A class of multivalent functions defined by generalized Ruscheweyh derivatives involving a general fractional derivative operator

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Abstract

The main aim of the present paper is to obtain a new class of multivalent functions which is defined by making use of the generalized Ruscheweyh derivatives involving a general fractional derivative operator. We study the region of starlikeness and convexity of the class $\Omega_p(\alpha, \beta, \gamma)$. Also we apply the Fractional calculus techniques to obtain the applications of the class $\Omega_p(\alpha, \beta, \gamma)$. Finally, the familiar concept of δ -neighborhoods of p -valent functions for above mentioned class are employed.

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1. Introduction

Let A denote the class of functions that are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$ and let A_p be the subclass of A consisting of the functions f of the form

$$(1.1) \quad f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad (n \in N)$$

where p is some positive integer and f is analytic and p -valent in U .

The generalized fractional derivative operator of order λ , introduced by Srivastava and Saxena [9], [10], is defined as

$$(1.2) \quad J_{0,z}^{\lambda,\mu,\nu} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \left\{ z^{\lambda-\mu} \int_0^z (z-\zeta)^{-\lambda} \right. \\ \quad \cdot {}_2F_1 \left(\mu-\lambda, 1-\nu; 1-\lambda; 1-\frac{\zeta}{z} \right) f(\zeta) d\zeta \Big\}, \\ \quad (0 \leq \lambda < 1) \\ \frac{d^n}{dz^n} J_{0,z}^{\lambda-n,\mu,\nu} f(z), \quad (n \leq \lambda < n+1, n \in N) \end{cases}$$

where f is an analytic function in a simply connected region of the z -plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$, provided further that,

$$(1.3) \quad f(z) = O(|z|^k), \quad (z \rightarrow 0)$$

In terms of gamma function, we have

$$(1.4) \quad J_{0,z}^{\lambda,\mu,\nu} z^\rho = \frac{\Gamma(\rho+1)\Gamma(\rho-\mu+\nu+2)}{\Gamma(\rho-\mu+1)\Gamma(\rho-\lambda+\nu+2)} z^{\rho-\mu},$$

$$(0 \leq \lambda < 1, \rho > \max\{0, \mu - \nu - 1\} - 1)$$

It follows at once from the above definition that

$$(1.5) \quad J_{0,z}^{\lambda,\lambda,\nu} f(z) = D_z^\lambda f(z) = \frac{1}{\Gamma\lambda} \int_0^z \frac{f(t)}{(z-t)^{1-\delta}} dt, \quad (0 \leq \lambda < 1).$$

where $D_z^\lambda f(z)$ is the fractional derivative operator of order λ . Furthermore, in terms of gamma function, we have

$$(1.6) \quad D_z^\lambda z^\rho = \frac{\Gamma(\rho+1)}{\Gamma(\rho-\delta+1)} z^{\rho-\lambda}, \quad (0 \leq \lambda)$$

Similary, the fractional integral operator of order λ is

$$(1.7) \quad D_z^{-\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\delta} dt,$$

where f is an analytic function in a simply connected region of the z -plane containing the origin, and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $\log(z-t)$ to be real when $z-t > 0$. In terms of gamma function,

$$(1.8) \quad D_z^{-\lambda} z^\rho = \frac{\Gamma(\rho+1)}{\Gamma(\rho+\delta+1)} z^{\rho+\lambda}.$$

The generalized Ruscheweyh derivatives $\mathbf{J}_p^{\lambda,\mu} f$, $\mu > -1$ of $f \in A_p$ is defined by Goyal and Goyal [2] as follows:

$$(1.9) \quad \begin{aligned} \mathbf{J}_p^{\lambda,\mu} f(z) &= \frac{\Gamma(\mu-\lambda+\nu+2)}{\Gamma(\nu+2)\Gamma(\mu+1)} z^p J_{0,z}^{\lambda,\mu,\nu}(z^{\mu-p} f(z)) \\ &= z^p - \sum_{k=n+p}^{\infty} a_k B_p^{\lambda,\mu}(k) z^k \end{aligned}$$

where

$$(1.10) \quad B_p^{\lambda,\mu}(k) = \frac{\Gamma(k-p+1+\mu)\Gamma(\nu+2+\mu-\lambda)\Gamma(k+\nu-p+2)}{\Gamma(k-p+1)\Gamma(k+\nu-p+2+\mu-\lambda)\Gamma(\nu+2)\Gamma(1+\mu)}$$

