Prime Submodules of Graded Modules

Rashid Abu-Dawwas
Yarmouk University, Jordan
Khaldoun Al-Zoubi
Jordan University of Science and Technology, Jordan
and
Malik Bataineh
Jordan University of Science and Technology, Jordan
Received: March 2012. Accepted: October 2012

Abstract

Let $G$ be a group, $R$ be a $G$-graded ring and $M$ be a $G$-graded $R$-module. Suppose $P$ is a prime ideal of $R_e$ and $g \in G$. In this article, we define

$$M_g(P) = \{ m \in M_g : Am \subseteq PM_g \}
for some ideal $A$ of $R_e$ satisfying $A \not\subseteq P\}$$

that is an $R_e$-submodule of $M_g$, and we investigate some results on this submodule. Also, we introduce a situation where if $N$ is a $gr$-prime $R$-submodule of $M$, then $(N_g : M_g)$ is a maximal ideal of $R_e$. We close this article by introducing a situation where if $N$ is a $gr$-$R$-submodule of $M$ such that $N_e$ is a weakly prime $R_e$-submodule of $M_e$, then $N_g$ is a prime $R_e$-submodule of $M_g$.

2010 AMS Subject Classifications: 13 A 02.

Keywords and Phrases: Graded rings, graded modules, prime submodules.
Introduction

Let $G$ be a group and $R$ be a commutative $G$-graded ring which is denoted by $(R, G)$. The elements of $R_g$ are called homogeneous of degree $g$ where $R_g$ are additive subgroups of $R$ indexed by the elements $g \in G$. Consider $\text{supp}(R, G) = \{g \in G : R_g \neq 0\}$. If $x \in R$, then $x$ can be written uniquely as $\sum_{g \in G} x_g$, where $x_g$ is the component of $x$ in $R_g$. Moreover, $R_e$ is a subring of $R$ and $1 \in R_e$. Further, if $r \in R_g$ and $r$ is a unit, then $r^{-1} \in R_g^{-1}$. Let, $h(R) = \bigcup_{g \in G} R_g$. Assume $M$ is a left $R$-module. Then $M$, denoted by $(M, G)$, is a $G$-graded $R$-module (for simplicity, we write $M$ is gr-$R$-module) if there exist additive subgroups $M_g$ of $M$ indexed by the elements $g \in G$ such that $M = \bigoplus_{g \in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Also, we consider $\text{supp}(M, G) = \{g \in G : M_g \neq 0\}$. It is clear that $M_g$ is an $R_e$-submodule of $M$ for all $g \in G$. For more details, one can look in [3,4,5]. Throughout this article, $R$ is commutative ring with unity 1 and $M$ is a left $R$-module.

A $G$-graded ring $R$ is said to be first strongly graded if $1 \in R_g R_{g^{-1}}$ for all $g \in \text{supp}(R, G)$, this is equivalent to say that $\text{supp}(R, G)$ is a subgroup of $G$ and $R_g R_h = R_{gh}$ for all $g, h \in \text{supp}(R, G)$. A $G$-graded $R$-module $M$ is said to be first strongly graded if $\text{supp}(R, G)$ is a subgroup of $G$ and $R_g M_h = M_{gh}$ for all $g \in \text{supp}(R, G), h \in G$. Clearly, $(R, G)$ is first strong if and only if every graded $R$-module is first strongly graded. For more details, one can look in [6]. $(R, G)$ is said to be crossed product over the support if $R_g$ contains a unit for all $g \in \text{supp}(R, G)$. It is not difficult to prove that if $(R, G)$ is crossed product over the support, then $(R, G)$ is first strong. Also, if $R_e$ is a field, then $(R, G)$ is crossed product over the support. For more details, it is nice to see [1]. An $R$-submodule $N$ of a $G$-gr-$R$-module $M$ is said to be graded if $N = \bigoplus_{g \in G} (N \cap M_g)$, a submodule of a graded module need not be graded. For more details, it is good to look quickly in [7].

For a gr-$R$-submodule $N$ of a gr-$R$-module $M$, we define $(N : M) = \{r \in R : rM \subseteq N\}$. Clearly, $(N : M)$ is a graded ideal of $R$, see [2]. A proper gr-$R$-submodule $N$ of a gr-$R$-module $M$ will be called a graded prime $R$-submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $m \in N$ or $r \in (N : M)$. Moreover, it is easy to prove that if $N$ is a graded prime $R$-submodule of $M$, then $(N : M)$ is a graded prime ideal of $R$. 


