\ \Delta^{\frac{n-1}{2}} \begin{bmatrix} j_{q,k(n+1)} & -q^k j_{q,kn} \\ j_{q,kn} & -q^k j_{q,k(n-1)} \end{bmatrix} & \text{if } n \text{ odd,} \end{cases}$$

where $J_{q,kn}$ and $j_{q,kn}$ are the kn -th q -Jacobsthal and q -Jacobsthal-Lucas numbers, respectively.

Proof. We use mathematical induction on n . First, we consider odd n . For $n = 1$,

$$j_q^1 = \begin{bmatrix} j_{q,2k} & -q^k j_{q,k} \\ j_{q,k} & -q^k j_{q,0} \end{bmatrix},$$

since $j_{q,2k} = j_{q,k}^2 - 2q^k$ and $j_{q,0} = 2$. So, (2.7) is indeed true for $n = 1$. Now we suppose it is true for $n = t$, that is

$$j_q^t = \Delta^{\frac{t-1}{2}} \begin{bmatrix} j_{q,k(t+1)} & -q^k j_{q,kt} \\ j_{q,kt} & -q^k j_{q,k(t-1)} \end{bmatrix}.$$

Using the induction hypothesis and j_q^2 by a direct computation. we can write

$$j_q^{t+2} = j_q^t j_q^2 = \Delta^{\frac{t+1}{2}} \begin{bmatrix} j_{q,k(t+3)} & -q^k j_{q,k(t+2)} \\ j_{q,k(t+2)} & -q^k j_{q,k(t+1)} \end{bmatrix},$$

as desired. Secondly, let us consider even n . For $n = 2$ we can write

$$j_q^2 = \Delta \begin{bmatrix} J_{q,3k} & -q^k J_{q,2k} \\ J_{q,2k} & -q^k J_{q,k} \end{bmatrix}.$$

So, (2.7) is true for $n = 2$. Now, we suppose it is true for $n = t$, using properties of the q -Jacobsthal numbers and the induction hypothesis, we can write

$$j_q^{t+2} = \Delta^{\frac{t+2}{2}} \begin{bmatrix} J_{q,k(t+3)} & -q^k J_{q,k(t+2)} \\ J_{q,k(t+2)} & -q^k J_{q,k(t+1)} \end{bmatrix},$$

as desired. Hence, (2.7) holds for all n . $\square$

If we use the equation (2.5), we can write $j_{q,k} A_q^{n+1} = j_{q,k}^2 A_q^n - q^k j_{q,k} A_q^{n-1}$
 $= j_{q,k}^2 A_q^n - q^k (A_q^n + q^k A_q^{n-2})$
 $= (j_{q,k}^2 - q^k) A_q^n - q^{2k} A_q^{n-2}.$

Comparing the entries in the first row and first column for the above matrix equation, we get

$$(2.8) \quad j_{q,k} = \frac{(j_{q,k}^2 - q^k) J_{q,k(n+1)} - q^{2k} J_{q,k(n-1)}}{J_{q,k(n+2)}}.$$

For $n \geq 0$, if we consider the fact that $\det(A_q^n) = (\det(A_q))^n$, then we obtain the generalized Cassini identity

$$(2.9) \quad J_{q,k(n+1)} J_{q,k(n-1)} - J_{q,kn}^2 = -q^{k(n-1)} J_{q,k}^2.$$

For example, for $k = 1$, we get $J_{q,n+1}J_{q,n-1} - J_{q,n}^2 = -q^{n-1}$, the generalized Cassini identity with q -Jacobsthal numbers. In this case, if $q = -1$, we get Cassini identity on classic Fibonacci sequence.

Now we shall derive some results for $\{J_{q,kn}\}$ by matrix methods.

Proposition 2.3. *For all $n, m \in \mathbb{Z}$*

$$(2.10) \quad J_{q,k}J_{q,k(n+m)} = J_{q,km}J_{q,k(n+1)} - q^k J_{q,k(m-1)}J_{q,kn}.$$

Proof. Since $A_q^{n+m} = A_q^n A_q^m$ and after some simplifications, we obtain

$$\begin{aligned} A_q^{n+m} &= \frac{1}{J_{q,k}^2} \begin{bmatrix} J_{q,k(n+1)} & -q^k J_{q,kn} \\ J_{q,kn} & -q^k J_{q,k(n-1)} \end{bmatrix} \begin{bmatrix} J_{q,k(m+1)} & -q^k J_{q,km} \\ J_{q,km} & -q^k J_{q,k(m-1)} \end{bmatrix} \\ &= \frac{J_{q,km}}{J_{q,k}} A_q^{n+1} - q^k \frac{J_{q,k(m-1)}}{J_{q,k}} A_q^n. \end{aligned}$$