For $\lambda = \mu$, this generalized Ruscheweyh derivatives get reduced to Ruscheweyh derivatives of $f(z)$ of order λ (see, e.g. [12]):

$$(1.11) \quad \begin{aligned} D^\lambda f(z) &= \frac{z^p}{\Gamma(\lambda+1)} \frac{d^\lambda}{dz^\lambda} (z^{\lambda-p} f(z)) \\ &= z^p + \sum_{k=n+p}^{\infty} a_k B_k(\lambda) z^k \end{aligned}$$

where

$$(1.12) \quad B_k(\lambda) = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda + p)\Gamma(k - p + 1)}$$

For $p=1$, (1.11) reduces to ordinary Ruscheweyh derivatives for univalent functions [8].

The operation $*$ is the convolution (Hadamard product) of two power series

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \text{ and } g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k$$

defined as

$$(1.13) \quad (f * g)(z) = z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k$$

A function $f \in \Omega_p$ is said to be in the class $\Omega_p(\alpha, \beta, \lambda)$ if and only if

$$(1.14) \quad \operatorname{Re} \left\{ \frac{z(\mathbf{J}_p^{\lambda, \mu} f(z))'}{(1 - \alpha)(\mathbf{J}_p^{\lambda, \mu} f(z)) + \alpha z^2 (\mathbf{J}_p^{\lambda, \mu} f(z))''} \right\} > \beta$$

for $z \in U$ and $0 \leq \alpha < 1$, $0 \leq \beta < p$ and $\lambda > -1$.

The class $\Omega_p(\alpha, \beta, \lambda)$ contains many well-known classes of analytic functions such as:

- For $\alpha = \lambda = 0$, $\Omega_p(\alpha, \beta, \lambda)$ reduces to the class $S^*(\beta)$ of starlike functions of order β .
- For $\alpha = \lambda = -1$, $\Omega_p(\alpha, \beta, \lambda)$ reduces to the class $K(\beta)$ of convex functions of order β .

2. Main Results

The coefficient bounds for the functions $f \in \Omega_p(\alpha, \beta, \lambda)$ are found in the following theorem:

Theorem 2.1. Let $f \in A_p$, $z \in U$ be of the form (1.1). Then $f \in \Omega_p(\alpha, \beta, \lambda)$ iff

$$(2.1) \quad \sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2) + k - \beta}{p - \beta + \alpha\beta(1+p-p^2)} B_p^{\lambda, \mu}(k) a_k < 1$$

where $0 \leq \alpha < 1$, $0 \leq \beta < p$ and $\lambda > -1$.

Proof. Since $f \in \Omega_p(\alpha, \beta, \lambda)$

$$(2.2) \quad \operatorname{Re} \left\{ \frac{z(\mathbf{J}_p^{\lambda, \mu} f(z))'}{(1 - \alpha)(\mathbf{J}_p^{\lambda, \mu} f(z)) + \alpha z^2 (\mathbf{J}_p^{\lambda, \mu} f(z))''} \right\} > \beta$$

Making use of equation (1.9) in the above inequality, we obtain

$$(2.3) \quad \operatorname{Re} \left\{ \frac{pz^{p-1} - \sum_{k=n+p}^{\infty} k a_k B_p^{\lambda, \mu}(k) z^{k-1}}{(1 - \alpha + \alpha p(p-1))z^p - \sum_{k=n+p}^{\infty} [(1 - \alpha) + \alpha(k(k-1))] a_k B_p^{\lambda, \mu}(k) z^k} \right\} > \beta$$

Therefore, we obtain

$$(2.4) \quad \sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2) + k - \beta}{p - \beta + \alpha\beta(1+p-p^2)} B_p^{\lambda, \mu}(k) a_k < 1.$$

In this theorem, we will show that this class is closed under linear combination.

