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Titchmarsh's Theorem for the Dunkl transform in the space $L^2(\mathbf{R}^d, w_k(x)dx)$

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Abstract

Using a generalized spherical mean operator, we obtain a generalization of Titchmarsh's theorem for the Dunkl transform for functions satisfying the (ψ, α, β) -Dunkl Lipschitz condition in $L^2(\mathbf{R}^d, w_k(x)dx)$.

Keywords : *Dunkl operator, Dunkl transform, generalized spherical mean operator.*

Mathematics Subject Classification : *47B48; 33C52.*

1. Introduction and preliminaries

In [10], E. C. Titchmarsh characterized the set of functions in $L^2(\mathbf{R})$ satisfying the Cauchy Lipschitz condition for the Fourier transform, namely we have

Theorem 1.1. *Let $\alpha \in (0, 1)$ and assume that $f \in L^2(\mathbf{R})$. Then the following are equivalents*

1. $\|f(x+h) - f(x)\|_{L^2(\mathbf{R})} = O(h^\alpha)$ as $h \rightarrow 0$,
2. $\int_{|\lambda| \geq r} |F(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ as $r \rightarrow +\infty$,

where F stands for the Fourier transform of f .

The main aim of this paper is to establish a generalization of Theorem 1.1 in the Dunkl transform setting by means of the generalized spherical mean operator.

In this paper we consider the Dunkl operators T_j , $j = 1, 2, \dots, d$, which are the differential-difference operators introduced by C.F. Dunkl in [3]. These operators are very important in pure mathematics and in physics.

In the first we collect some notations and results on Dunkl operators and the Dunkl kernel (see [3], [4], [6]).

We consider $\mathbf{R}^d$ with the Euclidean scalar product $\langle \cdot, \cdot \rangle$ and $|x| = \sqrt{\langle x, x \rangle}$. For $\alpha \in \mathbf{R}^d \setminus \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbf{R}^d$ orthogonal to α .
i.e.,

$$\sigma_\alpha(x) = x - 2 \frac{\langle \alpha, x \rangle}{|\alpha|^2} \alpha.$$

A finite set $R \subset \mathbf{R}^d \setminus \{0\}$ is called a root system, if $R \cap \mathbf{R} \cdot \alpha = \{\alpha, -\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflection σ_α , $\alpha \in R$, generate a finite group $W \subset O(d)$, the reflection group associated with R . We fix $\beta \in \mathbf{R}^d \setminus \bigcup_{\alpha \in R} H_\alpha$ and define a positive root system $R_+ = \{\alpha \in R / \langle \alpha, \beta \rangle > 0\}$.

A function $k : R \rightarrow \mathbf{C}$ on a root system R is called a multiplicity function, if it is invariant under the action of the associated reflection group W .

If one regards k as a function on the corresponding reflections, this means that k is constant on the conjugacy classes of reflections in W .

We consider the weight function

$$w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)},$$

where w_k is W -invariant and homogeneous of degree 2γ where

$$\gamma = \sum_{\alpha \in R_+} k(\alpha).$$

We let η be the normalized surface measure on the unit sphere $\mathbf{S}^{d-1}$ in $\mathbf{R}^d$ and set

$$d\eta_k(y) = w_k(y)d\eta(y).$$

Then η_k is a W -invariant measure on $\mathbf{S}^{d-1}$, we let $d_k = \eta_k(\mathbf{S}^{d-1})$.

Introduced by C.F. Dunkl in [3] the Dunkl operators T_j , $1 \leq j \leq d$, on $\mathbf{R}^d$ associated with the reflection group W and the multiplicity function k are the first-order differential-difference operators given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \quad f \in C^1(\mathbf{R}^d),$$

where $\alpha_j = \langle \alpha, e_j \rangle$; $(e_1, \dots, e_d)$ being the canonical basis of $\mathbf{R}^d$ and $C^1(\mathbf{R}^d)$ is the space of functions of class C^1 on $\mathbf{R}^d$.

