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On some seminormed sequence spaces defined by Orlicz function

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Abstract

The sequence space BV_σ was introduced and studied by Mursaleen [9]. In this paper we extend BV_σ to $BV_\sigma(M, p, q, r)$ on a seminormed complex linear space by using orlicz function. We give various properties and some inclusion relations on this space.

1. Introduction

Let ℓ_∞ , c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with $(x_k) \in \mathbf{R}$ or $\mathbf{C}$ the usual norm $\|x\| = \sup_k |x_k|$, where $k \in \mathbf{N} = 1, 2, 3, \dots$, the positive integers. Let σ be an injection of the set of positive integers $\mathbf{N}$ into itself having no finite orbit and T be the operator defined on ℓ_∞ by $T((x_n)_{n=1}^\infty) = (x_{\sigma(n)})_{n=1}^\infty$.

A positive linear functional ϕ with $\|\phi\| = 1$ is called a σ -mean or an invariant mean if $\phi(x) = \phi(Tx)$ for all $x \in \ell_\infty$.

A sequence x is said to be σ -convergent, denoted by $x \in V_\sigma$, if $\phi(x)$ takes the same value, called $\sigma - \lim x$, for all σ -means ϕ . (Schaefer [14])

$$V_\sigma = \left\{ x = (x_n) : \sum_{m=1}^{\infty} t_{m,n}(x) = L \text{ uniformly in } n, \ L = \sigma - \lim x \right\}.$$

Where $m \geq 0, n \geq 0$

$$t_{m,n}(x) = \frac{x_n + x_{\sigma(n)} + \dots + x_{\sigma^m(n)}}{m+1}, \quad \text{and } t_{-1,n} = 0.$$

Where $\sigma^m(n)$ denotes the m - iterative of σ at n . In particular, if σ is the translation a σ -mean is often called a Banach limit and V_σ reduces to f , the set of almost convergent sequence [5]. Subsequently invariant means have been studied by Ahmad and Mursaleen [1], Mursaleen [8], Rami [12] and many others.

The concept of paranormed is closely related to linear metric spaces. It is generalization of that of absolute value. Let X be a linear space. A function $g : X \rightarrow R$ is called paranorme, if

$$(P1) \ g(x) \geq 0, \text{ for all } x \in X,$$

$$(P2) \ g(-x) = g(x), \text{ for all } x \in X,$$

(P3) $g(x + y) \leq g(x) + g(y)$, for all $x, y \in X$,

(P4) if λ_n is a sequence of scalar with $\lambda_n \rightarrow \lambda$ ($n \rightarrow \infty$) and (x_n) is a sequence of vector with $g(x_n - x) \rightarrow 0$ ($n \rightarrow \infty$) then $g(\lambda_n x_n - \lambda x) \rightarrow 0$ ($n \rightarrow \infty$).

A paranormed g for which $g(x) = 0$ implies $x = 0$ is called a total paranorm on X and pair (X, g) is called a totally paranormed space. It is well known that the metric of any linear metric space is given by total paranorm (cf [15, Theorem 10.4.2, p-183]).

A map $M : R \rightarrow [0, +\infty]$ is called to be an orlicz function if M is even, convex left continuous on R_+ , continuous at zero, convex $M(0) = 0$ and $M(u) \rightarrow \infty$ as $u \rightarrow \infty$.

If M takes the value zero only at zero we write $M > 0$ and if M takes only finite value we will write $M < \infty$. [2,3,6,7,10,13]

W. Orlicz [11] used the idea of orlicz function to construct the space (L^M) Lindenstrauss and Tzafriri [4] use the idea of Orlicz function and defined the sequence space ℓ_M such as

$$\ell_M = \left\{ x = (x_i) : \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm $\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$ becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the sequence space ℓ_p , which is an Orlicz sequence spaces with $M(x) = x^p$ for $1 \leq p \leq \infty$.

The Δ_2 -condition is equivalent to $M(Lx) \leq KLM(x)$, for all $x \geq 0$ and for $L > 1$.

An Orlicz function M can be represented in the following integral from

$$M(x) = \int_0^x \eta(t) dt,$$

where η is known as the kernal of M is right differentiable for $t \geq 0$, $\eta(0) =$

0, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \rightarrow \infty$ as $t \rightarrow \infty$. Note that an Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1.$$

Let E be a sequence space. Then E is called

- (i) A sequence space E is said to be symmetric if $(x_n) \in E$ implies $(x_{\pi(n)}) \in E$, where $\pi(n)$ is a permutation of the elements of $\mathbf{N}$,
- (ii) Solid (or normal), if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ for all sequences of scalar (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbf{N}$.

Lemma 1.1. A sequence space E is solid implies E is monotone.

