Prime Submodules of Graded Modules

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Received : March 2012. Accepted : October 2012

Abstract

Let G be a group, R be a G -graded ring and M be a G -graded R -module. Suppose P is a prime ideal of R_e and $g \in G$. In this article, we define

$$M_g(P) = \{m \in M_g : Am \subseteq PM_g \\ \text{for some ideal } A \text{ of } R_e \text{ satisfying } A \not\subseteq P\}$$

that is an R_e -submodule of M_g , and we investigate some results on this submodule. Also, we introduce a situation where if N is a gr -prime R -submodule of M , then $(N_g : M_g)$ is a maximal ideal of R_e . We close this article by introducing a situation where if N is a gr - R -submodule of M such that N_e is a weakly prime R_e -submodule of M_e , then N_g is a prime R_e -submodule of M_g .

2010 AMS Subject Classifications : 13 A 02.

Keywords and Phrases : Graded rings, graded modules, prime submodules.

Introduction

Let G be a group and R be a commutative G -graded ring which is denoted by (R, G) . The elements of R_g are called homogeneous of degree g where R_g are additive subgroups of R indexed by the elements $g \in G$. Consider $\text{supp}(R, G) = \{g \in G : R_g \neq 0\}$. If $x \in R$, then x can be written uniquely as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g . Moreover, R_e is a subring of R and $1 \in R_e$. Further, if $r \in R_g$ and r is a unit, then $r^{-1} \in R_{g^{-1}}$. Let, $h(R) = \bigcup_{g \in G} R_g$. Assume M is a left R -module. Then M , denoted by (M, G) , is a G -graded R -module (for simplicity, we write M is gr- R -module) if there exist additive subgroups M_g of M indexed by the elements $g \in G$ such that $M = \bigoplus_{g \in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Also, we consider $\text{supp}(M, G) = \{g \in G : M_g \neq 0\}$. It is clear that M_g is an R_e -submodule of M for all $g \in G$. For more details, one can look in [3,4,5]. Throughout this article, R is commutative ring with unity 1 and M is a left R -module.

A G -graded ring R is said to be first strongly graded if $1 \in R_g R_{g^{-1}}$ for all $g \in \text{supp}(R, G)$, this is equivalent to say that $\text{supp}(R, G)$ is a subgroup of G and $R_g R_h = R_{gh}$ for all $g, h \in \text{supp}(R, G)$. A G -graded R -module M is said to be first strongly graded if $\text{supp}(R, G)$ is a subgroup of G and $R_g M_h = M_{gh}$ for all $g \in \text{supp}(R, G)$, $h \in G$. Clearly, (R, G) is first strong if and only if every graded R -module is first strongly graded. For more details, one can look in [6]. (R, G) is said to be crossed product over the support if R_g contains a unit for all $g \in \text{supp}(R, G)$. It is not difficult to prove that if (R, G) is crossed product over the support, then (R, G) is first strong. Also, if R_e is a field, then (R, G) is crossed product over the support. For more details, it is nice to see [1]. An R -submodule N of a G -gr- R -module M is said to be graded if $N = \bigoplus_{g \in G} (N \cap M_g)$, a submodule of a graded module need not be graded. For more details, it is good to look quickly in [7].

For a gr- R -submodule N of a gr- R -module M , we define $(N : M) = \{r \in R : rM \subseteq N\}$. Clearly, $(N : M)$ is a graded ideal of R , see [2]. A proper gr- R -submodule N of a gr- R -module M will be called a graded prime R -submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $m \in N$ or $r \in (N : M)$. Moreover, it is easy to prove that if N is a graded prime R -submodule of M , then $(N : M)$ is a graded prime ideal of R .

