On Contra $\beta\theta$-Continuous Functions

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Abstract

In this paper, we introduce and investigate the notion of contra $\beta\theta$-continuous functions by utilizing $\beta\theta$-closed sets. We obtain fundamental properties of contra $\beta\theta$-continuous functions and discuss the relationships between contra $\beta\theta$-continuity and other related functions.

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1. Introduction and Preliminaries

In 1996, Dontchev [9] introduced a new class of functions called contra-continuous functions. He defined a function \( f : X \rightarrow Y \) to be contra-continuous if the pre image of every open set of \( Y \) is closed in \( X \). In 2007, Caldas and Jafari [3] introduced and investigated the notion of contra-\( \beta \)-continuity. In this paper, we present a new notion of a contra-continuity called contra-\( \beta \theta \)-continuity which is a strong form of contra-\( \beta \)-continuity.

Throughout this paper \((X, \tau), (Y, \sigma)\) and \((Z, \gamma)\) will always denote topological spaces. Let \( S \) a subset of \( X \). Then we denote the closure and the interior of \( S \) by \( Cl(S) \) and \( Int(S) \) respectively. A subset \( S \) is said to be \( \beta \)-open \([1, 2]\) if \( S \subset Cl(\text{Int}(Cl(S))) \). The complement of a \( \beta \)-open set is said to be \( \beta \)-closed. The intersection of all \( \beta \)-closed sets containing \( S \) is called the \( \beta \)-closure of \( S \) and is denoted by \( \beta Cl(S) \). A subset \( S \) is said to be \( \beta \)-regular \([17]\) if it is both \( \beta \)-open and \( \beta \)-closed. The family of all \( \beta \)-open sets (resp. \( \beta \)-regular sets) of \((X, \tau)\) is denoted by \( \beta O(X, \tau) \) (resp. \( \beta R(X, \tau) \)). The \( \beta \theta \)-closure of a subset \( A \) of a topological space \((X, \tau)\), the following properties are hold:

**Lemma 1.1.** Let \( A \) be a subset of a topological space \((X, \tau)\).
(i) If \( A \in \beta O(X, \tau) \), then \( \beta Cl(A) \in \beta R(X) \).
(ii) \( A \in \beta R(X) \) if and only if \( A \in \beta \theta O(X) \cap \beta \theta C(X) \).

**Lemma 1.2.** For the \( \beta \theta \)-closure of a subset \( A \) of a topological space \((X, \tau)\), the following properties are hold:
(i) $A \subseteq \beta Cl(A) \subseteq \beta Cl_\emptyset(A)$ and $\beta Cl(A) = \beta Cl_\emptyset(A)$ if $A \in \beta O(X)$.

(ii) If $A \subseteq B$, then $\beta Cl_\emptyset(A) \subseteq \beta Cl_\emptyset(B)$.

(iii) If $A_\alpha \in \beta \theta C(X)$ for each $\alpha \in A$, then $\bigcap \{A_\alpha \mid \alpha \in A\} \in \beta \theta C(X)$.

(iv) If $A_\alpha \in \beta \theta O(X)$ for each $\alpha \in A$, then $\bigcup \{A_\alpha \mid \alpha \in A\} \in \beta \theta O(X)$.

(v) $\beta Cl_\emptyset(\beta Cl_\emptyset(A)) = \beta Cl_\emptyset(A)$ and $\beta Cl_\emptyset(A) \in \beta \theta C(X)$.

The union of two $\beta$-$\theta$-closed sets is not necessarily $\beta$-$\theta$-closed as showed in the following example.

**Example 1.3.** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. The subsets $\{a\}$ and $\{b\}$ are $\beta$-$\theta$-closed in $(X, \tau)$ but $\{a, b\}$ is not $\beta$-$\theta$-closed.

### 2. Contra $\beta$-$\theta$-continuous functions

**Definition 1.** A function $f : X \to Y$ is called contra $\beta$-$\theta$-continuous if $f^{-1}(V)$ is $\beta$-$\theta$-closed in $X$ for every open set $V$ of $Y$.

**Example 2.1.** ([11]) 1) Let $R$ be the set of real numbers, $\tau$ be the countable extension topology on $R$, i.e. the topology with subbase $\tau_1 \cup \tau_2$, where $\tau_1$ is the Euclidean topology of $R$ and $\tau_2$ is the topology of countable complements of $R$, and $\sigma$ be the discrete topology of $R$. Define a function $f : (R, \tau) \to (R, \sigma)$ as follows: $f(x) = 1$ if $x$ is rational, and $f(x) = 2$ if $x$ is irrational. Then $f$ is not contra $\beta$-$\theta$-continuous, since $\{1\}$ is closed in $(R, \sigma)$ and $f^{-1}(\{1\}) = Q$, where $Q$ is the set of rationals, is not $\beta$-$\theta$-open in $(R, \tau)$.

2) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. We have $\beta O(X, \tau) = \{X, \emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. The $\beta$-$\theta$-closed sets of $(X, \tau)$ are $\{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be defined by $f(a) = c, f(b) = b$ and $f(c) = a$. Then $f$ is contra $\beta$-$\theta$-continuous.

Let $A$ be a subset of a space $(X, \tau)$. The set $\bigcap \{U \in \tau \mid A \subseteq U\}$ is called the kernel of $A$ [15] and is denoted by $ker(A)$.

**Lemma 2.2.** [14]. The following properties hold for subsets $A, B$ of a space $X$:

1) $x \in ker(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in C(X, x)$. 


2) $A \subset \ker(A)$ and $A = \ker(A)$ if $A$ is open in $X$.
3) If $A \subset B$, then $\ker(A) \subset \ker(B)$.

**Theorem 2.3.** The following are equivalent for a function $f : X \to Y$:

1) $f$ is contra $\beta\theta$-continuous;
2) The inverse image of every closed set of $Y$ is $\beta\theta$-open in $X$;

3) For each $x \in X$ and each closed set $V$ in $Y$ with $f(x) \in V$, there exists a $\beta\theta$-open set $U$ in $X$ such that $x \in U$ and $f(U) \subset V$;
4) $f(\beta Cl_{\theta}(A)) \subset Ker(f(A))$ for every subset $A$ of $X$;
5) $\beta Cl_{\theta}(f^{-1}(B)) \subset f^{-1}(Ker(B))$ for every subset $B$ of $Y$.

**Proof.** (1) $\Rightarrow$ (2): Let $U$ be any closed set of $Y$. Since $Y \setminus U$ is open, then by (1), it follows that $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is $\beta\theta$-closed. This shows that $f^{-1}(U)$ is $\beta\theta$-open in $X$.

(1) $\Rightarrow$ (3): Let $x \in X$ and $V$ be a closed set in $Y$ with $f(x) \in V$. By (1), it follows that $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is $\beta\theta$-closed and so $f^{-1}(V)$ is $\beta\theta$-open. Take $U = f^{-1}(V)$ We obtain that $x \in U$ and $f(U) \subset V$.

(3) $\Rightarrow$ (2): Let $V$ be a closed set in $Y$ with $x \in f^{-1}(V)$. Since $f(x) \in V$, by (3) there exists a $\beta\theta$-open set $U$ in $X$ containing $x$ such that $f(U) \subset V$. It follows that $x \in U \subset f^{-1}(V)$. Hence $f^{-1}(V)$ is $\beta\theta$-open.

(2) $\Rightarrow$ (4): Let $A$ be any subset of $X$. Let $y \notin Ker(f(A))$. Then by Lemma 1.2, there exist a closed set $F$ containing $y$ such that $f(A) \cap F = \emptyset$. We have $A \cap f^{-1}(F) = \emptyset$ and since $f^{-1}(F)$ is $\beta\theta$-open then we have $\beta Cl_{\theta}(A) \cap f^{-1}(F) = \emptyset$. Hence we obtain $f(\beta Cl_{\theta}(A)) \cap F = \emptyset$ and $y \notin f(\beta Cl_{\theta}(A))$. Thus $f(\beta Cl_{\theta}(A)) \subset Ker(f(A))$.

(4) $\Rightarrow$ (5): Let $B$ be any subset of $Y$. By (4), $f(\beta Cl_{\theta}(f^{-1}(B))) \subset Ker(B)$ and $\beta Cl_{\theta}(f^{-1}(B)) \subset f^{-1}(Ker(B))$.

(5) $\Rightarrow$ (1): Let $B$ be any open set of $Y$. By (5), $\beta Cl_{\theta}(f^{-1}(B)) \subset f^{-1}(Ker(B)) = f^{-1}(B)$ and $\beta Cl_{\theta}(f^{-1}(B)) = f^{-1}(B)$. So we obtain that $f^{-1}(B)$ is $\beta\theta$-closed in $X$.

**Definition 2.** A function $f : X \to Y$ is said to be contra-continuous [9] (resp. contra-$\alpha$-continuous [12], contra-precontinuous [13], contra-semi-continuous [10], contra-$\beta$-continuous [3] if for each open set $V$ of $Y$, $f^{-1}(V)$ is closed (resp. $\alpha$-closed, preclosed, semi-closed, $\beta$-closed) in $X$. 
For the functions defined above, we have the following implications:

\[
\begin{align*}
A \Rightarrow B \Rightarrow C \\
\downarrow \quad \downarrow \\
E \Rightarrow F \Leftarrow G
\end{align*}
\]

Notation: \( A = \text{contra-continuity}, \ B = \text{contra } \alpha\text{-continuity}, \ C = \text{contra precontinuity}, \ E = \text{contra semi-continuity}, \ F = \text{contra } \beta\text{-continuity}, \ G = \text{contra } \beta\theta\text{-continuity.} \)

**Remark 2.4.** It should be mentioned that none of these implications is reversible as shown by the examples stated below.

