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On Contra $\beta\theta$-Continuous Functions

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Abstract

In this paper, we introduce and investigate the notion of contra $\beta\theta$-continuous functions by utilizing β-θ-closed sets. We obtain fundamental properties of contra $\beta\theta$-continuous functions and discuss the relationships between contra $\beta\theta$-continuity and other related functions.

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1. Introduction and Preliminaries

In 1996, Dontchev [9] introduced a new class of functions called contra-continuous functions. He defined a function $f : X \to Y$ to be contra-continuous if the pre image of every open set of Y is closed in X. In 2007, Caldas and Jafari [3] introduced and investigated the notion of contra β-continuity. In this paper, we present a new notion of a contra-continuity called contra $\beta\theta$-continuity which is a strong form of contra β-continuity.

Throughout this paper (X, τ), (Y, σ) and (Z, γ) will always denote topological spaces. Let S a subset of X. Then we denote the closure and the interior of S by $Cl(S)$ and $Int(S)$ respectively. A subset S is said to be β-open [1, 2] if $S \subset Cl(Int(Cl(S)))$. The complement of a β-open set is said to be β-closed. The intersection of all β-closed sets containing S is called the β-closure of S and is denoted by $\beta Cl(S)$. A subset S is said to be β-regular [17] if it is both β-open and β-closed. The family of all β-open sets (resp. β-regular sets) of (X, τ) is denoted by $\beta O(X, \tau)$ (resp. $\beta R(X, \tau)$). The β-θ-closure of S [17], denoted by $\beta Cl_\theta(S)$, is defined to be the set of all $x \in X$ such that $\beta Cl(O) \cap S \neq \emptyset$ for every $O \in \beta O(X, \tau)$ with $x \in O$. The set $\{x \in X : \beta Cl_\theta(O) \subset S \ for\ some\ O \in \beta(X, x)\}$ is called the β-θ- interior of S and is denoted by $\beta Int_\theta(S)$. A subset S is said to be β-θ-closed [17] if $S = \beta Cl_\theta(S)$. The complement of a β-θ-closed set is said to be β-θ-open. The family of all β-θ-open (resp. β-θ-closed) subsets of X is denoted by $\beta\theta O(X, \tau)$ or $\beta\theta O(X)$ (resp. $\beta\theta C(X, \tau)$). We set $\beta\theta O(X, x) = \{U : x \in U \in \beta\theta O(X, \tau)\}$ and $\beta\theta C(X, x) = \{U : x \in U \in \beta\theta C(X, \tau)\}$.

A function $f : (X, \tau) \to (Y, \sigma)$ is called, weakly β-irresolute [17] (resp. strongly β-irresolute [17]) if $f^{-1}(V)$ is β-θ-open (resp. β-θ-open) in X for every β-θ-open (resp. β-open) set V in Y.

We recall the following three results which were obtained by Noiri [17].

Lemma 1.1. *Let A be a subset of a topological space (X, τ).*
(i) If $A \in \beta O(X, \tau)$, then $\beta Cl(A) \in \beta R(X)$.
(ii) $A \in \beta R(X)$ if and only if $A \in \beta\theta O(X) \cap \beta\theta C(X)$.

Lemma 1.2. *For the β-θ-closure of a subset A of a topological space (X, τ), the following properties are hold:*

(i) $A \subset \beta Cl(A) \subset \beta Cl_\theta(A)$ *and* $\beta Cl(A) = \beta Cl_\theta(A)$ *if* $A \in \beta O(X)$.

(ii) If $A \subset B$, *then* $\beta Cl_\theta(A) \subset \beta Cl_\theta(B)$.

(iii) If $A_\alpha \in \beta\theta C(X)$ *for each* $\alpha \in A$, *then* $\bigcap\{A_\alpha \mid \alpha \in A\} \in \beta\theta C(X)$.

(iv) If $A_\alpha \in \beta\theta O(X)$ *for each* $\alpha \in A$, *then* $\bigcup\{A_\alpha \mid \alpha \in A\} \in \beta\theta O(X)$.

(v) $\beta Cl_\theta(\beta Cl_\theta(A)) = \beta Cl_\theta(A)$ *and* $\beta Cl_\theta(A) \in \beta\theta C(X)$.

The union of two β-θ-closed sets is not necessarily β-θ-closed as showed in the following example.