Results

We begin our article by introducing a situation where every ideal A of R_e has the form $(K : M_e)$ for some R_e -submodule K of M_e :

Proposition 0.1. *Let R be a first strongly G -graded ring and M be a gr - R -module. Suppose A is an ideal of R_e . Then there exists a proper gr - R -submodule N of M such that $A = (N_e : M_e)$ if and only if $AM_g \neq M_g$ and $A = (AM_g : M_g)$ for all $g \in \text{supp}(R, G)$.*

Proof. Suppose $A = (N_e : M_e)$ for some proper gr - R -submodule N of M . Then $AM_e \subseteq N_e$ and then $AM_e \neq M_e$. Let $g \in \text{supp}(R, G)$. If $AM_g = M_g$, then $M_e = R_{g^{-1}}M_g = R_{g^{-1}}AM_g = AR_{g^{-1}}M_g = AM_e$ that is a contradiction. Thus, $AM_g \neq M_g$. On the other hand, let $a \in A$. Then $aM_g \subseteq AM_g$, so $a \in (AM_g : M_g)$. Thus, $A \subseteq (AM_g : M_g)$. Let $x \in (AM_g : M_g)$. $xM_g \subseteq AM_g$ and then $xM_e = xR_{g^{-1}}M_g = R_{g^{-1}}xM_g \subseteq R_{g^{-1}}AM_g = AR_{g^{-1}}M_g = AM_e \subseteq N_e$, so $x \in (N_e : M_e) = A$. Thus $(AM_g : M_g) \subseteq A$ and hence $A = (AM_g : M_g)$. The converse is obvious. $\square$

A gr - R -module M will be called gr -weakly prime Noetherian if for every $a \in h(R)$ and every $m \in h(M)$, the gr - R -submodule $RaRm$ is finitely generated. Let P be a prime ideal of R_e . Then we define $M_g(P) = \{m \in M_g : Am \subseteq PM_g \text{ for some ideal } A \text{ of } R_e \text{ satisfying } A \not\subseteq P\}$, $g \in G$. It is clear that $M_g(P)$ is an R_e -submodule of M_g for all $g \in G$ and $PM_g \subseteq M_g(P)$. Now, we introduce the following results about $M_g(P)$.

Proposition 0.2. *Let R be a first strongly G -graded ring and M be a gr - R -module. If P is a prime ideal of R_e and K is a gr -prime R -submodule of M such that $(K_e : M_e) = P$, then $M_g(P) \subseteq K$ for all $g \in \text{supp}(R, G)$.*

Proof. Let $g \in \text{supp}(R, G)$. Suppose $m \in M_g(P)$. Then there exists an ideal of R_e such that AP and $Am \subseteq PM_g$. However, $PM_g = PR_gM_e = R_gPM_e \subseteq R_gK_e \subseteq K$ and hence $Am \subseteq K$. Since K is gr -prime, either $m \in K$ or $Am \subseteq K$. If $Am \subseteq K$, then $AM_e \subseteq K_e$, so $A \subseteq (K_e : M_e) = P$ that is a contradiction. Thus $m \in K$ and hence $M_g(P) \subseteq K$. $\square$

Proposition 0.3. *Let R be a G -graded ring and M be a gr - R -module. Suppose P is a prime ideal of R_e and $g \in G$ such that M_g/PM_g is weakly Noetherian R_e/P -module. If $N = M_g(P)$, then $N = M_g$ or N is a prime R_e -submodule of M_g such that $P = (N : M_g)$.*