**Example 2.5.** Let \( X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\} \) and \( \sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}. \) Then the identity function \( f : (X, \tau) \to (X, \sigma) \) is

1) contra \( \alpha\)-continuous but not contra-continuous [12].

2) contra \( \beta\)-continuous but not contra \( \beta\theta\)-continuous.

**Example 2.6.** ([10]) A contra semicontinuous function need not be contra precontinuous. Let \( f : R \to R \) be the function \( f(x) = [x], \) where \([x]\) is the Gaussian symbol. If \( V \) is a closed subset of the real line, its preimage \( U = f^{-1}(V) \) is the union of the intervals of the form \([n, n + 1], \) \( n \in \mathbb{Z}; \) hence \( U \) is semi-open being union of semi-open sets. But \( f \) is not contra precontinuous, since \( f^{-1}(0.5, 1.5) = [1, 2) \) is not preclosed in \( R. \)

**Example 2.7.** ([10]) A contra precontinuous function need not be contra semicontinuous. Let \( X = \{a, b\}, \tau = \{\emptyset, X\} \) and \( \sigma = \{\emptyset, \{a\}, X\}. \) The identity function \( f : (X, \tau) \to (Y, \sigma) \) is contra precontinuous as only the trivial subsets of \( X \) are open in \( (X, \tau). \) However, \( f^{-1}(\{a\}) = \{a\} \) is not semi-closed in \( (X, \tau); \) hence \( f \) is not contra semicontinuous.

**Example 2.8.** ([11]) Let \( R \) be the set of real numbers, \( \tau \) be the countable extension topology on \( R, \) i.e. the topology with subbase \( \tau_1 \cup \tau_2, \) where \( \tau_1 \) is the Euclidean topology of \( R \) and \( \tau_2 \) is the topology of countable complements of \( R, \) and \( \sigma \) be the discrete topology of \( R. \) Define a function \( f : (R, \tau) \to (R, \sigma) \) as follows: \( f(x) = 1 \) if \( x \) is rational, and \( f(x) = 2 \) if \( x \) is irrational. Then \( f \) is contra \( \delta\)-precontinuous but not contra \( \beta\)-continuous, since \( \{1\} \) is closed in \( (R, \sigma) \) and \( f^{-1}(\{1\}) = Q, \) where \( Q \) is the set of rationals, is not \( \beta\)-open in \( (R, \tau). \)
Example 2.9. ([3]) Let \( X = \{a, b, c\}, \tau = \emptyset, \{a\}, \{b\}, \{a, b\}, X \) and \( Y = \{p, q\}, \sigma = \emptyset, \{p\}, Y \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be defined by \( f(a) = p \) and \( f(b) = f(c) = q \). Then \( f \) is contra \( \beta \)-continuous but not contra-precontinuous since \( f^{-1}(\{q\}) = \{b, c\} \) is \( \beta \)-open but not preopen.

**Definition 3.** A function \( f : X \rightarrow Y \) is said to be
1) \( \beta \theta \)-semiopen if \( f(U) \in SO(Y) \) for every \( \beta \theta \)-open set \( U \) of \( X \);
2) contra \( I(\beta \theta) \)-continuous if for each \( x \in X \) and each \( F \in C(Y, f(x)) \), there exists \( U \in \beta \theta O(X, x) \) such that \( Int(f(U)) \subset F \);
3) \( \beta \theta \)-continuous [17] if \( f^{-1}(F) \) is \( \beta \theta \)-closed in \( X \) for every closed set \( F \) of \( Y \);
4) \( \beta \)-continuous [1] if \( f^{-1}(F) \) is \( \beta \)-closed in \( X \) for every closed set \( F \) of \( Y \).

We note that, every contra \( \beta \theta \)-continuous function is a contra \( I(\beta \theta) \)-continuous function but the converse need not be true as seen from the following example: Let \( X = \{a, b, c\}, \tau = \emptyset, \{a\}, X \) and \( \sigma = \emptyset, \{b\}, \{c\}, \{b, c\}, X \). Then the identity function \( f : (X, \tau) \rightarrow (X, \sigma) \) is contra \( I(\beta \theta) \)-continuous but not contra \( \beta \theta \)-continuous.

**Theorem 2.10.** If a function \( f : X \rightarrow Y \) is contra \( I(\beta \theta) \)-continuous and \( \beta \theta \)-semiopen, then \( f \) is contra \( \beta \theta \)-continuous.

