Some Umbral Calculus Presentations of the Chan-Chyan-Srivastava Polynomials and the Erkuş-Srivastava Polynomials

H. M. Srivastava

University of Victoria, Canada

K. S. Nisar

*Salman Bin Abdu-Aziz University, Saudi Arabia
and*

Mumtaz Ahmad Khan

Aligarh Muslim University, India

Received : February 2013. Accepted : March 2013

Abstract

In their recent investigation involving differential operators for the generalized Lagrange polynomials, Chan et. al. [3] encountered and proved a certain summation identity and several other results for the Lagrange polynomials in several variables, which are popularly known in the literature as the Chan-Chyan-Srivastava polynomials. These multivariable polynomials have been studied systematically and extensively in the literature ever since then (see, for example, [1], [4], [9], [11], [12] and [13]). In the present paper, we investigate umbral calculus presentations of the Chan-Chyan-Srivastava polynomials and also of their substantially more general form, the Erkuş-Srivastava polynomials [9]. Some other closely-related results are also considered.

2010 Mathematics Subject Classification : *Primary 33C45, 33C65; Secondary 33C05.*

Key Words and Phrases : *Lagrange polynomials; Hermite-Kampé de Fériet polynomials; Lagrange-Hermite polynomials; Chan-Chyan-Srivastava polynomials; Erkuş-Srivastava polynomials; Umbral calculus; Pochhammer symbol; Multinomial theorem and multinomial coefficients; Principle of monoumbrality; Monoumbral expansions.*

1. Introduction, Definitions and Notations

The familiar (two-variable) polynomials $g_n^{(\alpha, \beta)}(x, y)$, which are generated by

$$(1 - xz)^{-\alpha} (1 - yz)^{-\beta} = \sum_{n=0}^{\infty} g_n^{(\alpha, \beta)}(x, y) z^n \quad (1.1)$$

$$(\alpha, \beta \in \mathbf{C}; |z| < \min\{|x|^{-1}, |y|^{-1}\}),$$

are known as the Lagrange polynomials which occur in certain problems in statistics (*cf.*, *e.g.*, Erdélyi *et al.* [8, p. 267]; see also [15, p. 441 *et seq.*]).

The (three-variable) Lagrange polynomials $g_n^{(\alpha, \beta, \gamma)}(x, y, z)$, which are defined by means of the generating function:

$$(1 - xt)^{-\alpha} (1 - yt)^{-\beta} (1 - zt)^{-\gamma} = \sum_{n=0}^{\infty} g_n^{(\alpha, \beta, \gamma)}(x, y, z) \frac{t^n}{n!} \quad (1.2)$$

$$(\alpha, \beta, \gamma \in \mathbf{C}; |t| < \min\{|x|^{-1}, |y|^{-1}, |z|^{-1}\}),$$

were studied recently by Khan and Shukla [10]. Subsequently, Chan et al. [3] introduced and investigated the multivariable extension of the classical Lagrange polynomials $g_n^{(\alpha, \beta)}(x, y)$ generated by (1.1).

These multivariable Lagrange polynomials $g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$, which are popularly known in the literature as the *Chan-Chyan-Srivastava polynomials*, are generated by (see, for details, [3]; see also [4], [12] and [13])

$$\prod_{j=1}^r \{(1 - x_j z)^{-\alpha_j}\} = \sum_{n=0}^{\infty} g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) z^n \quad (1.3)$$

$$(\alpha_j \in \mathbf{C} \quad (j = 1, \dots, r); |z| < \min\{|x_1|^{-1}, \dots, |x_r|^{-1}\}),$$

so that, upon comparison with the generating function (1.2), we have the following relationship:

$$g_n^{(\alpha, \beta, \gamma)}(x, y, z) = n! g_n^{(\alpha, \beta, \gamma)}(x, y, z)$$

$$(n \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}; \mathbf{N} := \{1, 2, 3, \dots\})$$

with the (three-variable) Lagrange polynomials $g_n^{(\alpha, \beta, \gamma)}(x, y, z)$ studied by Khan and Shukla [10].