Mursaleen [9] defined the sequence space.

$$BV_\sigma = \left\{ x \in \ell_\infty : \sum_m |\phi_{m,n}(x)| < \infty \text{ uniformly in } n \right\},$$

Where

$$\phi_{m,n}(x) = t_{m,n}(x) - t_{m-1,n}(x)$$

assuming

$$t_{m,n}(x) = 0, \text{ for } m = -1.$$

A straight forward calculation shows that

$$\phi_{m,n} = \begin{cases} \frac{1}{m(m+1)} \sum_{j=1}^m j(x_{\sigma^j(n)} - x_{\sigma^{j-1}(n)}) & (m \geq 1) \\ x_n & (m = 0). \end{cases}$$

Note that for any sequence x, y and scalar λ we have

$$\phi_{m,n}(x + y) = \phi_{m,n}(x) + \phi_{m,n}(y) \text{ and } \phi_{m,n}(\lambda x) = \lambda \phi_{m,n}(x).$$

2. Main Results

Let M be an Orlicz function, $p = (p_m)$ be any sequence of strictly positive real numbers, $r \geq 0$ and (X, q) be a seminorm space over the field $\mathbf{C}$ of complex number with seminorm q . Now we define the following sequence spaces,

$$BV_\sigma(M, p, q, r) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right]^{p_m} < \infty, \text{ uniformly in } n \text{ and for some } \rho > 0 \right\}.$$

For $M(x) = x$, we get

$$BV_\sigma(p, q, r) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} \left[\left(q \left(|\phi_{m,n}(x)| \right) \right) \right]^{p_m} < \infty, \text{ uniformly in } n \text{ and for some } \rho > 0 \right\}.$$

For $p_m = 1$ for all m , we get

$$BV_\sigma(M, q, r) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right] < \infty, \text{ uniformly in } n \text{ and for some } \rho > 0 \right\}.$$

For $r = 0$, we get

$$BV_\sigma(M, p, q) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} \left[M \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right]^{p_m} \right. \\ \left. < \infty, \text{ uniformly in } n \text{ and for some } \rho > 0 \right\}.$$

For $M(x) = x$ and $r = 0$ we get

$$BV_\sigma(p, q) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} \left[\left(q \left(|\phi_{m,n}(x)| \right) \right) \right]^{p_m} \right. \\ \left. < \infty, \text{ uniformly in } n \text{ and for some } \rho > 0 \right\}.$$

For $p_m = 1$ for all m and $r = 0$ we get

$$BV_\sigma(M, q) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} \left[M \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right] \right. \\ \left. < \infty, \text{ uniformly in } n \text{ and for some } \rho > 0 \right\}.$$

For $M(x) = x$, $p_m = 1$ for all m , $r = 0$ and $q(x) = |x|$ we get

$$BV_\sigma = \left\{ x = (x_k) : \sum_{m=1}^{\infty} |\phi_{m,n}(x)| \right. \\ \left. < \infty, \text{ uniformly in } n \text{ and for some } \rho > 0 \right\}.$$

Theorem 2.1. The sequence space $BV_\sigma(M, p, q, r)$ is a linear space over the field $\mathbf{C}$ of complex numbers.

Proof. Let $x, y \in BV_\sigma(M, p, q, r)$ and $\alpha, \beta \in \mathbf{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sum_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho_1} \right) \right) \right]^{p_m} < \infty$$

and

$$\sum_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(q \left(\frac{|\phi_{m,n}(y)|}{\rho_2} \right) \right) \right]^{p_m} < \infty \quad \text{uniformly in } n.$$

Define $\rho_3 = \max \left(2|\alpha|\rho_1, 2|\beta|\rho_2 \right)$. Since M is non-decreasing and convex we have

$$\begin{aligned} & \sum_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(q \left(\frac{|\alpha\phi_{m,n}(x) + \beta\phi_{m,n}(y)|}{\rho_3} \right) \right) \right]^{p_m} \\ & \leq \sum_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(q \left(\frac{|\alpha\phi_{m,n}(x)|}{\rho_3} + \frac{|\beta\phi_{m,n}(y)|}{\rho_3} \right) \right) \right]^{p_m} \\ & \leq \sum_{m=1}^{\infty} \frac{1}{m^r} \frac{1}{2} \left[M \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho_3} \right) \right) + M \left(q \left(\frac{|\beta\phi_{m,n}(y)|}{\rho_3} \right) \right) \right] < \infty \quad \text{uniformly in } n. \end{aligned}$$

This proves that $BV_\sigma(M, p, q, r)$ is a linear space over the field $\mathbf{C}$ of complex numbers.