Proof. Suppose $N \neq M_g$. Let $r \in R_e$ and $m \in M_g$ such that $rm \in N$. If $r \in P$, then $rM_g \subseteq PM_g \subseteq N$. Suppose $r \notin P$. Let $A = R_e r R_e$. Then A is an ideal of R_e such that AP . Since M_g/PM_g is weakly Noetherian, $Am + PM_g = Am_1 + \dots + Am_k + PM_g$ for some positive integer k and $m_i \in Am_i$, $1 \leq i \leq k$. For each $1 \leq i \leq k$, $m_i \in Am \subseteq N$ and hence there exists ideal B_i of R_e such that $B_i P$ and $B_i m_i \subseteq PM_g$. Let $B = B_1 \cap \dots \cap B_k$. Then B is an ideal of R_e such that BP because P is prime. Moreover, $BAm \subseteq Bm_1 + \dots + Bm_k + PM_g \subseteq PM_g$. However, P is prime implies BAP . Thus $m \in N$. It follows that N is a prime R_e -submodule of M_g . Now, let $x \in P$. Then $xM_g \subseteq PM_g \subseteq N$, so $x \in (N : M_g)$. Thus $P \subseteq (N : M_g)$. Suppose $P \neq (N : M_g)$. Then there exists $\alpha \in (N : M_g)$ such that $\alpha \notin P$. Let $t \in M_g$. Then $R_e \alpha R_e t \subseteq N$. By above technique, $t \in N$. Thus $N = M_g$ that is a contradiction. Hence $P = (N : M_g)$. $\square$

By Proposition 0.2, $M_g(P) \neq M_g$ if M contains a gr-prime R -submodule K such that $(K_e : M_e) = P$ provided that (R, G) is first strong and $g \in \text{supp}(R, G)$. Now, we introduce another situation where $M_g(P) \neq M_g$.

Proposition 0.4. *Let R be a G -graded ring and M be a gr- R -module. Suppose P is a prime ideal of R_e and $g \in G$ such that M_g/PM_g is finitely generated and weakly Noetherian R_e/P -module. If $P = (PM_g : M_g)$, then $M_g(P) \neq M_g$.*

Proof. Suppose $M_g(P) = M_g$. Then there exists a positive integer k and elements $m_i \in M_g$, $1 \leq i \leq k$ such that $M_g = R_e m_1 + \dots + R_e m_k + PM_g$. For each $1 \leq i \leq k$, there exists an ideal A_i of R_e such that $A_i P$ and $A_i m_i \subseteq PM_g$. Let $A = A_1 \cap \dots \cap A_k$. Then A is an ideal of R_e such that AP and $AM_g \subseteq PM_g$, so $A \subseteq (PM_g : M_g) = P$ that is a contradiction. Hence $M_g(P) \neq M_g$. $\square$

It is easy to prove that if N is a gr-prime R -submodule of a gr- R -module M , then $(N : M)$ is a gr-prime ideal of R , see [2, Proposition 2.4]. Similarly, one can prove that if N is a gr-prime R -submodule of a gr- R -module M , then $(N_g : M_g)$ is a prime ideal of R_e for all $g \in G$. In this article, we introduce a situation where if N is a gr-prime R -submodule of a gr- R -module M , then $(N_g : M_g)$ is a maximal ideal of R_e , see Corollary 0.7.

Proposition 0.5. *Let R be a first strongly G -graded ring and M be a gr- R -module. If M_e is Artinian and prime, then $R_e/\text{Ann}(M_g)$ is a field for all $g \in \text{supp}(R, G)$.*

Proof. Let $T = \{N : N \text{ is a nontrivial } R_e\text{-submodule of } M_e\}$. Suppose that N_0 is a minimal element of T . Obviously N_0 is a non-zero simple module. Hence there exists a nonzero $a \in M_e$ such that $N_0 = R_e a \approx R_e / \text{Ann}(a)$ and $\text{Ann}(a)$ is a maximal ideal of R_e . Since M_e is prime, $\text{Ann}(a) = \text{Ann}(M_e)$. Let $r \in \text{Ann}(M_e)$ and $g \in \text{supp}(R, G)$. Then $rM_g = rR_g M_e = R_g r M_e = R_g \cdot \{0\} = \{0\}$, so $r \in \text{Ann}(M_g)$. Let $s \in \text{Ann}(M_g)$. Then $sM_e = sR_{g^{-1}} M_g = R_{g^{-1}} s M_g = R_{g^{-1}} \cdot \{0\} = \{0\}$, so $s \in \text{Ann}(M_e)$. Hence $\text{Ann}(M_e) = \text{Ann}(M_g)$. Consequently, $\text{Ann}(M_g)$ is a maximal ideal of R_e and then $R_e / \text{Ann}(M_g)$ is a field. $\square$

Corollary 0.6. Let R be a first strongly G -graded ring and M be a gr - R -module. If M_e is Artinian, faithful and prime, then R_e is a field.