Clearly, the defining generating function (1.3) yields the explicit representation given by [3, p. 140, Eq. (6)]

$$\begin{aligned}
 g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) &= \sum_{\substack{k_1, \dots, k_r \in \mathbf{N}_0 \\ (k_1 + \dots + k_r = n)}} (\alpha_1)_{k_1} \cdots (\alpha_r)_{k_r} \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!} \\
 &= \sum_{\substack{k_1 + \dots + k_{r-1} \leq n \\ k_1, \dots, k_{r-1} = 0}} \frac{(\alpha_1)_{n-k_1-\dots-k_{r-1}} (\alpha_2)_{k_1} \cdots (\alpha_r)_{k_{r-1}}}{(n-k_1-\dots-k_{r-1})! k_1! \cdots k_{r-1}!} \\
 &\quad \cdot x_1^{n-k_1-\dots-k_{r-1}} x_2^{k_1} \cdots x_r^{k_{r-1}}
 \end{aligned}
 \tag{1.4}$$

or, equivalently, by [11, p. 522, Eq. (17)]

$$\begin{aligned}
 g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) &= \sum_{n_{r-1}=0}^n \sum_{n_{r-2}=0}^{n_{r-1}} \cdots \sum_{n_1=0}^{n_2} \frac{(\alpha_1)_{n_1} (\alpha_2)_{n_2-n_1} \cdots (\alpha_r)_{n-n_{r-1}}}{n_1! (n_2-n_1)! \cdots (n-n_{r-1})!} \\
 &\quad \cdot x_1^{n_1} x_2^{n_2-n_1} \cdots x_r^{n-n_{r-1}} \\
 &= \sum_{k_1=0}^n \sum_{k_2=0}^{k_1} \cdots \sum_{k_{r-1}=0}^{k_{r-2}} \frac{(\alpha_1)_{n-k_1} (\alpha_2)_{k_2-k_1} \cdots (\alpha_r)_{k_{r-1}}}{(n-k_1)! (k_2-k_1)! \cdots k_{r-1}!} \\
 &\quad \cdot x_1^{n-k_1} x_2^{k_2-k_1} \cdots x_r^{k_{r-1}},
 \end{aligned}
 \tag{1.5}$$

where, as usual, $(\lambda)_n$ denotes the Pochhammer symbol given by

$$(\lambda)_0 := 1 \quad \text{and} \quad (\lambda)_n := \lambda(\lambda+1) \cdots (\lambda+n-1) \quad (n \in \mathbf{N}).$$

Altın and Erkuş [1] presented a multivariable extension of the so-called Lagrange-Hermite polynomials generated by (see [1, p. 239, Eq. (1.2)])

$$\begin{aligned}
 (1.6) \quad \prod_{j=1}^r \{(1 - x_j z^j)^{-\alpha_j}\} &= \sum_{n=0}^{\infty} h_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) z^n \\
 &\quad \left(\alpha_j \in \mathbf{C} \quad (j = 1, \dots, r); |z| < \min_{j \in \{1, \dots, r\}} \{|x_j|^{-1/j}\} \right).
 \end{aligned}$$

The case $r = 2$ of the polynomials given by (1.6) corresponds to the familiar (two-variable) Lagrange-Hermite polynomials considered by Dattoli *et al.* [6].

The multivariable (Erkuş-Srivastava) polynomials

$\mathcal{U}_{n;\ell_1,\dots,\ell_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$, which are defined by the following generating function [9, p. 268, Eq. (3)]:

$$(1.7) \quad \prod_{j=1}^r \{(1 - x_j z^{\ell_j})^{-\alpha_j}\} = \sum_{n=0}^{\infty} \mathcal{U}_{n;\ell_1,\dots,\ell_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r) z^n$$

$$(\alpha_j \in \mathbf{C} \quad (j = 1, \dots, r); \ell_j \in \mathbf{N} \quad (j = 1, \dots, r);$$

$$|z| < \min\{|x_1|^{-1/\ell_1}, \dots, |x_r|^{-1/\ell_r}\},$$

are a unification (and generalization) of several known families of multivariable polynomials including (for example) the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ defined by (1.3) (see, for details, [9]). Obviously, the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ follow as the special case of the Erkuş-Srivastava polynomials $\mathcal{U}_{n;\ell_1,\dots,\ell_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ when $\ell_j = 1$ ($j = 1, \dots, r$).

Moreover, the Lagrange-Hermite polynomials $h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ follow as a special case of the Erkuş-Srivastava polynomials $\mathcal{U}_{n;\ell_1,\dots,\ell_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ when $\ell_j = j$ ($j = 1, \dots, r$).