The Dunkl kernel E_k on $\mathbf{R}^d \times \mathbf{R}^d$ has been introduced by C.F. Dunkl in [5]. For $y \in \mathbf{R}^d$ the function $x \mapsto E_k(x, y)$ can be viewed as the solution on $\mathbf{R}^d$ of the following initial problem

$$\begin{cases} T_j u(x, y) = y_j u(x, y) & \text{for } 1 \leq j \leq d \\ u(0, y) = 1 & \text{for all } y \in \mathbf{R}^d \end{cases}$$

This kernel has unique holomorphic extension to $\mathbf{C}^d \times \mathbf{C}^d$.

M. Rösler has proved in [8] the following integral representation for the Dunkl kernel

$$E_k(x, z) = \int_{\mathbf{R}^d} e^{\langle y, z \rangle} d\mu_x(y), \quad x \in \mathbf{R}^d, \quad z \in \mathbf{C}^d,$$

where μ_x is a probability measure on $\mathbf{R}^d$ with support in the closed ball $B(0, |x|)$ of center 0 and radius $|x|$.

Proposition 1.2. [6]: Let $z, w \in \mathbf{C}^d$ and $\lambda \in \mathbf{C}$. Then

1. $E_k(z, 0) = 1$,
2. $E_k(z, w) = E_k(w, z)$,
3. $E_k(\lambda z, w) = E_k(z, \lambda w)$,
4. For all $\nu = (\nu_1, \dots, \nu_d) \in \mathbf{N}^d$, $x \in \mathbf{R}^d$, $z \in \mathbf{C}^d$, we have

$$|D_z^\nu E_k(x, z)| \leq |x|^{|\nu|} \exp(|x| |Re z|),$$

where

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \dots \partial z_d^{\nu_d}}; \quad |\nu| = \nu_1 + \dots + \nu_d.$$

In particulier

$$|D_z^\nu E_k(ix, z)| \leq |x|^{|\nu|},$$

for all $x, z \in \mathbf{R}^d$.

The Dunkl transform is defined for $f \in L_k^1(\mathbf{R}^d) = L^1(\mathbf{R}^d, w_k(x)dx)$ by

$$\widehat{f}(\xi) = c_k^{-1} \int_{\mathbf{R}^d} f(x) E_k(-i\xi, x) w_k(x) dx,$$

where the constant c_k is given by

$$c_k = \int_{\mathbf{R}^d} e^{-\frac{|z|^2}{2}} w_k(z) dz.$$

According to [4, 6, 9] we have the following results:

1. When both f and $\widehat{f}$ are in $L_k^1(\mathbf{R}^d)$, we have the inversion formula

$$f(x) = \int_{\mathbf{R}^d} \widehat{f}(\xi) E_k(ix, \xi) w_k(\xi) d\xi, \quad x \in \mathbf{R}^d,$$

2. (Plancherel's theorem) The Dunkl transform on $S(\mathbf{R}^d)$, the space of Schwartz functions, extends uniquely to an isometric isomorphism on $L_k^2(\mathbf{R}^d)$.

K. Trimèche has introduced [11] the Dunkl translation operators τ_x , $x \in \mathbf{R}^d$. For $f \in L_k^2(\mathbf{R}^d)$ and we have

$$(\widehat{\tau_x(f)})(\xi) = E_k(ix, \xi) \widehat{f}(\xi),$$

and

$$\tau_x(f)(y) = c_k^{-1} \int_{\mathbf{R}^d} \widehat{f}(\xi) E_k(ix, \xi) E_k(iy, \xi) w_k(\xi) d\xi.$$

The generalized spherical mean operator for $f \in L_k^2(\mathbf{R}^d)$ is defined by

$$M_h f(x) = \frac{1}{d_k} \int_{\mathbf{S}^{d-1}} \tau_x(hy) d\eta_k(y), \quad x \in \mathbf{R}^d, h > 0.$$

From [7], we have $M_h f \in L_k^2(\mathbf{R}^d)$ whenever $f \in L_k^2(\mathbf{R}^d)$ and

$$\|M_h f\|_{L_k^2} \leq \|f\|_{L_k^2}.$$

For $p \geq -\frac{1}{2}$, we introduce the normalized Bessel function j_p defined by

$$j_p(z) = \Gamma(p+1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n+p+1)}, \quad z \in \mathbf{C},$$

where Γ is the gamma-function.

Lemma 1.3. [1] *The following inequalities are fulfilled*

1. $|j_p(x)| \leq 1$,
2. $1 - j_p(x) = O(x^2)$; $0 \leq x \leq 1$.