Example 1.3. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. *The subsets* $\{a\}$ *and* $\{b\}$ *are* β-θ-*closed in* (X, τ) *but* $\{a, b\}$ *is not* β-θ-*closed.*

2. Contra $\beta\theta$-continuous functions

Definition 1. *A function* $f : X \to Y$ *is called contra* $\beta\theta$-*continuous if* $f^{-1}(V)$ *is* β-θ-*closed in* X *for every open set* V *of* Y.

Example 2.1. *([11]) 1) Let* R *be the set of real numbers,* τ *be the countable extension topology on* R, *i.e. the topology with subbase* $\tau_1 \cup \tau_2$, *where* τ_1 *is the Euclidean topology of* R *and* τ_2 *is the topology of countable complements of* R, *and* σ *be the discrete topology of* R. *Define a function* $f : (R, \tau) \to (R, \sigma)$ *as follows:* $f(x) = 1$ *if* x *is rational, and* $f(x) = 2$ *if* x *is irrational. Then* f *is not contra* $\beta\theta$-*continuous, since* $\{1\}$ *is closed in* (R, σ) *and* $f^{-1}(\{1\}) = Q$, *where* Q *is the set of rationals, is not* β-θ-*open in* (R, τ).
2) Let $X = \{a, b, c\}$ *and* $\tau = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. *We have*
$\beta O(X, \tau) = \{X, \emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. *The* β-θ-*closed sets of* (X, τ) *are*
$\{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. *Let* $f : (X, \tau) \to (X, \tau)$ *be defined by* $f(a) = c$, $f(b) = b$ *and* $f(c) = a$. *Then* f *is contra* $\beta\theta$-*continuous.*

Let A be a subset of a space (X, τ). The set $\bigcap\{U \in \tau | A \subset U\}$ is called the kernel of A [15] and is denoted by $ker(A)$.

Lemma 2.2. *[14]. The following properties hold for subsets* A, B *of a space* X:
1) $x \in ker(A)$ *if and only if* $A \cap F \neq \emptyset$ *for any* $F \in C(X, x)$.

2) $A \subset ker(A)$ and $A = ker(A)$ if A is open in X.
3) If $A \subset B$, then $ker(A) \subset ker(B)$.

Theorem 2.3. *The following are equivalent for a function $f : X \to Y$:*

1) f is contra $\beta\theta$-continuous;
2) The inverse image of every closed set of Y is β-θ-open in X;

3) For each $x \in X$ and each closed set V in Y with $f(x) \in V$, there exists a β-θ-open set U in X such that $x \in U$ and $f(U) \subset V$;
4) $f(\beta Cl_\theta(A)) \subset Ker(f(A))$ for every subset A of X;
5) $\beta Cl_\theta(f^{-1}(B)) \subset f^{-1}(Ker(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let U be any closed set of Y. Since $Y \backslash U$ is open, then by (1), it follows that $f^{-1}(Y \backslash U) = X \backslash f^{-1}(U)$ is β-θ-closed. This shows that $f^{-1}(U)$ is β-θ-open in X.

(1) $\Rightarrow$ (3): Let $x \in X$ and V be a closed set in Y with $f(x) \in V$. By (1), it follows that $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V)$ is β-θ-closed and so $f^{-1}(V)$ is β-θ-open. Take $U = f^{-1}(V)$ We obtain that $x \in U$ and $f(U) \subset V$.

(3) $\Rightarrow$ (2): Let V be a closed set in Y with $x \in f^{-1}(V)$. Since $f(x) \in V$, by (3) there exists a β-θ-open set U in X containing x such that $f(U) \subset V$. It follows that $x \in U \subset f^{-1}(V)$. Hence $f^{-1}(V)$ is β-θ-open.

(2) $\Rightarrow$ (4): Let A be any subset of X. Let $y \notin Ker(f(A))$. Then by Lemma 1.2, there exist a closed set F containing y such that $f(A) \cap F = \emptyset$. We have $A \cap f^{-1}(F) = \emptyset$ and since $f^{-1}(F)$ is β-θ-open then we have $\beta Cl_\theta(A) \cap f^{-1}(F) = \emptyset$. Hence we obtain $f(\beta Cl_\theta(A)) \cap F = \emptyset$ and $y \notin f(\beta Cl_\theta(A))$. Thus $f(\beta Cl_\theta(A)) \subset Ker(f(A))$.
(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4), $f(\beta Cl_\theta(f^{-1}(B))) \subset Ker(B)$ and $\beta Cl_\theta(f^{-1}(B)) \subset f^{-1}(Ker(B))$.