The generating function (1.7) yields the following *explicit* representation [9, p. 268, Eq. (4)]:

$$(1.8) \quad \mathcal{U}_{n;\ell_1,\dots,\ell_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r) = \sum_{\substack{k_1,\dots,k_r \in \mathbf{N}_0 \\ (\ell_1 k_1 + \dots + \ell_r k_r = n)}} (\alpha_1)_{k_1} \cdots (\alpha_r)_{k_r} \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_r^{k_r}}{k_r!},$$

which, in the special case when $\ell_j = 1$ ($j = 1, \dots, r$), corresponds to the *first* expression in (1.4).

Each of the above families of multivariable polynomials has been investigated systematically and extensively in the literature ever since the publication of the work by Chan *et al.* [9] (see, for example, [1], [4], [9], [11], [12] and [13]). The main objective of the present paper is to derive umbral calculus presentations of the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ generated by (1.3) and also of the substantially more general Erkuş-Srivastava polynomials $\mathcal{U}_{n;\ell_1,\dots,\ell_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ generated by (1.7). Upon suitable specialization, this last umbral calculus presentation is shown to yield the corresponding result for the polynomials $h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ generated by (1.6).

2. Umbral Calculus Presentations

The Chan-Chyan-Srivastava polynomials in (1.3) exhibits a structure which, according to the prescription provided in [7], can be viewed as the umbral image of ordinary monomials. Indeed, by using the generating function in (1.3) and an elementary integral identity in the form [2] :

$$(2.1) \quad \kappa^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\kappa t} t^{\nu-1} dt \quad \left(\min\{\Re(\kappa), \Re(\nu)\} > 0 \right),$$

we can easily derive the following integral representation of the Chan-Chyan-Srivastava polynomials:

$$(2.2) \quad g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) = \frac{1}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_r)} \int_0^\infty \cdots \int_0^\infty e^{-(\xi_1 + \cdots + \xi_r)} \cdot \xi_1^{\alpha_1-1} \cdots \xi_r^{\alpha_r-1} \mathcal{P}_n(x_1 \xi_1, \dots, x_r \xi_r) d\xi_1 \cdots d\xi_r,$$

where

$$(2.3) \quad \mathcal{P}_n(x_1, \dots, x_r) := \frac{(x_1 + \cdots + x_r)^n}{n!} = \frac{1}{n!} \sum_{\substack{k_1, \dots, k_r \in \mathbf{N}_0 \\ (k_1 + \cdots + k_r = n)}} \binom{n}{k_1, \dots, k_r} x_1^{k_1} \cdots x_r^{k_r}$$

in terms of the multinomial coefficients given by

$$\binom{n}{k_1, \dots, k_r} := \frac{n!}{k_1! \cdots k_r!} \quad (n, k_1, \dots, k_r \in \mathbf{N}_0).$$

Alternatively, the multinomial theorem (see, for example, [15, p. 87, Problem 5]) used in (2.3) can indeed be restated as follows:

$$(2.4) \quad \mathcal{P}_n(x_1, \dots, x_r) = \frac{1}{n!} \sum_{k_1=0}^n \sum_{k_2=0}^{n-k_1} \cdots \sum_{k_{r-1}=0}^{n-k_1-\cdots-k_{r-2}} \binom{n}{k_1} \binom{n-k_1}{k_2} \cdots \binom{n-k_1-\cdots-k_{r-2}}{k_{r-1}} \cdot x_1^{n-k_1-\cdots-k_{r-1}} x_2^{k_1} \cdots x_r^{k_{r-1}}.$$

It is immediately seen from the integral representation (2.2) that [see Eq. (1.4)]

$$\begin{aligned}
 g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) &= \sum_{\substack{k_1, \dots, k_r \in \mathbf{N}_0 \\ (k_1 + \dots + k_r = n)}} \binom{\alpha_1 + k_1 - 1}{k_1} \dots \binom{\alpha_r + k_r - 1}{k_r} x_1^{k_1} \dots x_r^{k_r} \\
 &= \sum_{k_1=0}^n \sum_{k_2=0}^{n-k_1} \dots \sum_{k_{r-1}=0}^{n-k_1-\dots-k_{r-2}} \frac{(\alpha_1)_{n-k_1-\dots-k_{r-1}} (\alpha_2)_{k_1} \dots (\alpha_r)_{k_{r-1}}}{(n-k_1-\dots-k_{r-1})! k_1! \dots k_{r-2}!} \\
 (2.5) \quad &\cdot x_1^{n-k_1-\dots-k_{r-1}} x_2^{k_1} \dots x_r^{k_{r-1}},
 \end{aligned}$$

which follows also from the *second* explicit expression in (1.4).