Lemma 1.4. *The following inequality is true*

$$|1 - j_p(x)| \geq c,$$

with $|x| \geq 1$, where $c > 0$ is a certain constant.

(Analog of lemma 2.9 in [2])

Proposition 1.5. *Let $f \in L_k^2(\mathbf{R}^d)$. Then*

$$(\widehat{M_h f})(\xi) = j_{\gamma+\frac{d}{2}-1}(h|\xi|)\widehat{f}(\xi).$$

(See [7])

For any function $f(x) \in L_k^2(\mathbf{R}^d)$ we define differences of the order m ($m \in \{1, 2, \dots\}$) with a step $h > 0$.

$$(1.1) \quad \Delta_h^m f(x) = (M_h - I)^m f(x),$$

here I is the unit operator.

Lemma 1.6. *Let $f \in L_k^2(\mathbf{R}^d)$. Then*

$$\|\Delta_h^m f(x)\|_{L_k^2}^2 = \int_{\mathbf{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi.$$

From formula (1.1) and proposition 1.5, we have

$$(\widehat{\Delta_h^m f})(\xi) = (j_{\gamma+\frac{d}{2}-1}(h|\xi|) - 1)^m \widehat{f}(\xi).$$

By Parseval's identity, we obtain

$$\|\Delta_h^m f(x)\|_{L_k^2}^2 = \int_{\mathbf{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi.$$

The lemma is proved

2. Main Results

In this section we give the main result of this paper. We need first to define (ψ, α, β) -Dunkl Lipschitz class.

Definition 2.1. Let $\alpha > 0$ and $\beta > 0$. A function $f \in L_k^2(\mathbf{R}^d)$ is said to be in the (ψ, α, β) -Dunkl Lipschitz class, denoted by $Lip(\psi, \alpha, \beta)$; if:

$$\|\Delta_h^m f(x)\|_{L_k^2} = O(h^\alpha \psi(h^\beta)) \text{ as } h \longrightarrow 0; \quad m \in \{1, 2, \dots\},$$

where $\psi(t)$ is a continuous increasing function on $[0, \infty)$, $\psi(0) = 0$ and $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in [0, \infty)$ and this function verify

$$\int_0^{1/h} r^{2m-2\alpha-1} \psi(r^{-2\beta}) dr = O(h^{2\alpha-2m} \psi(h^{2\beta})) \text{ as } h \longrightarrow 0.$$

Theorem 2.2. Let $f \in L_k^2(\mathbf{R}^d)$. Then the following are equivalents

1. $f \in Lip(\psi, \alpha, \beta)$,
 2. $\int_{|\xi| \geq s} |\hat{f}(\xi)|^2 w_k(\xi) d\xi = O(s^{-2\alpha} \psi(s^{-2\beta}))$ as $s \longrightarrow +\infty$.
- 1) $\implies$ 2) Assume that $f \in Lip(\psi, \alpha, \beta)$. Then we have

$$\|\Delta_h^m f(x)\|_{L_k^2} = O(h^\alpha \psi(h^\beta)) \text{ as } h \longrightarrow 0,$$

From lemma 1.6, we have

$$\|\Delta_h^m f(x)\|_{L_k^2}^2 = \int_{\mathbf{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m} |\hat{f}(\xi)|^2 w_k(\xi) d\xi.$$

If $|\xi| \in [\frac{1}{h}, \frac{2}{h}]$ then $h|\xi| \geq 1$ and lemma 1.4 implies that

$$1 \leq \frac{1}{c^{2m}} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m}.$$

Then

$$\begin{aligned} \int_{\frac{1}{h} \leq |\xi| \leq \frac{2}{h}} |\hat{f}(\xi)|^2 w_k(\xi) d\xi &\leq \frac{1}{c^{2m}} \int_{\frac{1}{h} \leq |\xi| \leq \frac{2}{h}} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m} |\hat{f}(\xi)|^2 w_k(\xi) d\xi \\ &\leq \frac{1}{c^{2m}} \int_{\mathbf{R}^d} |1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|)|^{2m} |\hat{f}(\xi)|^2 w_k(\xi) d\xi \\ &\leq C h^{2\alpha} (\psi(h^\beta))^2 = C h^{2\alpha} \psi(h^{2\beta}). \end{aligned}$$