(5) $\Rightarrow$ (1): Let B be any open set of Y. By (5), $\beta Cl_\theta(f^{-1}(B)) \subset f^{-1}(Ker(B)) = f^{-1}(B)$ and $\beta Cl_\theta(f^{-1}(B)) = f^{-1}(B)$. So we obtain that $f^{-1}(B)$ is β-θ-closed in X.

Definition 2. *A function $f : X \to Y$ is said to be contra-continuous [9] (resp. contra-α-continuous [12], contra-precontinuous [13], contra-semi-continuous [10], contra-β-continuous [3] if for each open set V of Y, $f^{-1}(V)$ is closed (resp. α-closed, preclosed, semi-closed, β-closed) in X.*

For the functions defined above, we have the following implications:

$$\begin{array}{ccccc} A & \Rightarrow & B & \Rightarrow & C & & \\ & & \Downarrow & & \Downarrow & & \\ & & E & \Rightarrow & F & \Leftarrow & G \end{array}$$

Notation: A = contra-continuity, B = contra α-continuity, C = contra precontinuity, E = contra semi-continuity, F = contra β-continuity, G = contra $\beta\theta$-continuity.

Remark 2.4. *It should be mentioned that none of these implications is reversible as shown by the examples stated below.*

Example 2.5. *Let* $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$ *and* $\sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. *Then the identity function* $f : (X, \tau) \to (X, \sigma)$ *is*

1) contra α*-continuous but not contra-continuous [12].*

2) contra β*-continuous but not contra* $\beta\theta$*-continuous.*

Example 2.6. *([10]) A contra semicontinuous function need not be contra precontinuous. Let* $f : R \to R$ *be the function* $f(x) = [x]$*, where* $[x]$ *is the Gaussian symbol. If* V *is a closed subset of the real line, its preimage* $U = f^{-1}(V)$ *is the union of the intervals of the form* $[n, n + 1]$, $n \in Z$*; hence* U *is semi-open being union of semi-open sets. But* f *is not contra precontinuous, since* $f^{-1}(0.5, 1.5) = [1, 2)$ *is not preclosed in* R*.*

Example 2.7. *([10]) A contra precontinuous function need not be contra semicontinuous. Let* $X = \{a, b\}$, $\tau = \{\emptyset, X\}$ *and* $\sigma = \{\emptyset, \{a\}, X\}$*. The identity function* $f : (X, \tau) \to (Y, \sigma)$ *is contra precontinuous as only the trivial subsets of* X *are open in* (X, τ)*. However,* $f^{-1}(\{a\}) = \{a\}$ *is not semi-closed in* (X, τ)*; hence* f *is not contra semicontinuous.*

Example 2.8. *([11]) Let* R *be the set of real numbers,* τ *be the countable extension topology on* R*, i.e. the topology with subbase* $\tau_1 \cup \tau_2$*, where* τ_1 *is the Euclidean topology of* R *and* τ_2 *is the topology of countable complements of* R*, and* σ *be the discrete topology of* R*. Define a function* $f : (R, \tau) \to (R, \sigma)$ *as follows:* $f(x) = 1$ *if* x *is rational, and* $f(x) = 2$ *if* x *is irrational. Then* f *is contra* δ*-precontinuous but not contra* β*-continuous, since* $\{1\}$ *is closed in* (R, σ) *and* $f^{-1}(\{1\}) = Q$*, where* Q *is the set of rationals, is not* β*-open in* (R, τ)*.*

Example 2.9. *([3]) Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ *and* $Y = \{p, q\}$, $\sigma = \{\emptyset, \{p\}, Y\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = p$ *and* $f(b) = f(c) = q$. *Then* f *is contra* β*-continuous but not contra-precontinuous since* $f^{-1}(\{q\}) = \{b, c\}$ *is* β*-open but not preopen.*

Definition 3. *A function* $f : X \to Y$ *is said to be*
1) $\beta\theta$*-semiopen if* $f(U) \in SO(Y)$ *for every* β*-*θ*-open set* U *of* X;
2) contra $I(\beta\theta)$*-continuous if for each* $x \in X$ *and each* $F \in C(Y, f(x))$, *there exists* $U \in \beta\theta O(X, x)$ *such that* $Int(f(U)) \subset F$;
3) $\beta\theta$*-continuous [17] if* $f^{-1}(F)$*is* β*-*θ*-closed in* X *for every closed set* F *of* Y;
4) β*-continuous [1] if* $f^{-1}(F)$*is* β*-closed in* X *for every closed set* F *of* Y.