**Proof.** Suppose that \( x \in X \) and \( F \in C(Y, f(x)) \). Since \( f \) is contra \( I(\beta \theta) \)-continuous, there exists \( U \in \beta \theta O(X, x) \) such that \( Int(f(U)) \subset F \). By hypothesis \( f \) is \( \beta \theta \)-semiopen, therefore \( f(U) \in SO(Y) \) and \( f(U) \subset Cl(Int(f(U))) \subset F \). This shows that \( f \) is contra \( \beta \theta \)-continuous.

**Theorem 2.11.** If a function \( f : X \rightarrow Y \) is contra \( \beta \theta \)-continuous and \( Y \) is regular, then \( f \) is \( \beta \theta \)-continuous.

**Proof.** Let \( x \) be an arbitrary point of \( X \) and \( V \) be an open set of \( Y \) containing \( f(x) \). Since \( Y \) is regular, there exists an open set \( W \) in \( Y \) containing \( f(x) \) such that \( Cl(W) \subset U \). Since \( f \) is contra \( \beta \theta \)-continuous, there exists \( U \in \beta \theta O(X, x) \) such that \( f(U) \subset Cl(W) \). Then \( f(U) \subset Cl(W) \subset V \). Hence \( f \) is \( \beta \theta \)-continuous.

**Theorem 2.12.** Let \( \{X_i : i \in \Omega\} \) be any family of topological spaces. If a function \( f : X \rightarrow \prod X_i \) is contra \( \beta \theta \)-continuous, then \( Pr_i \circ f : X \rightarrow X_i \) is contra \( \beta \theta \)-continuous for each \( i \in \Omega \), where \( Pr_i \) is the projection of \( \prod X_i \) onto \( X_i \).
Proof. For a fixed $i \in \Omega$, let $V_i$ be any open set of $X_i$. Since $Pr_i$ is continuous, $Pr_i^{-1}(V_i)$ is open in $\prod X_i$. Since $f$ is contra $\beta\theta$-continuous, $f^{-1}(Pr_i^{-1}(V_i)) = (Pr_i \circ f)^{-1}(V_i)$ is $\beta\theta$-closed in $X$. Therefore, $Pr_i \circ f$ is contra $\beta\theta$-continuous for each $i \in \Omega$.

Theorem 2.13. Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $g \circ f : X \rightarrow Z$ functions. Then the following hold:
1) If $f$ is contra $\beta\theta$-continuous and $g$ is continuous, then $g \circ f$ is contra $\beta\theta$-continuous;
2) If $f$ is $\beta\theta$-continuous and $g$ is contra-continuous, then $g \circ f$ is contra $\beta\theta$-continuous;
3) If $f$ is contra $\beta\theta$-continuous and $g$ is contra-continuous, then $g \circ f$ is $\beta\theta$-continuous;
4) If $f$ is weakly $\beta$-irresolute and $g$ is contra $\beta\theta$-continuous, then $g \circ f$ is contra $\beta\theta$-continuous;
5) If $f$ is strongly $\beta$-irresolute and $g$ is contra $\beta$-continuous, then $g \circ f$ is contra $\beta\theta$-continuous.

3. Properties of contra $\beta\theta$-continuous functions

Definition 4. [7, 5] A topological space $(X, \tau)$ is said to be:

1) $\beta\theta$-$T_0$ (resp. $\beta\theta$-$T_1$) if for any distinct pair of points $x$ and $y$ in $X$, there is a $\beta\theta$-open $U$ in $X$ containing $x$ but not $y$ or (resp. and) a $\beta\theta$-open set $V$ in $X$ containing $y$ but not $x$.
2) $\beta\theta$-$T_2$ (resp. $\beta$-$T_2$ [16]) if for every pair of distinct points $x$ and $y$, there exist two $\beta\theta$-open (resp. $\beta$-open) sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

From the definitions above, we obtain the following diagram:

$\beta\theta$-$T_2 \Rightarrow \beta\theta$-$T_1 \Rightarrow \beta\theta$-$T_0$.

Theorem 3.1. [8] If $(X, \tau)$ is $\beta\theta$-$T_0$, then $(X, \tau)$ is $\beta\theta$-$T_2$.

Proof. For any points $x \neq y$ let $V$ be a $\beta\theta$-open set that $x \in V$ and $y \notin V$. Then, there exists $U \in \beta O(X, \tau)$ such that $x \in U \subset \beta Cl_\theta(U) \subset V$. 
By Lemma 1.1 and 1.2, $\beta Cl_\theta(U) \in \beta R(X, \tau)$. Then $\beta Cl_\theta(U)$ is $\beta$-$\theta$-open and also $X \setminus \beta Cl_\theta(U)$ is a $\beta$-$\theta$-open set containing $y$. Therefore, $X$ is $\beta \theta$-$T_2$.

**Remark 3.2.** For a topological space $(X, \tau)$ the three properties in the diagram are equivalent.

**Theorem 3.3.** A topological space $(X, \tau)$ is $\beta \theta$-$T_2$ if and only if the singletons are $\beta$-$\theta$-closed sets.