Theorem 2.2. Let for $j \in \{1, 2, 3, \dots, m\}$

$$f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \in \Omega_p(\alpha, \beta, \lambda)$$

Then for $0 < P_j < 1$, $\sum_{j=1}^m P_j = 1$, the function $F(z)$ defined by

$$(2.5) \quad F(z) = \sum_{j=1}^m P_j f_j(z)$$

is also in $\Omega_p(\alpha, \beta, \lambda)$.

Proof. For every $j \in \{1, 2, 3, \dots, m\}$, we obtain

$$(2.6) \quad \sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2) + k - \beta}{p - \beta + \alpha\beta(1+p-p^2)} B_p^{\lambda, \mu}(k) a_{k,j} < 1.$$

Since

$$(2.7) \quad F(z) = \sum_{j=1}^m P_j \left(z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \right) = z^p - \sum_{k=n+p}^{\infty} \left(\sum_{j=1}^m P_j a_{k,j} \right) z^k$$

Therefore,

$$(2.8) \quad \sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k) \left(\sum_{j=1}^m P_j a_{k,j} \right) < \sum_{j=1}^m P_j = 1$$

which proves the Theorem.

Theorem 2.3. *Let*

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

and

$$g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k$$

belong to $\Omega_p(\alpha, \beta, \lambda)$. Then the function

$$G(z) = z^p - \sum_{k=n+p}^{\infty} (a_k^2 + b_k^2) z^k$$

is in $\Omega_p(\alpha, \beta, \lambda_1)$, where

$$(2.9) \quad \lambda_1 \leq \inf_k \left[\frac{(k-p)[\alpha\beta(1+k-k^2)+k-\beta]}{2[p-\beta+\alpha\beta(1+p-p^2)]} (B_p^{\lambda,\mu}(k))^2 - 1 \right].$$

Proof.

Since $f, g \in \Omega_p(\alpha, \beta, \lambda)$,

$$(2.10) \quad \begin{aligned} & \sum_{k=n+p}^{\infty} \left[\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k) \right]^2 a_k^2 \\ & \leq \left[\sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k) a_k \right]^2 < 1. \end{aligned}$$

Similarly,

$$\sum_{k=n+p}^{\infty} \left[\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k) \right]^2 b_k^2$$

$$(2.11) \quad \leq \left[\sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k) b_k \right]^2 < 1.$$

Therefore

$$(2.12) \quad \sum_{k=n+p}^{\infty} \frac{1}{2} \left[\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k) \right]^2 (a_k^2 + b_k^2) < 1$$

Now, we must show that

$$(2.13) \quad \sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda_1,\mu}(k) (a_k^2 + b_k^2) < 1$$

This inequality holds if

$$(2.14) \quad \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda_1,\mu}(k) \leq \frac{1}{2} \left[\frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} B_p^{\lambda,\mu}(k) \right]^2$$

which is equivalent to

$$(2.15) \quad B_p^{\lambda_1,\mu}(k) = \frac{1}{2} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} \left[B_p^{\lambda,\mu}(k) \right]^2$$

Since $\frac{\lambda_1+1}{k-p} \leq B_p^{\lambda_1,\mu}(k)$, we obtain

$$(2.16) \quad \frac{\lambda_1+1}{k-p} \leq \frac{1}{2} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} \left[B_p^{\lambda,\mu}(k) \right]^2$$

and this gives the required result.

A modified Komatu operator $K_c^\gamma : A \rightarrow A$ is defined for $\gamma \geq 0$ and $c > -p$ as

$$(2.17) \quad K_{c,p}^\gamma f(z) = \frac{(c+p)^\gamma}{\Gamma(\gamma)z^c} \int_0^1 t^c \left(\log \frac{1}{t}\right)^{\gamma-1} f(tz) dt$$

It can be easily verified that for $f \in A_p$

$$(2.18) \quad K_{c,p}^\gamma f(z) = z^p - \sum_{k=p+1}^{\infty} \left(\frac{c+p}{c+k} \right)^\gamma a_k z^k$$

Theorem 2.4. If $f \in \Omega_p(\alpha, \beta, \lambda)$, then $K_{c,p}^\gamma f \in \Omega_p(\alpha, \beta, \lambda)$.