Results

We begin our article by introducing a situation where every ideal $A$ of $R_e$ has the form $(K : M_e)$ for some $R_e$-submodule $K$ of $M_e$:

**Proposition 0.1.** Let $R$ be a first strongly $G$-graded ring and $M$ be a $gr$-$R$-module. Suppose $A$ is an ideal of $R_e$. Then there exists a proper $gr$-$R$-submodule $N$ of $M$ such that $A = (N_e : M_e)$ if and only if $AM_g = M_g$ and $A = (AM_g : M_g)$ for all $g \in \text{supp}(R,G)$.

**Proof.** Suppose $A = (N_e : M_e)$ for some proper $gr$-$R$-submodule $N$ of $M$. Then $AM_e \subseteq N_e$ and then $AM_e \neq M_e$. Let $g \in \text{supp}(R,G)$. If $AM_g = M_g$, then $M_e = R_g^{-1}M_g = R_g^{-1}AM_g = AR_g^{-1}M_g = AM_e$ that is a contradiction. Thus, $AM_g \neq M_g$. On the other hand, let $a \in A$. Then $aM_g \subseteq AM_g$, so $a \in (AM_g : M_g)$. Thus, $A \subseteq (AM_g : M_g)$. Let $x \in (AM_g : M_g)$. $xM_g \subseteq AM_g$ and then $xM_e = xR_g^{-1}M_g = R_g^{-1}xM_g \subseteq R_g^{-1}AM_g = AR_g^{-1}M_g = AM_e \subseteq N_e$, so $x \in (N_e : M_e) = A$. Thus $(AM_g : M_g) \subseteq A$ and hence $A = (AM_g : M_g)$. The converse is obvious. \[\square\]

A $gr$-$R$-module $M$ will be called $gr$-weakly prime Noetherian if for every $a \in h(R)$ and every $m \in h(M)$, the $gr$-$R$-submodule $RaRm$ is finitely generated. Let $P$ be a prime ideal of $R_e$. Then we define $M_g(P) = \{m \in M_g : Am \subseteq PM_g \text{ for some ideal } A \text{ of } R_e \text{ satisfying } A \not\subseteq P\}$, $g \in G$. It is clear that $M_g(P)$ is an $R_e$-submodule of $M_g$ for all $g \in G$ and $PM_g \subseteq M_g(P)$. Now, we introduce the following results about $M_g(P)$.

**Proposition 0.2.** Let $R$ be a first strongly $G$-graded ring and $M$ be a $gr$-$R$-module. If $P$ is a prime ideal of $R_e$ and $K$ is a $gr$-prime $R$-submodule of $M$ such that $(K_e : M_e) = P$, then $M_g(P) \subseteq K$ for all $g \in \text{supp}(R,G)$.

**Proof.** Let $g \in \text{supp}(R,G)$. Suppose $m \in M_g(P)$. Then there exists an ideal of $R_e$ such that $AP$ and $Am \subseteq PM_g$. However, $PM_g = PR_gM_e = R_gPM_e \subseteq R_gK_e \subseteq K$ and hence $Am \subseteq K$. Since $K$ is $gr$-prime, either $m \in K$ or $Am \subseteq K$. If $Am \subseteq K$, then $AM_e \subseteq K_e$, so $A \subseteq (K_e : M_e) = P$ that is a contradiction. Thus $m \in K$ and hence $M_g(P) \subseteq K$. \[\square\]

**Proposition 0.3.** Let $R$ be a $G$-graded ring and $M$ be a $gr$-$R$-module. Suppose $P$ is a prime ideal of $R_e$ and $g \in G$ such that $M_g/PM_g$ is weakly Noetherian $R_e/P$-module. If $N = M_g(P)$, then $N = M_g$ or $N$ is a prime $R_e$-submodule of $M_g$ such that $P = (N : M_g)$. 
Proof. Suppose $N \neq M_g$. Let $r \in R_e$ and $m \in M_g$ such that $rm \in N$. If $r \in P$, then $rM_g \subseteq PM_g \subseteq N$. Suppose $r \notin P$. Let $A = R_e r R_e$. Then $A$ is an ideal of $R_e$ such that $AP$. Since $M_g/PM_g$ is weakly Noetherian, $Am + PM_g = Am_1 + \ldots + Am_k + PM_g$ for some positive integer $k$ and $m_i \in Am_i, 1 \leq i \leq k$. For each $1 \leq i \leq k$, $m_i \in Am \subseteq N$ and hence there exists ideal $B_i$ of $R_e$ such that $B_i P$ and $B_i m_i \subseteq PM_g$. Let $B = B_1 \cap \ldots \cap B_k$. Then $B$ is an ideal of $R_e$ such that $BP$ because $P$ is prime. Moreover, $BAm \subseteq Bm_1 + \ldots + Bm_k + PM_g \subseteq PM_g$. However, $P$ is prime implies $BAP$. Thus $m \in N$. It follows that $N$ is a prime $R_e$-submodule of $M_g$. Now, let $x \in P$. Then $xM_g \subseteq PM_g \subseteq N$, so $x \in (N : M_g)$. Thus $P \subseteq (N : M_g)$. Suppose $P \neq (N : M_g)$. Then there exists $\alpha \in (N : M_g)$ such that $\alpha \notin P$. Let $t \in M_g$. Then $R_e \alpha P t \subseteq N$. By above technique, $t \in N$. Thus $N = M_g$ that is a contradiction. Hence $P = (N : M_g)$. \(\square\)