Corollary 0.7. Let R be a first strongly G -graded ring, M be a gr - R -module and N be a gr -prime R -submodule of M . If M_e is Artinian, then $(N_g : M_g)$ is a maximal ideal of R_e for all $g \in \text{supp}(R, G)$.

Proof. Since N is a gr -prime R -submodule of M , by [2, Proposition. 2.5], N_e is a prime R_e -submodule of M_e . Then M_e / N_e is an Artinian prime R_e -module, consequently, by Proposition 0.5, $R_e / \text{Ann}(M_g / N_g)$ is a field for all $g \in \text{supp}(R, G)$, and then $(N_g : M_g) = \text{Ann}(M_g / N_g)$ is a maximal ideal of R_e for all $g \in \text{supp}(R, G)$. $\square$

A proper R -submodule N of an R -module M is said to be weakly prime if whenever $a, b \in R$ and $x \in M$ such that $abx \in N$, then either $ax \in N$ or $bx \in N$. Obviously, any prime submodule is a weakly prime submodule, but the converse is not always correct. We close our article by introducing a situation where if N is a gr - R -submodule of M such that N_e is a weakly prime R_e -submodule of M_e , then N_g is a prime R_e -submodule of M_g .

Proposition 0.8. Let R be a G -graded ring and M be a gr - R -module such that (R, G) is crossed product over the support and M_e satisfies the DCC on cyclic R_e -submodules. Assume that M_e is R_e -torsion free. Suppose N is a gr - R -submodule of M such that N_e is weakly prime R_e -submodule of M_e . Then N_g is prime R_e -submodule of M_g for all $g \in \text{supp}(R, G)$, $g \neq e$.

Proof. Let $g \in \text{supp}(R, G)$, $g \neq e$. Suppose $r \in R_e$, $a \in M_g$ such that $ra \in N_g$. Assume $r \notin (N_g : M_g)$. Then there exists $b \in M_g$ such that $rb \notin N_g$. Since (R, G) is crossed product over the support, $R_{g^{-1}}$ contains a unit, say x . Consider the following chain of R_e -submodules of M_e : $\dots \subseteq R_e x r^3 (a + b) \subseteq R_e x r^2 (a + b) \subseteq R_e x r (a + b)$. For some positive

integer n , we have $R_e x r^{n+1}(a+b) = R_e x r^n(a+b)$, that is $x r^n(rt-1)(a+b) = 0 \in N_e$ for some $t \in R_e$. If $r^n(a+b) \in N_e$, then $r(a+b) \in N_e$ and then $0 \neq r(a+b) \in N_e \cap M_g \subseteq M_e \cap M_g$ that is a contradiction since $g \neq e$. So, $x(rt-1)(a+b) \in N_e$. Now, $x r t a - x a + x(rt-1)b = x(rt-1)(a+b) \in N_e$, on the other hand $x r t a = x t(r a) \in R_{g^{-1}} R_e N_g \subseteq N_e$, so

$$-x a + x(rt-1)b \in N_e \dots\dots\dots (*)$$

before and then $-a + (rt-1)b \in N_g$. We get that $-r a + r(rt-1)b \in N_g$, on the other hand, $-r a \in N_g$, so $r(rt-1)b \in N_g$ and then $x r(rt-1)b \in N_e$. If $r b \in N_e$, then $0 \neq r b \in N_e \cap M_g \subseteq M_e \cap M_g$ that is a contradiction, so $x(rt-1)b \in N_e$ and then by (*), $x a \in N_e$ and then $a \in N_g$. Hence N_g is a prime R_e -submodule of M_g . $\square$

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