Theorem 2.2. For any Orlicz function M and a bounded sequence $p = (p_m)$ of strictly positive real numbers, $BV_\sigma(M, p, q, r)$ is a paramormed (need not be total paranormed) space with

$$g(x) = \inf_{n \geq 1} \left\{ \rho^{p_n/k} : \left(\sum_{m=1}^{\infty} m^{-r} \left[M \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right]^{p_m} \right)^{1/K} \leq 1, \right. \\ \left. \text{uniformly in } n \right\}.$$

Where $K = \max \left(1, \sup p_m \right)$

Proof. It is clear that $g(x) = g(-x)$. Since $M(0) = 0$, we get

$$\inf \left\{ \rho^{p_n/K} \right\} = 0 \quad \text{for } x = 0$$

By using Theorem 1, for $\alpha = \beta = 1$, we get

$$g(x + y) \leq g(x) + g(y)$$

For the continuity of scalar multiplication let $l \neq 0$ be any complex number. Then by definition we have

$$g(lx) = \inf_{n \geq 1} \left\{ \rho^{p_n/k} : \left(\sum_{m=1}^{\infty} m^{-r} \left[M \left(q \left(\frac{|\phi_{m,n}(lx)|}{\rho} \right) \right) \right]^{p_m} \right)^{1/K} \leq 1, \right. \\ \left. \text{uniformly in } n \right\}$$

$$g(lx) = \inf_{n \geq 1} \left\{ \left(s|l| \right)^{p_n/k} : \left(\sum_{m=1}^{\infty} m^{-r} \left[M \left(q \left(\frac{|\phi_{m,n}(lx)|}{(s|l|)} \right) \right) \right]^{p_m} \right)^{1/K} \leq 1, \right.$$

uniformly in n $\Bigg\}$

Where $s = \frac{\rho}{|l|}$. Since $|l|^{p_n} \leq \max(1, |l|^H)$, we have

$$g(lx) \leq \max(1, |l|^H) \inf_{n \geq 1} \left\{ s^{p_n/k} : \left(\sum_{m=1}^{\infty} m^{-r} \left[M \left(q \left(\frac{|\phi_{m,n}(x)|}{s} \right) \right) \right]^{p_m} \right)^{1/K} \leq 1, \right.$$

uniformly in n $\Bigg\}$

$$= \max(1, |l|^H) g(x)$$

and therefore $g(lx)$ converges to zero when $g(x)$ converges to zero in $BV_{\sigma}(M, p, q, r)$.

Now let x be fixed element in $BV_{\sigma}(M, p, q, r)$. Then there exists $\rho > 0$ such that

$$g(x) = \inf_{n \geq 1} \left\{ \rho^{p_n/k} : \left(\sum_{m=1}^{\infty} m^{-r} \left[M \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right]^{p_m} \right)^{1/K} \leq 1, \right.$$

uniformly in n $\Bigg\}$

Now

$$g(lx) = \inf_{n \geq 1} \left\{ \rho^{p_n/k} : \left(\sum_{m=1}^{\infty} m^{-r} \left[M \left(q \left(\frac{|\phi_{m,n}(lx)|}{\rho} \right) \right) \right]^{p_m} \right)^{1/K} \leq 1, \right.$$

uniformly in n $\Bigg\} \rightarrow 0$

as $l \rightarrow 0$.

This completes the proof.

Theorem 2.3. Suppose that $0 < p_m \leq t_m < \infty$ for each $m \in \mathbf{N}$ and $r \geq 0$. Then

- (a) $BV_\sigma(M, p, q) \subseteq BV_\sigma(M, t, q)$,
- (b) $BV_\sigma(M, q) \subseteq BV_\sigma(M, q, r)$.

Proof.(i) suppose that $x \in BV_\sigma(M, p, q)$. This implies that

$$\left[M \left(q \left(\frac{|\phi_{i,n}(x)|}{\rho} \right) \right) \right]^{p_m} \leq 1$$
 for sufficiently large values of i , say that $i \geq m_0$ for some fixed $m_0 \in \mathbf{N}$. Since M is non- decreasing, we have

$$\sum_{m=m_0}^{\infty} \left[M \left(q \left(\frac{|\phi_{i,n}(x)|}{\rho} \right) \right) \right]^{t_m} \leq \sum_{m=m_0}^{\infty} \left[M \left(q \left(\frac{|\phi_{i,n}(x)|}{\rho} \right) \right) \right]^{p_m} < \infty.$$

Hence $x \in BV_\sigma(M, t, q)$.

The proof (ii) is trivial.

The following result is consequence of the above result.

Corollary 1. If $0 < p_m \leq 1$ for each m , then $BV_\sigma(M, p, q) \subseteq BV_\sigma(M, q)$.
If $p_m \geq 1$ for all m , then $BV_\sigma(M, q) \subseteq BV_\sigma(M, p, q)$.