We now define the umbral quantities $\langle \alpha_1 x_1 \rangle, \dots, \langle \alpha_r x_r \rangle$ together with their properties given by (see, for details, [5] ; see also [14])

$$\langle \alpha_j x_j \rangle^n = (\alpha_j)_n x_j^n \quad (j = 1, \dots, r; n \in \mathbf{N}_0)$$

and the pairs of operators $\hat{X}_j$ and $\hat{P}_{x_j}$ ($j = 1, \dots, r$) such that

$$\hat{X}_j \{ \langle \alpha_j x_j \rangle^n \} = \langle \alpha_j x_j \rangle^{n+1} \quad \text{and} \quad \hat{P}_{x_j} \{ \langle \alpha_j x_j \rangle^n \} = n \langle \alpha_j x_j \rangle^{n-1}$$

$$(j = 1, \dots, r),$$

so that, obviously,

$$\begin{aligned}
 \hat{X}_j \hat{P}_{x_j} \{ \langle \alpha_j x_j \rangle^n \} &= n \langle \alpha_j x_j \rangle^n \quad \text{and} \quad \hat{P}_{x_j} \hat{X}_j \{ \langle \alpha_j x_j \rangle^n \} = (n+1) \langle \alpha_j x_j \rangle^n \\
 (j = 1, \dots, r).
 \end{aligned}$$

We thus find that the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$ satisfy the following operational formula:

$$\begin{aligned}
 &\left(\hat{X}_1 \hat{P}_{x_1} + \dots + \hat{X}_r \hat{P}_{x_r} \right) \left\{ g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \right\} \\
 (2.6) \quad &= n g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r),
 \end{aligned}$$

which, in the special case when

$$r = 2, \quad \alpha_1 = \alpha, \quad \alpha_2 = \beta, \quad x_1 = x \quad \text{and} \quad x_2 = y,$$

would provide the *corrected* version of a result stated by Dattoli *et al.* [6, p. 182, Eq. (10)].

It is also clear that the umbral image of the generating function (1.3) is given by

$$(2.7) \quad e^{[\langle \alpha_1 x_1 \rangle + \dots + \langle \alpha_r x_r \rangle]z} = \sum_{n=0}^{\infty} g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) z^n.$$

In our next section (Section 3), we will show that the above notion and formalism of *monoumbrality* (that is, *monomiality* together with *umbrality*) can be extended to a large family of polynomials as well as of functions and that this principle of monoumbrality will provide a powerful tool for simplifying calculations (see also [5], [6] and [7]).

3. Applications of the Principle of Monoumbrality

In the preceding section, we have noted that the umbral image of the generating function (1.3) of the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$ is, in fact, of the exponential type given by (2.7). Here, in this section, we begin by recalling the following easily understandable decomposition rules for the umbral algebra (see [5], [6] and [7]):

$$(3.1) \quad e^{\langle \alpha_1 x_1 \rangle + \dots + \langle \alpha_r x_r \rangle} = e^{\langle \alpha_1 x_1 \rangle} \dots e^{\langle \alpha_r x_r \rangle},$$

and

$$(3.2) \quad e^{[\langle \alpha(x_1 + \dots + x_r) \rangle]} \neq e^{\langle \alpha x_1 \rangle} \dots e^{\langle \alpha x_r \rangle},$$

that is, more explicitly,

$$(3.3) \quad e^{[\langle \alpha(x_1 + \dots + x_r) \rangle]} \neq \left(\sum_{k_1=0}^{\infty} (\alpha)_{k_1} \frac{x_1^{k_1}}{k_1!} \right) \dots \left(\sum_{k_r=0}^{\infty} (\alpha)_{k_r} \frac{x_r^{k_r}}{k_r!} \right).$$

We also note that

$$(3.4) \quad \begin{aligned} e^{\langle \alpha_1 x_1 \rangle z} &= e^{\langle \alpha_1 x_1 \rangle z + \dots + \langle \alpha_r x_r \rangle z} \cdot e^{\langle -\alpha_2 x_2 \rangle z} \dots e^{\langle -\alpha_r x_r \rangle z} \\ &= e^{[\langle \alpha_1 x_1 \rangle + \dots + \langle \alpha_r x_r \rangle]z} \cdot e^{[\langle -\alpha_2 x_2 \rangle + \dots + \langle -\alpha_r x_r \rangle]z}. \end{aligned}$$