**Proof.** Suppose that $(X, \tau)$ is $\beta \theta$-$T_2$ and $x \in X$. Let $y \in X \setminus \{x\}$. Then $x \neq y$ and so there exists a $\beta$-$\theta$-open set $U_y$ such that $y \in U_y$ but $x \notin U_y$. Consequently $y \in U_y \subset X \setminus \{x\}$ i.e., $X \setminus \{x\} = \bigcup \{U_y/y \in X \setminus \{x\}\}$ which is $\beta$-$\theta$-open.

Conversely, Suppose that $\{p\}$ is $\beta$-$\theta$-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies that $y \in X \setminus \{x\}$. Hence $X \setminus \{x\}$ is a $\beta$-$\theta$-open set containing $y$ but not $x$. Similarly $X \setminus \{y\}$ is a $\beta$-$\theta$-open set containing $x$ but not $y$. From Remark 3.2, $X$ is a $\beta \theta$-$T_2$ space.

**Theorem 3.4.** For a topological space $(X, \tau)$, the following properties are equivalent:
1) $(X, \tau)$ is $\beta \theta$-$T_2$;
2) $(X, \tau)$ is $\beta$-$T_2$;
3) For every pair of distinct points $x, y \in X$, there exist $U, V \in \beta O(X)$ such that $x \in U$, $y \in V$ and $\beta Cl(U) \cap \beta Cl(V) = \emptyset$;
4) For every pair of distinct points $x, y \in X$, there exist $U, V \in \beta R(X)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.
5) For every pair of distinct points $x, y \in X$, there exist $U \in \beta \theta O(X, x)$ and $V \in \beta \theta O(X, y)$ such that $\beta Cl_\theta(U) \cap \beta Cl_\theta(V) = \emptyset$.

**Proof.** (1) $\Rightarrow$ (2): Since $\beta \theta O(X) \subset \beta O(X)$, the proof is obvious.
(2) $\Rightarrow$ (3): This follows from Lemma 5.2 of [17].

(3) $\Rightarrow$ (4): By Lemma 1.1, $\beta Cl(U) \in \beta R(X)$ for every $U \in \beta O(X)$ and the proof immediately follows.

(4) $\Rightarrow$ (5): By Lemma 1.1, every $\beta$-regular set is $\beta$-$\theta$-open and $\beta$-$\theta$-closed. Hence the proof is obvious.

(5) $\Rightarrow$ (1): This is obvious.
**Theorem 3.5.** Let $X$ be a topological space. Suppose that for each pair of distinct points $x_1$ and $x_2$ in $X$, there exists a function $f$ of $X$ into a Urysohn space $Y$ such that $f(x_1) \neq f(x_2)$. Moreover, let $f$ be contra $\beta\theta$-continuous at $x_1$ and $x_2$. Then $X$ is $\beta\theta$-$T_2$.

**Proof.** Let $x_1$ and $x_2$ be any distinct points in $X$. Then suppose that there exist an Urysohn space $Y$ and a function $f : X \to Y$ such that $f(x_1) \neq f(x_2)$ and $f$ is contra $\beta\theta$-continuous at $x_1$ and $x_2$. Let $w = f(x_1)$ and $z = f(x_2)$. Then $w \neq z$. Since $Y$ is Urysohn, there exist open sets $U$ and $V$ containing $w$ and $z$, respectively such that $\text{Cl}(U) \cap \text{Cl}(V) = \emptyset$. Since $f$ is contra $\beta\theta$-continuous at $x_1$ and $x_2$, then there exist $\beta\theta$-open sets $A$ and $B$ containing $x_1$ and $x_2$, respectively such that $f(A) \subset \text{Cl}(U)$ and $f(B) \subset \text{Cl}(V)$. So we have $A \cap B = \emptyset$ since $\text{Cl}(U) \cap \text{Cl}(V) = \emptyset$. Hence, $X$ is $\beta\theta$-$T_2$.

**Corollary 3.6.** If $f$ is a contra $\beta\theta$-continuous injection of a topological space $X$ into a Urysohn space $Y$, then $X$ is $\beta\theta$-$T_2$.

**Proof.** For each pair of distinct points $x_1$ and $x_2$ in $X$ and $f$ is a contra $\beta\theta$-continuous function of $X$ into a Urysohn space $Y$ such that $f(x_1) \neq f(x_2)$ because $f$ is injective. Hence by Theorem 3.5, $X$ is $\beta\theta$-$T_2$.

Recall, that a space $X$ is said to be
1) weakly Hausdorff [18] if each element of $X$ is an intersection of regular closed sets.
2) Ultra Hausdorff [19] if for each pair of distinct points $x$ and $y$ in $X$, there exist clopen sets $A$ and $B$ containing $x$ and $y$, respectively such that $A \cap B = \emptyset$.