Thus we obtain

$$(2.11) \quad J_{q,k} A_q^{n+m} = J_{q,km} A_q^{n+1} - q^k J_{q,k(m-1)} A_q^n,$$

which, Comparing the entries in the second row and first column for the matrix equation (2.11), gives the conclusion. $\square$

When $m = n$ in (2.10), we obtain

$$(2.12) \quad J_{q,k}J_{q,2kn} = J_{q,kn}(J_{q,k(n+1)} - q^k J_{q,k(n-1)}),$$

and the follow equality for $J_{q,kn}(x) \neq 0$

$$(2.13) \quad J_{q,k}J_{q,kn} = J_{q,k(n+1)} - q^k J_{q,k(n-1)}.$$

Comparing the entries in the first row and first column of the equality (2.11) and by taking $m = n$, we obtain

$$(2.14) \quad J_{q,k}J_{q,k(2n+1)} = J_{q,k(n+1)}^2 - q^k J_{q,kn}^2,$$

a particular case of the next equality

$$(2.15) \quad J_{q,k} A_q^{2n} = J_{q,kn} A_q^{n+1} - q^k J_{q,k(n-1)} A_q^n.$$

Corollary 2.4. *For $k \geq 1$ and $n \in \mathbb{Z}$,*

$$(2.16) \quad J_{q,k(2n+1)} + J_{q,k(2n-1)} = \frac{1}{J_{q,k}} \left(J_{q,k(n+1)}^2 + (1 - q^k) J_{q,kn}^2 - J_{q,k(n+1)}^2 \right).$$

Proof. Considering the first row and first column of the matrix $A_q^{2n} = (A_q^n)^2$, we get $J_{q,k}J_{q,k(2n+1)} = J_{q,k(n+1)}^2 - q^k J_{q,kn}^2$, and the second row and second column of the matrix A_q^{2n} , $J_{q,k}J_{q,k(2n-1)} = J_{q,kn}^2 - q^k J_{q,k(n-1)}^2$. By adding side by side in the above equations, we have the conclusion. $\square$

For any integer m , we have $A_q^{2n} = A_q^{n+m} A_q^{n-m}$. Here if we consider the entries in the second row and first column in the product $A_q^{n+m} A_q^{n-m}$ and the matrix A_q^{2n} , we get

$$(2.17) \quad J_{q,k}J_{q,2kn} = J_{q,k(n+m)}J_{q,k(n-m+1)} - q^k J_{q,k(n+m-1)}J_{q,k(n-m)}.$$

3. sums with q -Jacobsthal numbers

We define the q -Jacobsthal G_q -matrix by

$$(3.1) \quad G_q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & j_{q,k} & -q^k \\ 0 & 1 & 0 \end{bmatrix}.$$

By an inductive argument and using (1.9), we get

Proposition 3.1. For any integer $n \geq 1$ holds:

$$(3.2) \quad G_q^n = \frac{1}{J_{q,k}} \begin{bmatrix} 1 & 0 & 0 \\ J_{q,kn}^s & J_{q,k(n+1)} & -q^k J_{q,kn} \\ J_{q,k(n-1)}^s & J_{q,kn} & -q^k J_{q,k(n-1)} \end{bmatrix},$$

where $J_{q,kn}^s$ is defined such that $J_{q,kn}^s = \sum_{i=1}^n J_{q,ki}$.

Proof. (By induction). For $n = 1$:

$$(3.3) \quad G_q^1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & j_{q,k} & -q^k \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{J_{q,k}} \begin{bmatrix} 1 & 0 & 0 \\ J_{q,k}^s & J_{q,2k} & -q^k J_{q,k} \\ J_{q,0}^s & J_{q,k} & -q^k J_{q,0} \end{bmatrix}$$

since $J_{q,0} = 0$ and $J_{q,2k} = j_{q,k}J_{q,k}$. Let us suppose that the formula is true for $n - 1$:

$$(3.4) \quad G_q^{n-1} = \frac{1}{J_{q,k}} \begin{bmatrix} 1 & 0 & 0 \\ J_{q,k(n-1)}^s & J_{q,kn} & -q^k J_{q,k(n-1)} \\ J_{q,k(n-2)}^s & J_{q,k(n-1)} & -q^k J_{q,k(n-2)} \end{bmatrix}$$