Proof. Since $f \in \Omega_p(\alpha, \beta, \lambda)$ and $\left\{\frac{c+p}{c+k}\right\}^\gamma < 1$, we have

$$(2.19) \quad \sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} \left(\frac{c+p}{c+k}\right)^\gamma B_p^{\lambda,\mu}(k)a_k < 1$$

This completes the proof.

3. Radius of starlikeness and convexity

Now we obtain the radii of starlikeness and convexity for the functions $K_{c,p}^\gamma f$.

Theorem 3.1. The function $K_{c,p}^\gamma f$ is starlike of order η in

$$|z| < r_1(\alpha, \beta, \lambda, c, \gamma, \eta), \text{ where}$$

$$r_1(\alpha, \beta, \lambda, c, \gamma, \eta)$$

$$(3.1) \quad = \inf_k \left[\frac{1-\eta}{k-\eta-p+1} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} \left(\frac{c+p}{c+k}\right)^\gamma B_p^{\lambda,\mu}(k)a_k \right]^{\frac{1}{k-p}}$$

Proof. We must show that

$$(3.2) \quad \left| \frac{z(K_{c,p}^\gamma f(z))'}{K_{c,p}^\gamma f(z)} - p \right| < 1 - \eta.$$

i.e.

$$(3.3) \quad \left| \frac{z(K_{c,p}^\gamma f(z))' - pK_{c,p}^\gamma f(z)}{K_{c,p}^\gamma f(z)} \right| \leq \frac{\sum_{k=n+p}^{\infty} \left(\frac{c+p}{c+k}\right)^\gamma (k-p)a_k |z|^{k-p}}{1 - \sum_{k=n+p}^{\infty} \left(\frac{c+p}{c+k}\right)^\gamma a_k |z|^{k-p}} < 1 - \eta$$

or to show that

$$\sum_{k=n+p}^{\infty} \left(\frac{c+p}{c+k}\right)^\gamma (k-p)a_k |z|^{k-p} + (1-\eta) \sum_{k=n+p}^{\infty} \left(\frac{c+p}{c+k}\right)^\gamma a_k |z|^{k-p} < 1 - \eta$$

(3.4)

Now, in view of (2.1), the theorem holds if

$$(3.5) \quad |z|^{k-p} < \frac{1-\eta}{k-p+1-\eta} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} \left(\frac{c+p}{c+k}\right)^\gamma B_p^{\lambda,\mu}(k)$$

This proves the result.

Theorem 3.2. *The function $K_{c,p}^\gamma f$ is convex of order η in*

$|z| < r_2(\alpha, \beta, \lambda, c, \gamma, \eta)$, where

$$(3.6) \quad r_2(\alpha, \beta, \lambda, c, \gamma, \eta) = \inf_k \left[\frac{p(p-\eta)}{k(k-\eta)} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} \left(\frac{c+p}{c+k}\right)^\gamma B_p^{\lambda,\mu}(k) a_k \right]^{\frac{1}{k-p}}$$

Proof. Noting the fact that $K_{c,p}^\gamma f$ is convex iff $z(K_{c,p}^\gamma f)'$ is starlike. Therefore, we must show that,

$$(3.7) \quad \left| \frac{z(K_{c,p}^\gamma f(z))''}{(K_{c,p}^\gamma f(z))'} \right| < 1 - \eta.$$

i.e.

$$(3.8) \quad \frac{p(p-1)z^{p-1} - \sum_{k=n+p}^{\infty} \left(\frac{c+p}{c+k}\right)^\gamma (k-p)a_k |z|^{k-1}}{pz^{p-1} - \sum_{k=n+p}^{\infty} \left(\frac{c+p}{c+k}\right)^\gamma a_k |z|^{k-1}} < 1 - \eta$$

or to show that

$$(3.9) \quad \sum_{k=n+p}^{\infty} \left(\frac{c+p}{c+k}\right)^\gamma [k(k-\eta)] a_k |z|^{k-p} < p(p-\eta)$$