Therefore

$$\int_{s \leq |\xi| \leq 2s} |\hat{f}(\xi)|^2 w_k(\xi) d\xi \leq C s^{-2\alpha} \psi(s^{-2\beta}).$$

Furthermore, we have

$$\begin{aligned}
& \int_{|\xi| \geq s} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi \\
&= \left(\int_{s \leq |\xi| \leq 2s} + \int_{2s \leq |\xi| \leq 4s} + \int_{4s \leq |\xi| \leq 8s} + \dots \right) |\widehat{f}(\xi)|^2 w_k(\xi) d\xi \\
&\leq C \left(s^{-2\alpha} \psi(s^{-2\beta}) + (2s)^{-2\alpha} \psi((2s)^{-2\beta}) + (2^2 s)^{-2\alpha} \psi((2^2 s)^{-2\beta}) + \dots \right) \\
&\leq C \left(s^{-2\alpha} \psi(s^{-2\beta}) + 2^{-2\alpha} s^{-2\alpha} \psi(s^{-2\beta}) + (2^{-2\alpha})^2 s^{-2\alpha} \psi(s^{-2\beta}) + \dots \right) \\
&\leq C s^{-2\alpha} \psi(s^{-2\beta}) (1 + 2^{-2\alpha} + (2^{-2\alpha})^2 + (2^{-2\alpha})^3 + \dots) \\
&\leq K_\alpha s^{-2\alpha} \psi(s^{-2\beta}), \\
&\text{where } K_\alpha = C(1 - 2^{-2\alpha})^{-1}.
\end{aligned}$$

This proves that

$$\int_{|\xi| \geq s} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi = O(s^{-2\alpha} \psi(s^{-2\beta})) \text{ as } s \longrightarrow +\infty$$

2) $\implies$ 1) Suppose now that

$$\int_{|\xi| \geq s} |\widehat{f}(\xi)|^2 w_k(\xi) d\xi = O(s^{-2\alpha} \psi(s^{-2\beta})) \text{ as } s \longrightarrow +\infty.$$

We have to show that

$$\int_0^\infty r^{2\gamma+d-1} |1 - j_{\gamma+\frac{d}{2}-1}(hr)|^{2m} \phi(r) dr = O(h^{2\alpha} \psi(h^{2\beta})),$$

where we have set

$$\phi(r) = \int_{\mathbf{S}^{d-1}} |\widehat{f}(ry)|^2 w_k(y) dy.$$

We write

$$I_1 = \int_0^{1/h} r^{2\gamma+d-1} |1 - j_{\gamma+\frac{d}{2}-1}(hr)|^{2m} \phi(r) dr,$$

and

$$I_2 = \int_{1/h}^\infty r^{2\gamma+d-1} |1 - j_{\gamma+\frac{d}{2}-1}(hr)|^{2m} \phi(r) dr.$$

Firstly, from (1) in lemma 1.3 we see that

$$I_2 \leq 4^m \int_{1/h}^\infty r^{2\gamma+d-1} \phi(r) dr = O(h^{2\alpha} \psi(h^{2\beta})) \text{ as } h \longrightarrow 0.$$

Set

$$g(r) = \int_r^\infty x^{2\gamma+d-1} \phi(x) dx.$$

From (2) in lemma 1.3, an integration by parts yields

$$\begin{aligned} I_1 &\leq -C_1 h^{2m} \int_0^{1/h} r^{2m} g'(r) dr \\ &\leq -C_1 g(1/h) + 2m C_1 h^{2m} \int_0^{1/h} r^{2m-1} g(r) dr \\ &\leq 2m C_1 h^{2m} \int_0^{1/h} r^{2m-1} r^{-2\alpha} \psi(r^{-2\beta}) dr \\ &\leq 2m C_1 h^{2m} \int_0^{1/h} r^{2m-2\alpha-1} \psi(r^{-2\beta}) dr \\ &\leq C_2 h^{2m} h^{2\alpha-2m} \psi(h^{2\beta}) \\ &\leq C_2 h^{2\alpha} \psi(h^{2\beta}), \end{aligned}$$

where C_1 and C_2 are positive constants, and this ends the proof

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