By applying this last umbral relationship (3.4) in conjunction with (2.5) and (2.7), we find that

$$\begin{aligned}
 \sum_{n=0}^{\infty} (\alpha_1)_n \frac{(x_1 z)^n}{n!} &= \left(\sum_{n=0}^{\infty} g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) z^n \right) \\
 &\quad \left(\sum_{n=0}^{\infty} g_n^{(-\alpha_2, \dots, -\alpha_r)}(x_2, \dots, x_r) z^n \right) \\
 (3.5) \quad &= \sum_{n=0}^{\infty} z^n \sum_{k=0}^n g_k^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) g_{n-k}^{(-\alpha_2, \dots, -\alpha_r)}(x_2, \dots, x_r),
 \end{aligned}$$

so that, upon equating the coefficients of z^n from both sides in (3.5), we finally obtain

$$\begin{aligned}
 x_1^n &= \frac{1}{(\alpha_1)_n} \sum_{k_1=0}^n \sum_{k_2=0}^{n-k_1} \cdots \sum_{k_{r-1}=0}^{n-k_1-\cdots-k_{r-2}} k_1! \binom{n}{k_1} \binom{n-k_1}{k_2} \\
 &\quad \cdots \binom{n-k_1-\cdots-k_{r-2}}{k_{r-1}} \\
 &\quad \cdot g_{k_1}^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \langle -\alpha_2 x_2 \rangle^{n-k_1-\cdots-k_{r-1}} \langle -\alpha_3 x_3 \rangle^{k_2} \cdots \langle -\alpha_r x_r \rangle^{k_{r-1}}, \\
 (3.6) \quad &
 \end{aligned}$$

where, just as in the preceding section,

$$\begin{aligned}
 \langle -\alpha_2 x_2 \rangle^{n-k_1-\cdots-k_{r-1}} &= \frac{\Gamma(-\alpha_2 + n - k_1 - \cdots - k_{r-1})}{\Gamma(-\alpha_2)} x_2^{n-k_1-\cdots-k_{r-1}} \\
 (3.7) \quad &= (-\alpha_2)_{n-k_1-\cdots-k_{r-1}} x_2^{n-k_1-\cdots-k_{r-1}}
 \end{aligned}$$

and

$$\begin{aligned}
 \langle -\alpha_j x_j \rangle^{k_{j-1}} &= \frac{\Gamma(-\alpha_j + k_{j-1})}{\Gamma(-\alpha_j)} x_j^{k_{j-1}} = (-\alpha_j)_{k_{j-1}} x_j^{k_{j-1}} \quad (j = 3, \dots, r). \\
 (3.8) \quad &
 \end{aligned}$$

Formula (3.6) provides the expansion of an ordinary monomial x_1^n in terms of the Chan-Chyan-Srivastava polynomials

$$g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$$

involving such umbral quantities as those specified in (3.7) and (3.8).

Next, for Erkuş-Srivastava polynomials $\mathcal{U}_{n;\ell_1,\dots,\ell_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ generated by (1.7), it is easily seen that

$$(3.9) \quad \begin{aligned} & \mathcal{U}_{n;\ell_1,\dots,\ell_r}^{(\alpha_1+\beta_1,\dots,\alpha_r+\beta_r)}(x_1,\dots,x_r) \\ &= \sum_{k=0}^n \mathcal{U}_{n-k;\ell_1,\dots,\ell_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r) \mathcal{U}_{k;\ell_1,\dots,\ell_r}^{(\beta_1,\dots,\beta_r)}(x_1,\dots,x_r), \end{aligned}$$

which, for $\ell_j = 1$ ($j = 1, \dots, r$), was given by Chan *et al.* [3, p. 147, Eq. (35)]. Moreover, the generating function (1.7) together with the integral formula (2.1) would yield the following analogue of the integral representation (2.2):

$$(3.10) \quad \begin{aligned} & \mathcal{U}_{n;\ell_1,\dots,\ell_r}^{(\alpha_1+\beta_1,\dots,\alpha_r+\beta_r)}(x_1,\dots,x_r) = \frac{1}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_r)} \int_0^\infty \cdots \int_0^\infty e^{-(\xi_1+\cdots+\xi_r)} \\ & \cdot \xi_1^{\alpha_1-1} \cdots \xi_r^{\alpha_r-1} \mathcal{Q}_n(x_1\xi_1, \dots, x_r\xi_r) d\xi_1 \cdots d\xi_r, \end{aligned}$$