**Theorem 3.7.** 1) If $f : X \to Y$ is a contra $\beta\theta$-continuous injection and $Y$ is $T_0$, then $X$ is $\beta\theta$-$T_1$.
2) If $f : X \to Y$ is a contra $\beta\theta$-continuous injection and $Y$ is Ultra Hausdorff, then $X$ is $\beta\theta$-$T_2$.

**Proof.** 1) Let $x_1$, $x_2$ be any distinct points of $X$, then $f(x_1) \neq f(x_2)$. There exists an open set $V$ such that $f(x_1) \in V$, $f(x_2) \notin V$ (or $f(x_2) \in V$, $f(x_1) \notin V$). Then $f(x_1) \notin Y \setminus V$, $f(x_2) \in Y \setminus V$ and $Y \setminus V$ is closed. By Theorem 2.3 $f^{-1}(Y \setminus V) \in \beta\theta O(X, x_2)$ and $x_1 \notin f^{-1}(Y \setminus V)$. Therefore $X$ is $\beta\theta$-$T_0$ and by Theorem 3.1 $X$ is $\beta\theta$-$T_2$.

2) By Remark 3.2, (2) is an immediate consequence of (1).
Definition 5. A space \((X, \tau)\) is said to be \(\beta\theta\)-connected if \(X\) cannot be expressed as the disjoint union of two non-empty \(\beta\theta\)-open sets.

Theorem 3.8. If \(f : X \to Y\) is a contra \(\beta\theta\)-continuous surjection and \(X\) is \(\beta\theta\)-connected, then \(Y\) is connected which is not a discrete space.

Proof. Suppose that \(Y\) is not a connected space. There exist non-empty disjoint open sets \(U_1\) and \(U_2\) such that \(Y = U_1 \cup U_2\). Therefore \(U_1\) and \(U_2\) are clopen in \(Y\). Since \(f\) is contra \(\beta\theta\)-continuous, \(f^{-1}(U_1)\) and \(f^{-1}(U_2)\) are \(\beta\theta\)-open in \(X\). Moreover, \(f^{-1}(U_1)\) and \(f^{-1}(U_2)\) are non-empty disjoint and \(X = f^{-1}(U_1) \cup f^{-1}(U_2)\). This shows that \(X\) is not \(\beta\theta\)-connected. This contradicts that \(Y\) is not connected assumed. Hence \(Y\) is connected.

By other hand, Suppose that \(Y\) is discrete. Let \(A\) be a proper non-empty open and closed subset of \(Y\). Then \(f^{-1}(A)\) is a proper non-empty \(\beta\theta\)-regular subset of \(X\) which is a contradiction to the fact that \(X\) is \(\beta\theta\)-connected.

A topological space \(X\) is said to be \(\beta\theta\)-normal if for each pair of non-empty disjoint closed sets can be separated by disjoint \(\beta\theta\)-open sets.

Theorem 3.9. If \(f : X \to Y\) is a contra \(\beta\theta\)-continuous, closed injection and \(Y\) is normal, then \(X\) is \(\beta\theta\)-normal.

Proof. Let \(F_1\) and \(F_2\) be disjoint closed subsets of \(X\). Since \(f\) is closed and injective, \(f(F_1)\) and \(f(F_2)\) are disjoint closed subsets of \(Y\). Since \(Y\) is normal, \(f(F_1)\) and \(f(F_2)\) are separated by disjoint open sets \(V_1\) and \(V_2\). Since \(Y\) is normal, for each \(i = 1, 2\), there exists an open set \(G_i\) such that \(f(F_i) \subset G_i \subset \text{Cl}(G_i) \subset V_i\). Hence \(F_i \subset f^{-1}(\text{Cl}(G_i))\), \(f^{-1}((\text{Cl}(G_i)) \in \beta\theta O(X)\) for \(i = 1, 2\) and \(f^{-1}((\text{Cl}(G_1)) \cap f^{-1}((\text{Cl}(G_2)) = \emptyset\). Thus \(X\) is \(\beta\theta\)-normal.

Definition 6. The graph \(G(f)\) of a function \(f : X \to Y\) is said to be contra \(\beta\theta\)-closed if for each \((x, y) \in (X \times Y) \setminus G(f)\), there exist a \(\beta\theta\)-open set \(U\) in \(X\) containing \(x\) and a closed set \(V\) in \(Y\) containing \(y\) such that \((U \times V) \cap G(f) = \emptyset\).

Lemma 3.10. A graph \(G(f)\) of a function \(f : X \to Y\) is contra \(\beta\theta\)-closed in \(X \times Y\) if and only if for each \((x, y) \in (X \times Y) \setminus G(f)\), there exists \(U \in \beta\theta O(X)\) containing \(x\) and \(V \in C(Y)\) containing \(y\) such that \(f(U) \cap V = \emptyset\).