By Proposition 0.2, $M_g(P) \neq M_g$ if $M$ contains a gr-prime $R$-submodule $K$ such that $(K_e : M_e) = P$ provided that $(R, G)$ is first strong and $g \in \text{supp}(R, G)$. Now, we introduce another situation where $M_g(P) \neq M_g$.

**Proposition 0.4.** Let $R$ be a $G$-graded ring and $M$ be a gr-$R$-module. Suppose $P$ is a prime ideal of $R_e$ and $g \in G$ such that $M_g/PM_g$ is finitely generated and weakly Noetherian $R_e/P$-module. If $P = (PM_g : M_g)$, then $M_g(P) \neq M_g$.

**Proof.** Suppose $M_g(P) = M_g$. Then there exists a positive integer $k$ and elements $m_i \in M_g, 1 \leq i \leq k$ such that $M_g = R_e m_1 + \ldots + R_e m_k + PM_g$. For each $1 \leq i \leq k$, there exists an ideal $A_i$ of $R_e$ such that $A_i P$ and $A_i m_i \subseteq PM_g$. Let $A = A_1 \cap \ldots \cap A_k$. Then $A$ is an ideal of $R_e$ such that $AP$ and $AM_g \subseteq PM_g$, so $A \subseteq (PM_g : M_g) = P$ that is a contradiction. Hence $M_g(P) \neq M_g$. \(\square\)

It is easy to prove that if $N$ is a gr-prime $R$-submodule of a gr-$R$-module $M$, then $(N : M)$ is a gr-prime ideal of $R$, see [2, Proposition 2.4]. Similarly, one can prove that if $N$ is a gr-prime $R$-submodule of a gr-$R$-module $M$, then $(N_g : M_g)$ is a prime ideal of $R_e$ for all $g \in G$. In this article, we introduce a situation where if $N$ is a gr-prime $R$-submodule of a gr-$R$-module $M$, then $(N_g : M_g)$ is a maximal ideal of $R_e$, see Corollary 0.7.

**Proposition 0.5.** Let $R$ be a first strongly $G$-graded ring and $M$ be a gr-$R$-module. If $M_e$ is Artinian and prime, then $R_e/\text{Ann}(M_g)$ is a field for all $g \in \text{supp}(R, G)$.
Proof. Let $T = \{N : N$ is a nontrivial $R_e$-submodule of $M_e\}$. Suppose that $N_0$ is a minimal element of $T$. Obviously $N_0$ is a non-zero simple module. Hence there exists a nonzero $a \in M_e$ such that $N_0 = R_e a \cong R_e/\text{Ann}(a)$ and $\text{Ann}(a)$ is a maximal ideal of $R_e$. Since $M_e$ is prime, $\text{Ann}(a) = \text{Ann}(M_e)$. Let $r \in \text{Ann}(M_e)$ and $g \in \text{supp}(R,G)$. Then $rM_g = rR_gM_e = R_g rM_e = R_g.\{0\} = \{0\}$, so $r \in \text{Ann}(M_g)$. Let $s \in \text{Ann}(M_g)$. Then $sM_e = sR_0^{-1}M_g = R_0^{-1}sM_g = R_0^{-1}.\{0\} = \{0\}$, so $s \in \text{Ann}(M_e)$. Hence $\text{Ann}(M_e) = \text{Ann}(M_g)$. Consequently, $\text{Ann}(M_g)$ is a maximal ideal of $R_e$ and then $R_e/\text{Ann}(M_g)$ is a field. \qed

Corollary 0.6. Let $R$ be a first strongly $G$-graded ring and $M$ be a gr-$R$-module. If $M_e$ is Artinian, faithful and prime, then $R_e$ is a field.