where the polynomials $\mathcal{Q}_n(x_1, \dots, x_r)$ are essentially the same as the multivariable Hermite-Kampé de Fériet polynomials given by

$$(3.11) \quad \mathcal{Q}_n(x_1, \dots, x_r) := \frac{1}{n!} \sum_{\substack{k_1, \dots, k_r \in \mathbf{N}_0 \\ (\ell_1 k_1 + \cdots + \ell_r k_r = n)}} \binom{n}{k_1, \dots, k_r} x_1^{k_1} \cdots x_r^{k_r}$$

in terms of the multinomial coefficients involved also in (2.3).

If we now make use of the principle of monounbrality as detailed above, we can show similarly that

$$(3.12) \quad \begin{aligned} x_1^n &= \frac{n!}{(\alpha_1)_n} \sum_{k_1=0}^{\ell_1 n} \sum_{\substack{k_2, \dots, k_r \in \mathbf{N}_0 \\ (\ell_2 k_2 + \cdots + \ell_r k_r = \ell_1 n - k_1)}} \mathcal{U}_{k_1;\ell_1,\dots,\ell_r}^{(\alpha_1,\dots,\alpha_r)}(x_1, \dots, x_r) \\ & \cdot \frac{\langle -\alpha_2 x_2 \rangle^{k_2}}{k_2!} \cdots \frac{\langle -\alpha_r x_r \rangle^{k_r}}{k_r!}, \end{aligned}$$

which, in the special case when $\ell_j = 1$ ($j = 1, \dots, r$), the monounbrl expansion given by (3.6). Moreover, if we simply assume that $\ell_r = 1$, we can easily rewrite the monounbrl expansion (3.12) as follows:

$$(3.13) \quad x_1^n = \frac{n!}{(\alpha_1)_n} \sum_{k_1=0}^{\ell_1 n} \sum_{k_2=0}^{\left\lfloor \frac{\ell_1 n - k_1}{\ell_2} \right\rfloor} \cdots \sum_{k_{r-1}=0}^{\left\lfloor \frac{\ell_1 n - k_1 - \ell_2 k_2 - \cdots - \ell_{r-2} k_{r-2}}{\ell_{r-1}} \right\rfloor} \cdot \mathcal{U}_{k_1; \ell_1, \dots, \ell_{r-1}, 1}^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \\ \cdot \frac{\langle -\alpha_2 x_2 \rangle^{\ell_1 n - k_1 - \ell_2 k_2 - \cdots - \ell_{r-1} k_{r-1}}}{(\ell_1 n - k_1 - \ell_2 k_2 - \cdots - \ell_{r-1} k_{r-1})!} \frac{\langle -\alpha_3 x_3 \rangle^{k_2}}{k_2!} \cdots \frac{\langle -\alpha_r x_r \rangle^{k_{r-1}}}{k_{r-1}!}$$

or, equivalently,

$$(3.14) \quad x_1^n = \frac{n!}{(\alpha_1)_n} \sum_{k_1=0}^{\ell_1 n} \sum_{k_2=0}^{\left\lfloor \frac{\ell_1 n - k_1}{\ell_2} \right\rfloor} \cdots \sum_{k_{r-1}=0}^{\left\lfloor \frac{\ell_1 n - k_1 - \ell_2 k_2 - \cdots - \ell_{r-2} k_{r-2}}{\ell_{r-1}} \right\rfloor} \cdot \mathcal{U}_{k_1; \ell_1, \dots, \ell_{r-1}, 1}^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \\ \cdot \frac{k_1! (\ell_2 k_2)! \cdots (\ell_{r-1} k_{r-1})!}{k_2! \cdots k_{r-1}!} \binom{\ell_1 n}{k_1} \binom{\ell_1 n - k_1}{\ell_2 k_2} \\ \cdots \binom{\ell_1 n - k_1 - \ell_2 k_2 - \cdots - \ell_{r-2} k_{r-2}}{\ell_{r-1} k_{r-1}} \\ \cdot \langle -\alpha_2 x_2 \rangle^{\ell_1 n - k_1 - \ell_2 k_2 - \cdots - \ell_{r-1} k_{r-1}} \langle -\alpha_3 x_3 \rangle^{k_2} \cdots \langle -\alpha_r x_r \rangle^{k_{r-1}}$$

in terms of the notations and conventions given by (3.7) and (3.8). Indeed, when we *further* set $\ell_j = 1$ ($j = 1, \dots, r-1$), this last result (3.14) would correspond *precisely* to the monounbrl expansion (3.6).