Theorem 3.11. If \(f : X \to Y\) is contra \(\beta\theta\)-continuous and \(Y\) is Urysohn, \(G(f)\) is contra \(\beta\theta\)-closed in \(X \times Y\).
Proof. Let \((x, y) \in (X \times Y)\setminus G(f)\). It follows that \(f(x) \neq y\). Since \(Y\) is Urysohn, there exist open sets \(V\) and \(W\) such that \(f(x) \in V\), \(y \in W\) and \(\text{Cl}(V) \cap \text{Cl}(W) = \emptyset\). Since \(f\) is contra \(\beta\theta\)-continuous, there exist a \(U \in \beta\theta O(X, x)\) such that \(f(U) \subset \text{Cl}(V)\) and \(f(U) \cap \text{Cl}(W) = \emptyset\). Hence \(G(f)\) is contra \(\beta\theta\)-closed in \(X \times Y\).

**Theorem 3.12.** Let \(f: X \to Y\) be a function and \(g: X \to X \times Y\) the graph function of \(f\), defined by \(g(x) = (x, f(x))\) for every \(x \in X\). If \(g\) is contra \(\beta\theta\)-continuous, then \(f\) is contra \(\beta\theta\)-continuous.

**Proof.** Let \(U\) be an open set in \(Y\), then \(X \times U\) is an open set in \(X \times Y\). It follows that \(f^{-1}(U) = g^{-1}(X \times U) \in \beta\theta C(X)\). Thus \(f\) is contra \(\beta\theta\)-continuous.

**Theorem 3.13.** Let \(f: X \to Y\) have a contra \(\beta\theta\)-closed graph. If \(f\) is injective, then \(X\) is \(\beta\theta\)-\(T_1\).

**Proof.** Let \(x_1\) and \(x_2\) be any two distinct points of \(X\). Then, we have \((x_1, f(x_2)) \in (X \times Y)\setminus G(f)\). Then, there exist a \(\beta\theta\)-open set \(U\) in \(X\) containing \(x_1\) and \(F \in C(Y, f(x_2))\) such that \(f(U) \cap F = \emptyset\). Hence \(U \cap f^{-1}(F) = \emptyset\). Therefore, we have \(x_2 \notin U\). This implies that \(X\) is \(\beta\theta\)-\(T_1\).

**Definition 7.** A topological space \((X, \tau)\) is said to be
1) Strongly \(S\)-closed [9] if every closed cover of \(X\) has a finite subcover.
2) Strongly \(\beta\theta\)-closed if every \(\beta\theta\)-closed cover of \(X\) has a finite subcover.
3) \(\beta\theta\)-compact [4] if every \(\beta\theta\)-open cover of \(X\) has a finite subcover.
4) \(\beta\theta\)-space [8] if every \(\beta\theta\)-closed set is closed.

**Theorem 3.14.** Let \((X, \tau)\) be a \(\beta\theta\)-space. If \(f: X \to Y\) has a contra-\(\beta\theta\)-closed graph, then the inverse image of a strongly \(S\)-closed set \(K\) of \(Y\) is closed in \((X, \tau)\).

**Proof.** Let \(K\) be a strongly \(S\)-closed set of \(Y\) and \(x \notin f^{-1}(K)\). For each \(k \in K\), \((x, k) \notin G(f)\). By Lemma 3.10, there exists \(U_k \in \beta\theta O(X, x)\) and \(V_k \in C(Y, k)\) such that \(f(U_k) \cap V_k = \emptyset\). Since \(\{K \cap V_k/k \in K\}\) is a closed cover of the subspace \(K\), there exists a finite subset \(K_0 \subset K\) such that \(K \subset \cup\{V_k/k \in K_0\}\). Set \(U = \cap\{U_k/k \in K_0\}\), then \(U\) is open since \(X\) is a \(\beta\theta\)-space. Therefore \(f(U) \cap K = \emptyset\) and \(U \cap f^{-1}(K) = \emptyset\). This shows that \(f^{-1}(K)\) is closed in \((X, \tau)\).

**Theorem 3.15.** Contra \(\beta\theta\)-continuous image of strongly \(\beta\theta\)-closed spaces are compact.
Proof. Suppose that $f : X \to Y$ is a contra $\beta\theta$-continuous surjection. Let $\{V_\alpha/\alpha \in I\}$ be any open cover of $Y$. Since $f$ is contra $\beta\theta$-continuous, then $\{f^{-1}(V_\alpha)/\alpha \in I\}$ is a $\beta\theta$-closed cover of $X$. Since $X$ is strongly $\beta\theta$-compact, then there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha)/\alpha \in I_0\}$. Thus, we have $Y = \bigcup \{V_\alpha/\alpha \in I_0\}$ and $Y$ is strongly $\beta\theta$-closed.