We note that, every contra $\beta\theta$-continuous function is a contra $I(\beta\theta)$-continuous function but the converse need not be true as seen from the following example: Let $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$ and
$\sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is contra $I(\beta\theta)$-continuous but not contra $\beta\theta$-continuous.

Theorem 2.10. *If a function* $f : X \to Y$ *is contra* $I(\beta\theta)$*-continuous and* $\beta\theta$*-semiopen, then* f *is contra* $\beta\theta$*-continuous.*

Proof. Suppose that $x \in X$ and $F \in C(Y, f(x))$. Since f is contra $I(\beta\theta)$-continuous, there exists $U \in \beta\theta O(X, x)$ such that $Int(f(U)) \subset F$. By hypothesis f is $\beta\theta$-semiopen, therefore $f(U) \in SO(Y)$ and $f(U) \subset Cl(Int(f(U))) \subset F$. This shows that f is contra $\beta\theta$-continuous.

Theorem 2.11. *If a function* $f : X \to Y$ *is contra* $\beta\theta$*-continuous and* Y *is regular, then* f *is* $\beta\theta$*-continuous.*

Proof. Let x be an arbitrary point of X and V be an open set of Y containing $f(x)$. Since Y is regular, there exists an open set W in Y containing $f(x)$ such that $Cl(W) \subset U$. Since f is contra $\beta\theta$-continuous, there exists $U \in \beta\theta O(X, x)$ such that $f(U) \subset Cl(W)$. Then $f(U) \subset Cl(W) \subset V$. Hence f is $\beta\theta$-continuous.

Theorem 2.12. *Let* $\{X_i : i \in \Omega\}$ *be any family of topological spaces. If a function* $f : X \to \prod X_i$ *is contra* $\beta\theta$*-continuous, then* $Pr_i \circ f : X \to X_i$ *is contra* $\beta\theta$*-continuous for each* $i \in \Omega$, *where* Pr_i *is the projection of* $\prod X_i$ *onto* X_i.

Proof. For a fixed $i \in \Omega$, let V_i be any open set of X_i. Since Pr_i is continuous, $Pr_i^{-1}(V_i)$ is open in $\prod X_i$. Since f is contra $\beta\theta$-continuous, $f^{-1}(Pr_i^{-1}(V_i)) = (Pr_i \circ f)^{-1}(V_i)$ is β-θ-closed in X. Therefore, $Pr_i \circ f$ is contra $\beta\theta$-continuous for each $i \in \Omega$.

Theorem 2.13. *Let $f : X \to Y$, $g : Y \to Z$ and $g \circ f : X \to Z$ functions. Then the following hold:*
1) If f is contra $\beta\theta$-continuous and g is continuous, then $g \circ f$ is contra $\beta\theta$-continuous;

2) If f is $\beta\theta$-continuous and g is contra-continuous, then $g \circ f$ is contra $\beta\theta$-continuous;

3) If f is contra $\beta\theta$-continuous and g is contra-continuous, then $g \circ f$ is $\beta\theta$-continuous;

4) If f is weakly β-irresolute and g is contra $\beta\theta$-continuous, then $g \circ f$ is contra $\beta\theta$-continuous;
5) If f is strongly β-irresolute and g is contra β-continuous, then $g \circ f$ is contra $\beta\theta$-continuous.

3. Properties of contra $\beta\theta$-continuous functions

Definition 4. *[7, 5] A topological space (X, τ) is said to be:*

1) $\beta\theta$-T_0 (resp. $\beta\theta$-T_1) if for any distinct pair of points x and y in X, there is a β-θ-open U in X containing x but not y or (resp. and) a β-θ-open set V in X containing y but not x.
2) $\beta\theta$-T_2 (resp. β-T_2 [16]) if for every pair of distinct points x and y, there exist two β-θ-open (resp. β-open) sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

From the definitions above, we obtain the following diagram:

$\beta\theta$-$T_2 \Rightarrow \beta\theta$-$T_1 \Rightarrow \beta\theta$-$T_0$.