Finally, upon setting $\ell_j = j$ ($j = 1, \dots, r$) in our general result (3.12), we are led at once to the following monounbrl expansion for the multivariable Lagrange-Hermite polynomials $h_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$ in (1.6), which were studied by (for example) Altın and Erkuş [1] :

$$x_1^n = \frac{n!}{(\alpha_1)_n} \sum_{k_1=0}^n \sum_{\substack{k_2, \dots, k_r \in \mathbf{N}_0 \\ (2k_2 + \cdots + rk_r = n - k_1)}} h_{k_1}^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$$

$$(3.15) \quad \cdot \frac{\langle -\alpha_2 x_2 \rangle^{k_2}}{k_2!} \dots \frac{\langle -\alpha_r x_r \rangle^{k_r}}{k_r!}.$$

4. Concluding Remarks and Observations

In this work, we have investigated the umbral calculus presentations of the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$ generated by (1.3) and also of the substantially more general Erkuş-Srivastava polynomials $\mathcal{U}_{n; \ell_1, \dots, \ell_r}^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$ generated by (1.7). One of our main monoumbral expansions asserted by (3.12) has been shown to yield the corresponding monoumbral expansion (3.15) for the multivariable Lagrange-Hermite polynomials $h_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$ in (1.6), which were studied by (for example) Altın and Erkuş [1].

We need hardly emphasize upon the fact that the notion and formalism of monoumbrality can be extended to a large family of polynomials as well as of functions and that the underlying principle of monoumbrality would provide a powerful tool for simplifying calculations.

We conclude our present investigation by remarking further that, in the special case when $\ell_1 = 1$, we find from (1.8) that

$$\begin{aligned} \mathcal{U}_{n; 1, \ell_2, \dots, \ell_r}^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) &= \sum_{\substack{k_1, \dots, k_r \in \mathbf{N}_0 \\ (k_1 + \ell_2 k_2 + \dots + \ell_r k_r = n)}} \frac{\langle \alpha_1 x_1 \rangle^{k_1}}{k_1!} \dots \frac{\langle \alpha_r x_r \rangle^{k_r}}{k_r!} \\ &= \sum_{k_2, \dots, k_r=0}^{l_2 k_2 + \dots + l_r k_r \leq n} \frac{\langle \alpha_1 x_1 \rangle^{n - \ell_2 k_2 - \dots - \ell_r k_r}}{(n - \ell_2 k_2 - \dots - \ell_r k_r)!} \frac{\langle \alpha_2 x_2 \rangle^{k_2}}{k_2!} \dots \frac{\langle \alpha_r x_r \rangle^{k_r}}{k_r!} \\ &= \sum_{k_1=0}^{\left\lfloor \frac{n}{\ell_2} \right\rfloor} \sum_{k_2=0}^{\left\lfloor \frac{n - \ell_2 k_1}{\ell_3} \right\rfloor} \dots \sum_{k_{r-1}=0}^{\left\lfloor \frac{n - \ell_2 k_1 - \dots - \ell_{r-1} k_{r-2}}{\ell_r} \right\rfloor} \\ (4.1) \quad &\cdot \frac{\langle \alpha_1 x_1 \rangle^{n - \ell_2 k_1 - \dots - \ell_r k_{r-1}}}{(n - \ell_2 k_1 - \dots - \ell_r k_{r-1})!} \frac{\langle \alpha_2 x_2 \rangle^{k_1}}{k_1!} \dots \frac{\langle \alpha_r x_r \rangle^{k_{r-1}}}{k_{r-1}!}, \end{aligned}$$

which readily yields the following generalization of the operational formula

(2.6):

$$\left(\hat{X}_1 \hat{P}_{x_1} + \ell_2 \hat{X}_2 \hat{P}_{x_2} + \dots + \ell_r \hat{X}_r \hat{P}_{x_r} \right) \left\{ \mathcal{U}_{n; 1, \ell_2, \dots, \ell_r}^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \right\}$$

$$(4.2) \quad = n\mathcal{U}_{n;1,\ell_2,\dots,\ell_r}^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r).$$