Theorem 2.4. The sequence space $BV_\sigma(M, p, q, r)$ is solid.

Proof. Let $x \in BV_\sigma(M, p, q, r)$. This implies that

$$\sum_{m=1}^{\infty} m^{-r} \left[M \left(q \left(\frac{|\phi_{k,n}(x)|}{\rho} \right) \right) \right]^{p_m} < \infty.$$

Let (α_m) be sequence of scalars such that $|\alpha_m| \leq 1$ for all $m \in \mathbf{N}$. Then the result follows from the following inequality.

$$\sum_{m=1}^{\infty} m^{-r} \left[M \left(q \left(\frac{|\alpha_m \phi_{k,n}(x)|}{\rho} \right) \right) \right]^{p_m} \leq \sum_{m=1}^{\infty} m^{-r} \left[M \left(q \left(\frac{|\phi_{k,n}(x)|}{\rho} \right) \right) \right]^{p_m} < \infty.$$

Hence $\alpha x \in BV_{\sigma}(M, p, q, r)$ for all sequences of scalar (α_m) with $|\alpha_m| \leq 1$ for all $m \in \mathbf{N}$, whenever $x \in BV_{\sigma}(M, p, q, r)$.

From Theorem 4 and Lemma (1.1) we have:

Corollary 2. The sequence spaces $BV_{\sigma}(M, p, q, r)$ is monotone.

Theorem 2.5. Let M_1 and M_2 be Orlicz functions satisfying Δ_2 -condition and $r, r_1, r_2 \geq 0$. Then we have

- (i) If $r > 1$ then $BV_{\sigma}(M_1, p, q, r) \subseteq BV_{\sigma}(M_1 M_2, p, q, r)$,
- (ii) $BV_{\sigma}(M_1, p, q, r) \cap BV_{\sigma}(M_2, p, q, r) \subseteq BV_{\sigma}(M_1 + M_2, p, q, r)$,
- (iii) If $r_1 \leq r_2$ then $BV_{\sigma}(M, p, q, r_1) \subseteq BV_{\sigma}(M, p, q, r_2)$.

Proof. (i). Since M is continuous at 0 from right, for $\epsilon > 0$ there exist $0 < \delta < 1$ such that $0 \leq c \leq \delta$ implies $M(c) < \epsilon$. If we define

$$I_1 = \left\{ m \in \mathbf{N} : M_1 \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \leq \delta \text{ for some } \rho > 0 \right\},$$

$$I_2 = \left\{ m \in \mathbf{N} : M_1 \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) > \delta \text{ for some } \rho > 0 \right\}.$$

then, when $M_1 \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) > \delta$, we get

$$M \left[M_1 \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right] \leq \left(\frac{2M(1)}{\delta} \right) \left[M_1 \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right].$$

Hence for $x \in BV_\sigma(M_1, p, q, r)$ and $r > 1$

$$\begin{aligned} & \sum_{m=1}^{\infty} m^{-r} \left[MoM_1 \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right]^{p_m} \\ = & \sum_{m \in I_1} m^{-r} \left[MoM_1 \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right]^{p_m} + \sum_{m \in I_2} m^{-r} \left[MoM_1 \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right]^{p_m} \\ \leq & \sum_{m \in I_1} m^{-r} [\epsilon]^{p_m} + \sum_{m \in I_2} \left(\frac{2M(1)}{\delta} \right) \left[M_1 \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right]^{p_m} \\ \leq & \max(\epsilon^h, \epsilon^H) \sum_{m=1}^{\infty} m^{-r} + \max \left\{ \left[\frac{2M(1)}{\delta} \right]^h, \left[\frac{2M(1)}{\delta} \right]^H \right\} \end{aligned}$$

where $0 < h = \inf p_m \leq p_m \leq H = \sup_m p_m < \infty$.

(ii) The proof follows from the following inequality:

$$\begin{aligned} m^{-r} \left[(M_1 + M_2) \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right]^{p_m} & \leq C m^{-r} \left[M_1 \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right]^{p_m} + \\ & C m^{-r} \left[M_2 \left(q \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \right]^{p_m}. \end{aligned}$$

(iii) The proof is straight forward.

Corollary 3. Let M be an Orlicz function satisfying Δ_2 - condition. Then we have

(i) If $r > 1$ then $BV_\sigma(p, q, r) \subseteq BV_\sigma(M, p, q, r)$,

- (ii) $BV_{\sigma}(M, p, q) \subseteq BV_{\sigma}(M, p, q, r)$,
- (iii) $BV_{\sigma}(p, q) \subseteq BV_{\sigma}(p, q, r)$,
- (iv) $BV_{\sigma}(M, q) \subseteq BV_{\sigma}(M, q, r)$.

The proof is straight forward.

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