Theorem 3.1. *[8] If (X, τ) is $\beta\theta$-T_0, then (X, τ) is $\beta\theta$-T_2.*

Proof. For any points $x \neq y$ let V be a β-θ-open set that $x \in V$ and $y \notin V$. Then, there exists $U \in \beta O(X, \tau)$ such that $x \in U \subset \beta Cl_\theta(U) \subset V$.

By Lemma 1.1 and 1.2 $\beta Cl_\theta(U) \in \beta R(X,\tau)$. Then $\beta Cl_\theta(U)$ is β-θ-open and also $X \backslash \beta Cl_\theta(U)$ is a β-θ-open set containing y. Therefore, X is $\beta\theta$-T_2.

Remark 3.2. *For a topological space (X,τ) the three properties in the diagram are equivalent.*

Theorem 3.3. *A topological space (X,τ) is $\beta\theta$-T_2 if and only if the singletons are β-θ-closed sets.*

Proof. Suppose that (X,τ) is $\beta\theta$-T_2 and $x \in X$. Let $y \in X\backslash\{x\}$. Then $x \neq y$ and so there exists a β-θ-open set U_y such that $y \in U_y$ but $x \notin U_y$. Consequently $y \in U_y \subset X\backslash\{x\}$ i.e., $X\backslash\{x\} = \bigcup\{U_y/y \in X\backslash\{x\}\}$ which is β-θ -open.

Conversely. Suppose that $\{p\}$ is β-θ-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies that $y \in X\backslash\{x\}$. Hence$X\backslash\{x\}$ is a β-θ-open set containing y but not x. Similarly $X\backslash\{y\}$ is a β-θ-open set containing x but not y. From Remark 3.2, X is a $\beta\theta$-T_2 space.

Theorem 3.4. *For a topological space (X,τ), the following properties are equivalent:*
1) (X,τ) is $\beta\theta$-T_2;
2) (X,τ) is β-T_2;
3) For every pair of distinct points $x, y \in X$, there exist $U, V \in \beta O(X)$ such that $x \in U$, $y \in V$ and $\beta Cl(U) \cap \beta Cl(V) = \emptyset$;
4) For every pair of distinct points $x, y \in X$, there exist $U, V \in \beta R(X)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.
5) For every pair of distinct points $x, y \in X$, there exist $U \in \beta\theta O(X, x)$ and $V \in \beta\theta O(X, y)$ such that $\beta Cl_\theta(U) \cap \beta Cl_\theta(V) = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Since $\beta\theta O(X) \subset \beta O(X)$, the proof is obvious.
(2) $\Rightarrow$ (3): This follows from Lemma 5.2 of [17].

(3) $\Rightarrow$ (4): By Lemma 1.1, $\beta Cl(U) \in \beta R(X)$ for every $U \in \beta O(X)$ and the proof immediately follows.

(4) $\Rightarrow$ (5): By Lemma 1.1, every β-regular set is β-θ-open and β-θ-closed. Hence the proof is obvious.

(5) $\Rightarrow$ (1): This is obvious.

Theorem 3.5. *Let X be a topological space. Suppose that for each pair of distinct points x_1 and x_2 in X, there exists a function f of X into a Urysohn space Y such that $f(x_1) \neq f(x_2)$. Moreover, let f be contra $\beta\theta$-continuous at x_1 and x_2. Then X is $\beta\theta$-T_2.*

Proof. Let x_1 and x_2 be any distinct points in X. Then suppose that there exist an Urysohn space Y and a function $f : X \to Y$ such that $f(x_1) \neq f(x_2)$ and f is contra $\beta\theta$-continuous at x_1 and x_2. Let $w = f(x_1)$ and $z = f(x_2)$. Then $w \neq z$. Since Y is Urysohn, there exist open sets U and V containing w and z, respectively such that $Cl(U) \cap Cl(V) = \emptyset$. Since f is contra $\beta\theta$-continuous at x_1 and x_2, then there exist β-θ-open sets A and B containing x_1 and x_2, respectively such that $f(A) \subset Cl(U)$ and $f(B) \subset Cl(V)$. So we have $A \cap B = \emptyset$ since $Cl(U) \cap Cl(V) = \emptyset$. Hence, X is $\beta\theta$-T_2.