Now, by (2.1), the last inequality holds if

$$(3.10) \quad |z|^{k-p} < \frac{p(p-\eta)}{k(k-\eta)} \frac{\alpha\beta(1+k-k^2)+k-\beta}{p-\beta+\alpha\beta(1+p-p^2)} \left(\frac{c+p}{c+k}\right)^\gamma B_p^{\lambda,\mu}(k)$$

This complete the proof.

Theorem 3.3. If $f \in \Omega_p(\alpha, \beta, \lambda)$, then the function $F_p^\mu(z)$, $z \in U$ defined by

$$(3.11) \quad F_p^\mu(z) = (1 - \mu)z^p + p\mu \int_0^z \frac{f(t)}{t} dt, \quad 0 \leq \mu < \frac{2}{p}$$

is in $\Omega_p(\alpha, \beta, \lambda)$.

Proof. We have

$$(3.12) \quad F_p^\mu(z) = (1 - \mu)z^p + p\mu \left[\int_0^z t^{p-1} dt - \sum_{k=n+p}^{\infty} \int_0^z a_k t^{k-1} dt \right] = z^p - \sum_{k=n+p}^{\infty} a_k \frac{p\mu}{k} z^k$$

Now, by (2.1), we obtain

$$(3.13) \quad \begin{aligned} & \sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2) + k - \beta}{p - \beta + \alpha\beta(1+p-p^2)} \left(\frac{p\mu}{k} \right) B_p^{\lambda, \mu}(k) a_k \\ & \leq \sum_{k=n+p}^{\infty} \frac{\alpha\beta(1+k-k^2) + k - \beta}{p - \beta + \alpha\beta(1+p-p^2)} \left(\frac{2}{k} \right) B_p^{\lambda, \mu}(k) a_k < 1 \end{aligned}$$

and this proves the theorem.

Remark 3.4. $F_p^\mu(z)$ is starlike of order η in $|z| < r_1^p(\alpha, \beta, \lambda, \eta, \mu)$, where

$$(3.14) \quad \begin{aligned} & r_1^p(\alpha, \beta, \lambda, \eta, \mu) \\ & = \inf_k \left[\frac{k(1-\eta)}{p\mu(p-k+1-\eta)} \frac{\alpha\beta(1+k-k^2) + k - \beta}{p - \beta + \alpha\beta(1+p-p^2)} B_p^{\lambda, \mu}(k) a_k \right]^{\frac{1}{k-p}}. \end{aligned}$$

Also, $F_p^\mu(z)$ is convex of order η in $|z| < r_1^p(\alpha, \beta, \lambda, \eta, \mu)$, where

$$(3.15) \quad r_2^p(\alpha, \beta, \lambda, \eta, \mu) = \inf_k \left[\frac{(p-\eta)}{p\mu(k-\eta)} \frac{\alpha\beta(1+k-k^2) + k - \beta}{p - \beta + \alpha\beta(1+p-p^2)} B_p^{\lambda, \mu}(k) a_k \right]^{\frac{1}{k-p}}.$$

The proof of the above remark is made by similar arguments of the Theorems 3.1 and 3.2.

4. Fractional Calculus on $\Omega_p(\alpha, \beta, \lambda)$

In this section, we apply the fractional calculus techniques and discuss the properties of the family $\Omega_p(\alpha, \beta, \lambda)$ (see [10]). In this theorem, we find the distortion bounds for $f(z)$.