Theorem 3.16. 1) Contra $\beta\theta$-continuous image of $\beta\theta$-compact spaces are strongly $S$-closed. 2) Contra $\beta\theta$-continuous image of a $\beta\theta$-compact space in any $\beta\theta$-space is strongly $\beta\theta$-closed.

Proof. 1) Suppose that $f : X \to Y$ is a contra $\beta\theta$-continuous surjection. Let $\{V_\alpha/\alpha \in I\}$ be any closed cover of $Y$. Since $f$ is contra $\beta\theta$-continuous, then $\{f^{-1}(V_\alpha)/\alpha \in I\}$ is a $\beta\theta$-open cover of $X$. Since $X$ is $\beta\theta$-compact, then there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha)/\alpha \in I_0\}$. Thus, we have $Y = \bigcup \{V_\alpha/\alpha \in I_0\}$ and $Y$ is strongly $S$-closed.

2) Suppose that $f : X \to Y$ is a contra $\beta\theta$-continuous surjection. Let $\{V_\alpha/\alpha \in I\}$ be any $\beta\theta$-closed cover of $Y$. Since $Y$ is a $\beta\theta$-space, then $\{V_\alpha/\alpha \in I\}$ is a closed cover of $Y$. Since $f$ is contra $\beta\theta$-continuous, then $\{f^{-1}(V_\alpha)/\alpha \in I\}$ is a $\beta\theta$-open cover of $X$. Since $X$ is $\beta\theta$-compact, then there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha)/\alpha \in I_0\}$. Thus, we have $Y = \bigcup \{V_\alpha/\alpha \in I_0\}$ and $Y$ is strongly $\beta\theta$-closed.

Theorem 3.17. If $f : X \to Y$ is a weakly $\beta$-irresolute surjective function and $X$ is strongly $\beta\theta$-closed then $Y = f(X)$ is strongly $\beta\theta$-closed.

Proof. Suppose that $f : X \to Y$ is a weakly $\beta$-irresolute surjection. Let $\{V_\alpha/\alpha \in I\}$ be any $\beta\theta$-closed cover of $Y$. Since $f$ is a weakly $\beta$-irresolute, then $\{f^{-1}(V_\alpha)/\alpha \in I\}$ is a $\beta\theta$-closed cover of $X$. Since $X$ is strongly $\beta\theta$-closed, then there exists a finite subset $I_0$ of $I$ such that $X = \bigcup \{f^{-1}(V_\alpha)/\alpha \in I_0\}$. Thus, we have $Y = \bigcup \{V_\alpha/\alpha \in I_0\}$ and $Y$ is strongly $\beta\theta$-closed.

Theorem 3.18. Let $f : X_1 \to Y$ and $g : X_2 \to Y$ be two functions where $Y$ is a Urysohn space and $f$ and $g$ are contra $\beta\theta$-continuous functions. Assume that the product of two $\beta\theta$-open sets is $\beta\theta$-open. Then $\{(x_1,x_2)/f(x_1) = g(x_2)\}$ is $\beta\theta$-closed in the product space $X_1 \times X_2$.

Proof. Let $V$ denote the set $\{(x_1,x_2)/f(x_1) = g(x_2)\}$. In order to show that $V$ is $\beta\theta$-closed, we show that $(X_1 \times X_2) \setminus V$ is $\beta\theta$-open. Let $(x_1,x_2) \notin
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Then $f(x_1) \neq g(x_2)$, Since $Y$ is Uryshon, there exist open sets $U_1$ and $U_2$ of $Y$ containing $f(x_1)$ and $g(x_2)$ respectively, such that $Cl(U_1) \cap Cl(U_2) = \emptyset$. Since $f$ and $g$ are contra $\beta\theta$-continuous, $f^{-1}(Cl(U_1))$ and $g^{-1}(Cl(U_2))$ are $\beta\theta$-open sets containing $x_1$ and $x_2$ in $X_i(i = 1, 2)$. Hence by hypothesis, $f^{-1}(Cl(U_1)) \times g^{-1}(Cl(U_2))$ is $\beta\theta$-open. Further $(x_1, x_2) \in f^{-1}(Cl(U_1)) \times g^{-1}(Cl(U_2)) \subset (X_1 \times X_2) \setminus V$. It follows that $(X_1 \times X_2) \setminus V$ is $\beta\theta$-open. Thus, $V$ is $\beta\theta$-closed in the product space $X_1 \times X_2$.

Corollary 3.19. If $f : X \to Y$ is contra $\beta\theta$-continuous, $Y$ is a Urysohn space and the product of two $\beta\theta$-open sets is $\beta\theta$-open, then $V = \{(x_1, x_2) / f(x_1) = f(x_2)\}$ is $\beta\theta$-closed in the product space $X \times X$.

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