Corollary 3.6. *If f is a contra $\beta\theta$-continuous injection of a topological space X into a Urysohn space Y, then X is $\beta\theta$-T_2.*

Proof. For each pair of distinct points x_1 and x_2 in X and f is a contra $\beta\theta$-continuous function of X into a Urysohn space Y such that $f(x_1) \neq f(x_2)$ because f is injective. Hence by Theorem 3.5, X is $\beta\theta$-T_2.

Recall, that a space X is said to be
1) weakly Hausdorff [18] if each element of X is an intersection of regular closed sets.
2) Ultra Hausdorff [19] if for each pair of distinct points x and y in X, there exist clopen sets A and B containing x and y, respectively such that $A \cap B = \emptyset$.

Theorem 3.7. *1) If $f : X \to Y$ is a contra $\beta\theta$-continuous injection and Y is T_0, then X is $\beta\theta$-T_1.*
2) If $f : X \to Y$ is a contra $\beta\theta$-continuous injection and Y is Ultra Hausdorff, then X is $\beta\theta$-T_2.

Proof. 1) Let x_1, x_2 be any distinct points of X, then $f(x_1) \neq f(x_2)$. There exists an open set V such that $f(x_1) \in V$, $f(x_2) \notin V$ (or $f(x_2) \in V$, $f(x_1) \notin V$). Then $f(x_1) \notin Y \backslash V$, $f(x_2) \in Y \backslash V$ and $Y \backslash V$ is closed. By Theorem 2.3 $f^{-1}(Y \backslash V) \in \beta\theta O(X, x_2)$ and $x_1 \notin f^{-1}(Y \backslash V)$. Therefore X is $\beta\theta$-T_0 and by Theorem 3.1 X is $\beta\theta$-T_2.

2) By Remark 3.2, (2) is an immediate consequence of (1).

Definition 5. *A space (X,τ) is said to be $\beta\theta$-connected if X cannot be expressed as the disjoint union of two non-empty β-θ-open sets.*

Theorem 3.8. *If $f : X \to Y$ is a contra $\beta\theta$-continuous surjection and X is $\beta\theta$-connected, then Y is connected which is not a discrete space*

Proof. Suppose that Y is not a connected space. There exist non-empty disjoint open sets U_1 and U_2 such that $Y = U_1 \cup U_2$. Therefore U_1 and U_2 are clopen in Y. Since f is contra $\beta\theta$-continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are β-θ-open in X. Moreover, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are non-empty disjoint and $X = f^{-1}(U_1) \cup f^{-1}(U_2)$. This shows that X is not $\beta\theta$-connected. This contradicts that Y is not connected assumed. Hence Y is connected.
By other hand, Suppose that Y is discrete. Let A be a proper non-empty open and closed subset of Y. Then $f^{-1}(A)$ is a proper non-empty β-regular subset of X which is a contradiction to the fact that X is $\beta\theta$-connected.

A topological space X is said to be $\beta\theta$-normal if for each pair of non-empty disjoint closed sets can be separated by disjoint β-θ-open sets.

Theorem 3.9. *If $f : X \to Y$ is a contra $\beta\theta$-continuous, closed injection and Y is normal, then X is $\beta\theta$-normal.*

Proof. Let F_1 and F_2 be disjoint closed subsets of X. Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of Y. Since Y is normal, $f(F_1)$ and $f(F_2)$ are separated by disjoint open sets V_1 and V_2. Since that Y is normal, for each $i = 1, 2$, There exists an open set G_i such that $f(F_i) \subset G_i \subset Cl(G_i) \subset V_i$. Hence $F_i \subset f^{-1}(Cl(G_i))$, $f^{-1}(Cl(G_i)) \in \beta\theta O(X)$ for $i = 1, 2$ and $f^{-1}(Cl(G_1)) \cap f^{-1}(Cl(G_2)) = \emptyset$. Thus X is $\beta\theta$-normal.