Then,

$$\begin{aligned}
 G_q^n &= G_q^{n-1} G_q^1 = \frac{1}{J_{q,k}} \begin{bmatrix} 1 & 0 & 0 \\ J_{q,k(n-1)}^s & J_{q,kn} & -q^k J_{q,k(n-1)} \\ J_{q,k(n-2)}^s & J_{q,k(n-1)} & -q^k J_{q,k(n-2)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & j_{q,k} & -q^k \\ 0 & 1 & 0 \end{bmatrix} \\
 &= \frac{1}{J_{q,k}} \begin{bmatrix} 1 & 0 & 0 \\ J_{q,k(n-1)}^s + J_{q,kn} & j_{q,k} J_{q,kn} - q^k J_{q,k(n-1)} & -q^k J_{q,kn} \\ J_{q,k(n-2)}^s + J_{q,k(n-1)} & j_{q,k} J_{q,k(n-1)} - q^k J_{q,k(n-2)} & -q^k J_{q,k(n-1)} \end{bmatrix} \\
 &= \frac{1}{J_{q,k}} \begin{bmatrix} 1 & 0 & 0 \\ J_{q,kn}^s & J_{q,k(n+1)} & -q^k J_{q,kn} \\ J_{q,k(n-1)}^s & J_{q,kn} & -q^k J_{q,k(n-1)} \end{bmatrix}. \quad \square
 \end{aligned}$$

Corollary 3.2. If $J_{q,kn}$ is the kn -th q -Jacobsthal number, then

$$\begin{aligned}
 J_{q,k} J_{q,k(n+m)}^s &= J_{q,kn}^s + J_{q,k(m-1)}^s (J_{q,k(n+1)} - q^k J_{q,kn}) + J_{q,k(n+1)} J_{q,km}. \\
 (3.5)
 \end{aligned}$$

Proof. The second row and first column in the matrix G_q^{n+m} is equal to $J_{q,k(n+m)}^s$. Considering the $(2,1)$ -entries of the matrix equation $G_q^n G_q^m$, $J_{q,k} J_{q,k(n+m)}^s = J_{q,kn}^s + J_{q,k(n+1)} J_{q,km}^s - q^k J_{q,kn} J_{q,k(m-1)}^s$
 $= J_{q,kn}^s + J_{q,k(n+1)} (J_{q,k(m-1)}^s + J_{q,km}) - q^k J_{q,kn} J_{q,k(m-1)}^s$
 $= J_{q,kn}^s + J_{q,k(m-1)}^s (J_{q,k(n+1)} - q^k J_{q,kn}) + J_{q,k(n+1)} J_{q,km}$. Thus, the proof is completed. $\square$

If $q = -2$ and $k = 1$, the classic Jacobsthal sequence satisfy

$$(3.6) \quad J_{n+m}^s = J_n^s + J_{m-1}^s J_{n+2} + J_{n+1} J_{q,m}.$$

4. conclusion

In this paper, we consider the amazing relationships between the q -Jacobsthal numbers and matrices. Sum formula involving the terms of q -Jacobsthal numbers is one of the most important results obtained in this study.

Note that the q -Jacobsthal numbers represent a generalization of the classical Fibonacci sequence. In particular, we have obtained new results by matrix method in these numbers.

References

- [C1] G. Cerda-Morales, On generalized Fibonacci and Lucas numbers by matrix methods, Hacettepe J. Math. Stat. (to appear).
- [C2] G. Cerda-Morales, Matrix methods in Horadam Sequences, Boletín de matemáticas, Universidad Nacional de Colombia, Vol. 19, no. 1, pp. 55-64, (2012).
- [H1] A.F. Horadam, Basic properties of a certain generalized sequence of numbers, Fibonacci Quarterly, Vol. 3, pp. 161-176, (1965).
- [H2] A. F. Horadam, Jacobsthal Representation Numbers, The Fibonacci Quarterly, Vol. 34, no. 1, pp. 40-53, (1996).
- [KB1] F. Köken, D. Bozkurt, On Lucas numbers by the matrix method, Hacettepe Journal of Mathematics and Statistics, Vol. 39, no. 4, pp. 471-475, (2010).
- [KB2] F. Köken, D. Bozkurt, On The Jacobsthal numbers by matrix methods, Int. J. Contemp. Math. Sciences, Vol. 3, no. 13, pp. 605-614, (2008).
- [KB3] F. Köken, D. Bozkurt, On The Jacobsthal-Lucas Numbers by Matrix Methods, Int. J. Contemp. Math. Sciences, Vol. 3, no. 33, pp. 1629-1633, (2008).
- [KM] D. Kalman, R. Mena, The Fibonacci numbers- exposed, Math. Mag., no. 76, pp. 81-167, 2003.
- [KS] E. Kilic, P. Stanica, Factorizations and representations of second linear recurrences with indices in arithmetic progressions, Bol. Mex. Math. Soc., Vol. 15, no. 1, pp. 23-36, (2009).
- [Ko] T. Koshy, Fibonacci and Lucas Numbers with Applications, A Wiley-Interscience Publications, New York, (2002).

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