In particular, if we set $\ell_j = j$ ($j = 2, \dots, r$) in this last result (4.2), we immediately obtain

$$(4.3) \quad \begin{aligned} & \left(\hat{X}_1 \hat{P}_{x_1} + 2\hat{X}_2 \hat{P}_{x_2} + \dots + r\hat{X}_r \hat{P}_{x_r} \right) \left\{ h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r) \right\} \\ & = nh_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r) \end{aligned}$$

for the multivariable Lagrange-Hermite polynomials $h_n^{(\alpha_1,\dots,\alpha_r)}(x_1,\dots,x_r)$ in (1.6), which were studied by (for example) Altın and Erkuş [1]. Indeed, in its *further* special case when

$$r = 2, \quad \alpha_1 = \alpha, \quad \alpha_2 = \beta, \quad x_1 = x \quad \text{and} \quad x_2 = y,$$

the operational formula (4.3) would provide the *corrected* version of another result stated by Dattoli *et al.* [6, p. 184, Eq. (22)].

References

- [1] A. Altın and E. Erkuş, On a multivariable extension of the Lagrange-Hermite polynomials, *Integral Transforms Spec. Funct.* **17**, pp. 239–244, (2006).
- [2] L. C. Andrews, *Special Functions for Engineers and Applied Mathematicians*, Macmillan Company, New York, (1985).
- [3] W.-C. C. Chan, C.-J. Chyan and H. M. Srivastava, The Lagrange polynomials in several variables, *Integral Transforms Spec. Funct.* **12**, pp. 139–148, (2001).
- [4] K.-Y. Chen, S.-J. Liu and H. M. Srivastava, Some new results for the Lagrange polynomials in several variables, *ANZIAM J.* **49**, pp. 243–258, (2007).
- [5] G. Dattoil, P. E. Ricci and C. Cesarano, Monumbral polynomials and the associated formalism, *Integral Transforms Spec. Funct.* **13**, pp. 155–162, (2002).

- [6] G. Dattoil, P. E. Ricci and C. Cesarano, The Lagrange polynomials, the associated generalizations, and the umbral calculus, *Integral Transforms Spec. Funct.* **14**, pp. 181–186, (2003).
- [7] G. Dattoil, P. E. Ricci and C. Cesarano, Beyond the monomiality: The monumbrality principle, *J. Comput. Anal. Appl.* **6**, pp. 77–83, (2004).
- [8] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. **III**, McGraw-Hill Book Company, New York, Toronto and London, (1955).
- [9] E. Erkuş and H. M. Srivastava, A unified presentation of some families of multivariable polynomials, *Integral Transforms Spec. Funct.* **17**, pp. 267–273, (2006).
- [10] M. A. Khan and A. K. Shukla, On Lagrange’s polynomials of three variables, *Proyecciones J. Math.* **17**, pp. 227–235, (1998).
- [11] S.-J. Liu, Bilateral generating functions for the Lagrange polynomials and the Lauricella functions, *Integral Transforms Spec. Funct.* **20**, pp. 519–527, (2009).
- [12] S.-J. Liu, C.-J. Chyan, H.-C. Lu and H. M. Srivastava, Bilateral generating functions for the Chan-Chyan-Srivastava polynomials and the generalized Lauricella functions, *Integral Transforms Spec. Funct.* **23**, pp. 539–549, (2012).
- [13] S.-J. Liu, S.-D. Lin, H. M. Srivastava and M.-M. Wong, Bilateral generating functions for the Erkuş-Srivastava polynomials and the generalized Lauricella functions, *Appl. Math. Comput.* **218**, pp. 7685–7693, (2012).
- [14] S. Roman, *The Umbral Calculus*, Academic Press, New York and London, (1984).
- [15] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, (1984).

H. M. Srivastava

Department of Mathematics and Statistics,
University of Victoria,
Victoria,
British Columbia V8W 3R4,
Canada
e-mail : harimsri@math.uvic.ca

K. S. Nisar

Department of Mathematics,
College of Engineering,
Salman Bin Abdul-Aziz University,
Wadi Dawaser,
Kingdom of Saudi Arabia
e-mail : ksnisar1@gmail.com

and

Mumtaz Ahmad Khan

Department of Applied Mathematics,
Faculty of Engineering and Technology,
Aligarh Muslim University,
Aligarh 202002, Uttar Pradesh,
India
e-mail : mumtaz_ahmad_khan_2008@yahoo.com