Corollary 0.7. Let $R$ be a first strongly $G$-graded ring, $M$ be a gr-$R$-module and $N$ be a gr-prime $R$-submodule of $M$. If $M_e$ is Artinian, then $(N_g : M_g)$ is a maximal ideal of $R_e$ for all $g \in \text{supp}(R,G)$.

Proof. Since $N$ is a gr-prime $R$-submodule of $M$, by [2, Proposition 2.5], $N_e$ is a prime $R_e$-submodule of $M_e$. Then $M_e/N_e$ is an Artinian prime $R_e$-module, consequently, by Proposition 0.5, $R_e/\text{Ann}(M_g/N_g)$ is a field for all $g \in \text{supp}(R,G)$, and then $(N_g : M_g) = \text{Ann}(M_g/N_g)$ is a maximal ideal of $R_e$ for all $g \in \text{supp}(R,G)$. \qed

A proper $R$-submodule $N$ of an $R$-module $M$ is said to be weakly prime if whenever $a, b \in R$ and $x \in M$ such that $abx \in N$, then either $ax \in N$ or $bx \in N$. Obviously, any prime submodule is a weakly prime submodule, but the converse is not always correct. We close our article by introducing a situation where if $N$ is a gr-$R$-submodule of $M$ such that $N_e$ is a weakly prime $R_e$-submodule of $M_e$, then $N_g$ is a prime $R_e$-submodule of $M_g$.

Proposition 0.8. Let $R$ be a $G$-graded ring and $M$ be a gr-$R$-module such that $(R,G)$ is crossed product over the support and $M_e$ satisfies the DCC on cyclic $R_e$-submodules. Assume that $M_e$ is $R_e$-torsion free. Suppose $N$ is a gr-$R$-submodule of $M$ such that $N_e$ is weakly prime $R_e$-submodule of $M_e$. Then $N_g$ is prime $R_e$-submodule of $M_g$ for all $g \in \text{supp}(R,G), g \neq e$.

Proof. Let $g \in \text{supp}(R,G), g \neq e$. Suppose $r \in R_e, a \in M_g$ such that $ra \in N_g$. Assume $r \notin (N_g : M_g)$. Then there exists $b \in M_g$ such that $rb \notin N_g$. Since $(R,G)$ is crossed product over the support, $R_g^{-1}$ contains a unit, say $x$. Consider the following chain of $R_e$-submodules of $M_e$: $\ldots \subseteq R_e x r^3(a + b) \subseteq R_e x r^2(a + b) \subseteq R_e x r(a + b)$. For some positive
integer $n$, we have $R_e x r^{n+1}(a+b) = R_e x r^n(a+b)$, that is $x r^n(rt-1)(a+b) = 0 \in N_e$ for some $t \in R_e$. If $r^n(a+b) \in N_e$, then $r(a+b) \in N_e$ and then $0 \neq r(a+b) \in N_e \cap M_g \subseteq M_e \cap M_g$ that is a contradiction since $g \neq e$. So, $x(rt-1)(a+b) \in N_e$. Now, $xrta - xa + x(rt-1)b = x(rt-1)(a+b) \in N_e$, on the other hand $xrta = xt(ra) \in R_{g-1}R_e M_g \subseteq N_e$, so

$$-xa + x(rt-1)b \in N_e............(*)$$

before and then $-a + (rt-1)b \in N_g$. We get that $-ra + r(rt-1)b \in N_g$, on the other hand, $-ra \in N_g$, so $r(rt-1)b \in N_g$ and then $xr(rt-1)b \in N_e$. If $rb \in N_e$, then $0 \neq rb \in N_e \cap M_g \subseteq M_e \cap M_g$ that is a contradiction, so $x(rt-1)b \in N_e$ and then by $(*)$, $xa \in N_e$ and then $a \in N_g$. Hence $N_g$ is a prime $R_e$-submodule of $M_g$. \qed

References


Rashid Abu-Dawwas  
Department of Mathematics  
Yarmouk University  
Irbid, Jordan  
e-mail: rrashid@yu.edu.jo

Khaldoun Al-Zoubi  
Department of Mathematics and Statistics  
Jordan University of Science and Technology  
Irbid, Jordan  
e-mail: kfzoubi@just.edu.jo

and

Malik Bataineh  
Department of Mathematics and Statistics  
Jordan University of Science and Technology  
Irbid, Jordan  
e-mail: msbataineh@just.edu.jo