Theorem 4.1. *Let $f \in \Omega_p(\alpha, \beta, \lambda)$, $\lambda \geq 0$. Then*

$$(4.1) \quad |z|^{p+\delta} \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} [1 - M|z|^n] \leq |D_z^{-\delta} f(z)| \leq |z|^{p+\delta} \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} [1 + M|z|^n]$$

where

$$M = \frac{(p+1)_n [p - \beta + \alpha\beta(1 + p - p^2)] (\nu + \mu - \lambda + 2)_n \Gamma(n+1)}{(p+\delta+1)_n [p + n + \beta\{\alpha(1 + p + n - (p+n)^2) - 1\}] (\mu+1)_n (\nu+2)_n}$$

and $f(z)$ is analytic function

Proof.

By equation (1.8), we have

$$(4.2) \quad \frac{\Gamma(\delta + p + 1)}{\Gamma(p + 1)} z^{-\delta} D_z^{-\delta} f(z) = z^p - \sum_{k=n+p}^{\infty} a_k H_p(k, \delta) z^k$$

where

$$(4.3) \quad H_p(k, \delta) = \frac{\Gamma(\delta + p + 1) \Gamma(k + 1)}{\Gamma(k + \delta + 1) \Gamma(p + 1)}$$

But $H_p(k, \delta)$ is a decreasing function for $k \geq n + p$ and also $B_p^{\lambda, \mu}(k)$ is increasing function of k , thus, we have

$$(4.4) \quad H_p(k, \delta) \leq \frac{\Gamma(\delta + p + 1) \Gamma(n + p + 1)}{\Gamma(n + p + \delta + 1) \Gamma(p + 1)} = \frac{(p+1)_n}{(\delta + p + 1)_n}$$

and

$$(4.5) \quad B_p^{\lambda, \mu}(k) \geq \frac{(\mu+1)_n (\nu+2)_n}{(\mu - \lambda + \nu + 2)_n \Gamma(n+1)}$$

So, we conclude that

$$\begin{aligned} & \left| \frac{\Gamma(\delta + p + 1)}{\Gamma(p + 1)} z^{-\delta} D_z^{-\delta} f(z) \right| \\ & \leq |z|^p + \frac{(p+1)_n}{(\delta + p + 1)_n} |z|^{n+p} \sum_{k=n+p}^{\infty} a_k \end{aligned}$$

$$(4.6) \quad \leq |z|^p \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} [1 + M|z|^{n+p}]$$

where M is defined in the theorem statement. Thus, we get

$$(4.7) \quad \left| D_z^{-\delta} f(z) \right| \leq |z|^{p+\delta} \frac{\Gamma(p+1)}{\Gamma(\delta+p+1)} [1 + M|z|^n]$$

Also, we have

$$(4.8) \quad \begin{aligned} & \left| \frac{\Gamma(\delta+p+1)}{\Gamma(p+1)} z^{-\delta} D_z^{-\delta} f(z) \right| \\ & \geq |z|^p - \frac{(p+1)_n}{(\delta+p+1)_n} |z|^{n+p} \sum_{k=n+p}^{\infty} a_k \\ & \geq |z|^p \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} [1 - M|z|^n] \end{aligned}$$

Then

$$(4.9) \quad \left| D_z^{-\delta} f(z) \right| \geq |z|^{p+\delta} \frac{\Gamma(p+1)}{\Gamma(\delta+p+1)} [1 - M|z|^n]$$

This completes the proof of the theorem.

Theorem 4.2. *Let $f \in \Omega_p(\alpha, \beta, \lambda)$, $\lambda \geq 0$. Then*

$$(4.10) \quad |z|^{p-\delta} \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} [1 - N|z|^n] \leq |D_z^{\delta} f(z)| \leq |z|^{p-\delta} \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} [1 + N|z|^n]$$

where

$$N = \frac{(p+1)_n [p - \beta + \alpha\beta(1 + p - p^2)] (\nu + \mu - \lambda + 2)_n \Gamma(n+1)}{(p-\delta+1)_n [p + n + \beta\{\alpha(1 + p + n - (p+n)^2) - 1\}] (\mu+1)_n (\nu+2)_n}$$

and $f(z)$ is analytic function

Proof. By equation (1.6), we have

$$(4.11) \quad \frac{\Gamma(p-\delta+1)}{\Gamma(p+1)} z^{\delta} D_z^{\delta} f(z) = z^p - \sum_{k=n+p}^{\infty} a_k R_p(k, \delta) z^k$$

where

$$(4.12) \quad R_p(k, \delta) = \frac{\Gamma(p - \delta + 1)\Gamma(k + 1)}{\Gamma(k - \delta + 1)\Gamma(p + 1)}$$