Definition 6. *The graph $G(f)$ of a function $f : X \to Y$ is said to be contra $\beta\theta$-closed if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist a β-θ-open set U in X containing x and a closed set V in Y containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Lemma 3.10. *A graph $G(f)$ of a function $f : X \to Y$ is contra $\beta\theta$-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exists $U \in \beta\theta O(X)$ containing x and $V \in C(Y)$ containing y such that $f(U) \cap V = \emptyset$.*

Theorem 3.11. *If $f : X \to Y$ is contra $\beta\theta$-continuous and Y is Urysohn, $G(f)$ is contra $\beta\theta$-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$. It follows that $f(x) \neq y$. Since Y is Urysohn, there exist open sets V and W such that $f(x) \in V$, $y \in W$ and $Cl(V) \cap Cl(W) = \emptyset$. Since f is contra $\beta\theta$-continuous, there exist a $U \in \beta\theta O(X, x)$ such that $f(U) \subset Cl(V)$ and $f(U) \cap Cl(W) = \emptyset$. Hence $G(f)$ is contra $\beta\theta$-closed in $X \times Y$.

Theorem 3.12. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is contra $\beta\theta$-continuous, then f is contra $\beta\theta$-continuous.*

Proof. Let U be an open set in Y, then $X \times U$ is an open set in $X \times Y$. It follows that $f^{-1}(U) = g^{-1}(X \times U) \in \beta\theta C(X)$. Thus f is contra $\beta\theta$-continuous.

Theorem 3.13. *Let $f : X \to Y$ have a contra $\beta\theta$-closed graph. If f is injective, then X is $\beta\theta$-T_1.*

Proof. Let x_1 and x_2 be any two distinct points of X. Then, we have $(x_1, f(x_2)) \in (X \times Y)\backslash G(f)$. Then, there exist a β-θ-open set U in X containing x_1 and $F \in C(Y, f(x_2))$ such that $f(U) \cap F = \emptyset$. Hence $U \cap f^{-1}(F) = \emptyset$. Therefore, we have $x_2 \notin U$. This implies that X is $\beta\theta$-T_1.

Definition 7. *A topological space (X, τ) is said to be*
1) Strongly S-closed [9] if every closed cover of X has a finite subcover.
2) Strongly $\beta\theta$-closed if every β-θ-closed cover of X has a finite subcover.
3) $\beta\theta$-compact [4] if every β-θ-open cover of X has a finite subcover.
4) $\beta\theta$-space [8] if every β-θ-closed set is closed.

Theorem 3.14. *Let (X, τ) be a $\beta\theta$-space. If $f : X \to Y$ has a contra-$\beta\theta$-closed graph, then the inverse image of a strongly S-closed set K of Y is closed in (X, τ).*

Proof. Let K be a strongly S-closed set of Y and $x \notin f^{-1}(K)$. For each $k \in K, (x, k) \notin G(f)$. By Lemma 3.10, there exists $U_k \in \beta\theta O(X, x)$ and $V_k \in C(Y, k)$ such that $f(U_k) \cap V_k = \phi$. Since $\{K \cap V_k / k \in K\}$ is a closed cover of the subspace K, there exists a finite subset $K_0 \subset K$ such that $K \subset \cup\{V_k / k \in K_0\}$. Set $U = \cap\{U_k / k \in K_0\}$, then U is open since X is a $\beta\theta$-space. Therefore $f(U) \cap K = \phi$ and $U \cap f^{-1}(K) = \phi$. This shows that $f^{-1}(K)$ is closed in (X, τ).

Theorem 3.15. *Contra $\beta\theta$-continuous image of strongly $\beta\theta$-closed spaces are compact.*

Proof. Suppose that $f : X \to Y$ is a contra $\beta\theta$-continuous surjection. Let $\{V_\alpha/\alpha \in I\}$ be any open cover of Y. Since f is contra $\beta\theta$-continuous, then $\{f^{-1}(V_\alpha)/\alpha \in I\}$ is a β-θ-closed cover of X. Since X is strongly $\beta\theta$-closed, then there exists a finite subset $I_\circ$ of I such that $X = \cup\{f^{-1}(V_\alpha)/\alpha \in I_\circ\}$. Thus, we have $Y = \cup\{V_\alpha/\alpha \in I_\circ\}$ and Y is compact.

Theorem 3.16. *1) Contra $\beta\theta$-continuous image of $\beta\theta$-compact spaces are strongly S-closed.*
2) Contra $\beta\theta$-continuous image of a $\beta\theta$-compact space in any $\beta\theta$-space is strongly $\beta\theta$-closed.