But $R_p(k, \delta)$ is a decreasing function for $k \geq n + p$ and also $B_p^{\lambda, \mu}(k)$ is increasing function of k , thus, we have

$$(4.13) \quad R_p(k, \delta) \leq \frac{\Gamma(p - \delta + 1)\Gamma(n + p + 1)}{\Gamma(n + p - \delta + 1)\Gamma(p + 1)} = \frac{(p + 1)_n}{(p - \delta + 1)_n}$$

and

$$(4.14) \quad B_p^{\lambda, \mu}(k) \geq \frac{(\mu + 1)_n(\nu + 2)_n}{(\mu - \lambda + \nu + 2)_n\Gamma(n + 1)}$$

So, we conclude that

$$(4.15) \quad \begin{aligned} & \left| \frac{\Gamma(p - \delta + 1)}{\Gamma(p + 1)} z^\delta D_z^\delta f(z) \right| \\ & \leq |z|^p + \frac{(p+1)_n}{(p-\delta+1)_n} |z|^{n+p} \sum_{k=n+p}^{\infty} a_k \\ & \leq |z|^p \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} [1 + N|z|^{n+p}] \end{aligned}$$

where N is defined in the theorem statement. Then, we get

$$(4.16) \quad \left| D_z^\delta f(z) \right| \leq |z|^{p-\delta} \frac{\Gamma(p + 1)}{\Gamma(p - \delta + 1)} [1 + N|z|^n]$$

Also, we have

$$(4.17) \quad \begin{aligned} & \left| \frac{\Gamma(p - \delta + 1)}{\Gamma(p + 1)} z^\delta D_z^\delta f(z) \right| \\ & \geq |z|^p - \frac{(p+1)_n}{(p-\delta+1)_n} |z|^{n+p} \sum_{k=n+p}^{\infty} a_k \\ & \geq |z|^p \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} [1 - N|z|^n] \end{aligned}$$

Then

$$(4.18) \quad \left| D_z^\delta f(z) \right| \geq |z|^{p-\delta} \frac{\Gamma(p + 1)}{\Gamma(p - \delta + 1)} [1 - N|z|^n]$$

This completes the proof.

Letting $\delta = 1$ in Theorem 4.1, we obtain

Corollary 4.3. Let $f \in \Omega_p(\alpha, \beta, \lambda)$, $\lambda \geq 0$. Then

$$(4.19) \quad \frac{|z|^{p+1}}{(p+1)} [1 - M|z|^n] \leq \left| \int_0^z f(t) dt \right| \leq \frac{|z|^{p+1}}{(p+1)} [1 + M|z|^n]$$

where

$$M = \frac{(p+1)_n [p - \beta + \alpha\beta(1 + p - p^2)] (\nu + \mu - \lambda + 2)_n \Gamma(n+1)}{(p+n+1)_n [p + n + \beta\{\alpha(1 + p + n - (p+n)^2) - 1\}] (\mu+1)_n (\nu+2)_n}$$

and $f(z)$ is analytic function

Letting $\delta = 0$ in Theorem 4.2, we obtain

Corollary 4.4. Let $f \in \Omega_p(\alpha, \beta, \lambda)$, $\lambda \geq 0$. Then

$$(4.20) \quad |z|^p [1 - N|z|^n] \leq |f(z)| \leq |z|^p [1 + N|z|^n]$$

where

$$N = \frac{[p - \beta + \alpha\beta(1 + p - p^2)] (\nu + \mu - \lambda + 2)_n \Gamma(n+1)}{[p + n + \beta\{\alpha(1 + p + n - (p+n)^2) - 1\}] (\mu+1)_n (\nu+2)_n}$$

and $f(z)$ is analytic function

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