Proof. 1) Suppose that $f : X \to Y$ is a contra $\beta\theta$-continuous surjection. Let $\{V_\alpha/\alpha \in I\}$ be any closed cover of Y. Since f is contra $\beta\theta$-continuous, then $\{f^{-1}(V_\alpha)/\alpha \in I\}$ is a β-θ-open cover of X. Since X is $\beta\theta$-compact, then there exists a finite subset $I_\circ$ of I such that $X = \cup\{f^{-1}(V_\alpha)/\alpha \in I_\circ\}$. Thus, we have $Y = \cup\{V_\alpha/\alpha \in I_\circ\}$ and Y is strongly S-closed.
2) Suppose that $f : X \to Y$ is a contra $\beta\theta$-continuous surjection. Let $\{V_\alpha/\alpha \in I\}$ be any β-θ-closed cover of Y. Since Y is a $\beta\theta$-space, then $\{V_\alpha/\alpha \in I\}$ is a closed cover of Y. Since f is contra $\beta\theta$-continuous, then $\{f^{-1}(V_\alpha)/\alpha \in I\}$ is a β-θ-open cover of X. Since X is $\beta\theta$-compact, then there exists a finite subset $I_\circ$ of I such that $X = \cup\{f^{-1}(V_\alpha)/\alpha \in I_\circ\}$. Thus, we have Y $= \cup\{V_\alpha/\alpha \in I_\circ\}$ and Y is strongly $\beta\theta$-closed.

Theorem 3.17. *If $f : X \to Y$ is a weakly β-irresolute surjective function and X is strongly $\beta\theta$-closed then $Y = f(X)$ is strongly $\beta\theta$-closed.*

Proof. Suppose that $f : X \to Y$ is a weakly β-irresolute surjection. Let $\{V_\alpha/\alpha \in I\}$ be any β-θ-closed cover of Y. Since f is a weakly β-irresolute,then $\{f^{-1}(V_\alpha)/\alpha \in I\}$ is a β-θ-closed cover of X. Since X is strongly $\beta\theta$-closed, then there exists a finite subset $I_\circ$ of I such that $X = \cup\{f^{-1}(V_\alpha)/\alpha \in I_\circ\}$. Thus, we have $Y = \cup\{V_\alpha/\alpha \in I_\circ\}$ and Y is strongly $\beta\theta$-closed.

Theorem 3.18. *Let $f : X_1 \to Y$ and $g : X_2 \to Y$ be two functions where Y is a Urysohn space and f and g are contra $\beta\theta$-continuous functions. Assume that the product of two β-θ-open sets is β-θ-open. Then $\{(x_1, x_2)/f(x_1) = g(x_2)\}$ is β-θ-closed in the product space $X_1 \times X_2$.*

Proof. Let V denote the set $\{(x_1, x_2)/f(x_1) = g(x_2)\}$. In order to show that V is β-θ-closed,we show that $(X_1 \times X_2)\backslash V$ is β-θ-open. Let $(x_1, x_2) \notin$

V. Then $f(x_1) \neq g(x_2)$, Since Y is Uryshon, there exist open sets U_1 and U_2 of Y containing $f(x_1)$ and $g(x_2)$ respectively, such that $Cl(U_1) \cap Cl(U_2) = \phi$. Since f and g are contra $\beta\theta$-continuous, $f^{-1}(Cl(U_1))$ and $g^{-1}(Cl(U_2))$ are β-θ-open sets containing x_1 and x_2 in $X_i (i = 1, 2)$. Hence by hypothesis, $f^{-1}(Cl(U_1)) \times g^{-1}(Cl(U_2))$ is β-θ-open. Further $(x_1, x_2) \in f^{-1}(Cl(U_1)) \times g^{-1}(Cl(U_2)) \subset (X_1 \times X_2) \backslash V$. It follows that $(X_1 \times X_2) \backslash V$ is β-θ-open. Thus, V is β-θ-closed in the product space $X_1 \times X_2$.

Corollary 3.19. *If $f : X \to Y$ is contra $\beta\theta$-continuous, Y is a Urysohn space and the product of two β-θ-open sets is β-θ-open, then $V = \{(x_1, x_2)/f(x_1) = f(x_2)\}$ is β-θ-closed in the product space $X \times X$.*

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