A class of multivalent functions defined by generalized Ruscheweyh derivatives involving a general fractional derivative operator

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Abstract

The main aim of the present paper is to obtain a new class of multivalent functions which is defined by making use of the generalized Ruscheweyh derivatives involving a general fractional derivative operator. We study the region of starlikeness and convexity of the class $\Omega_p(\alpha, \beta, \gamma)$. Also we apply the Fractional calculus techniques to obtain the applications of the class $\Omega_p(\alpha, \beta, \gamma)$. Finally, the familiar concept of $\delta$-neighborhoods of p-valent functions for above mentioned class are employed.

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1. Introduction

Let \( A \) denote the class of functions that are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and let \( A_p \) be the subclass of \( A \) consisting of the functions \( f \) of the form

\[
\begin{align*}
  f(z) &= z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad (n \in \mathbb{N}) \\
  \text{where } p &\text{ is some positive integer and } f \text{ is analytic and } p\text{-valent in } U.
\end{align*}
\]

The generalized fractional derivative operator of order \( \lambda \), introduced by Srivastava and Saxena [9], [10], is defined as

\[
\begin{align*}
  J_{0,z}^{\lambda,\mu,\nu} f(z) &= \begin{cases} \\
    \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \left\{ z^{\lambda-\mu} \int_0^z (z-\zeta)^{-1} f(\zeta) d\zeta \right\}, \\
    _2F_1 \left( \mu - \lambda, 1 - \nu; 1 - \lambda; 1 - \frac{z}{\zeta} \right) f(\zeta) d\zeta, \\
    0 \leq \lambda < 1 \\
    \frac{d^n}{dz^n} J_{0,z}^{\lambda-n,\mu,\nu} f(z), & (n \leq \lambda < n+1, n \in \mathbb{N})
\end{cases}
\end{align*}
\]

where \( f \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin, and the multiplicity of \( (z - \zeta)^{-\lambda} \) is removed by requiring \( \log(z - \zeta) \) to be real when \( z - \zeta > 0 \), provided further that,

\[
  f(z) = O(|z|^k), \quad (z \to 0)
\]

In terms of gamma function, we have

\[
\begin{align*}
  J_{0,z}^{\lambda,\mu,\nu} z^\rho &= \frac{\Gamma(\rho+1)\Gamma(\rho-\mu+\nu+2)}{\Gamma(\rho-\mu+1)\Gamma(\rho+\nu+2)} z^{\rho-\mu}, \\
  (0 \leq \lambda < 1, \rho &> \max\{0, \mu - \nu - 1 \} - 1)
\end{align*}
\]

It follows at once from the above definition that

\[
\begin{align*}
  J_{0,z}^{\lambda,\mu,\nu} f(z) &= D_z^\lambda f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \quad (0 \leq \lambda < 1).
\end{align*}
\]
where $D^\lambda_z f(z)$ is the fractional derivative operator of order $\lambda$. Furthermore, in terms of gamma function, we have

$$D^\lambda_z z^\rho = \frac{\Gamma(\rho + 1)}{\Gamma(\rho - \delta + 1)} z^{\rho - \lambda}, \quad (0 \leq \lambda) \tag{1.6}$$

Similarly, the fractional integral operator of order $\lambda$ is

$$D^{-\lambda}_z f(z) = \frac{1}{\Gamma(1 - \lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z - t)^\delta} dt, \tag{1.7}$$

where $f$ is an analytic function in a simply connected region of the $z$-plane containing the origin, and the multiplicity of $(z - t)^{-\lambda}$ is removed by requiring $\log(z - t)$ to be real when $z - t > 0$. In terms of gamma function,

$$D^{-\lambda}_z z^\rho = \frac{\Gamma(\rho + 1)}{\Gamma(\rho + \delta + 1)} z^{\rho + \lambda}. \tag{1.8}$$

The generalized Ruscheweyh derivatives $J^\lambda_{p,\mu} f$, $\mu > -1$ of $f \in A_p$ is defined by Goyal and Goyal [2] as follows:

$$J^\lambda_{p,\mu} f(z) = \frac{\Gamma(\mu - \lambda + \nu + 2)}{\Gamma(\nu + 2)\Gamma(\mu + 1)} z^\mu J^\lambda_{0,z} (z^{\mu - p} f(z)) \tag{1.9}$$

$$= z^p - \sum_{k=n+p}^{\infty} a_k B^{\lambda,\mu}_{p}(k) z^k$$

where

$$B^{\lambda,\mu}_{p}(k) = \frac{\Gamma(k - p + 1 + \mu)\Gamma(\nu + 2 + \mu - \lambda)\Gamma(k + \nu - p + 2)}{\Gamma(k - p + 1)\Gamma(k + \nu - p + 2 + \mu - \lambda)\Gamma(\nu + 2)\Gamma(1 + \mu)} \tag{1.10}$$

For $\lambda = \mu$, this generalized Ruscheweyh derivatives get reduced to Ruscheweyh derivatives of $f(z)$ of order $\lambda$ (see, e.g. [12]):

$$D^\lambda f(z) = \frac{z^p}{\Gamma(\lambda + 1)} \frac{d^\lambda}{dz^\lambda} (z^{\lambda - p} f(z))$$

$$= z^p + \sum_{k=n+p}^{\infty} a_k B(k) z^k \tag{1.11}$$
where
\[
B_k(\lambda) = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda + p)\Gamma(k - p + 1)}
\]  

For \( p = 1 \), (1.11) reduces to ordinary Ruscheweyh derivatives for univalent functions [8].

The operation \( * \) is the convolution (Hadamard product) of two power series

\[
f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k
\]

defined as
\[
(f * g)(z) = z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k
\]

A function \( f \in \Omega_p \) is said to be in the class \( \Omega_p(\alpha, \beta, \lambda) \) if and only if
\[
\text{Re} \left\{ \frac{z^{(J^p_\lambda)^\mu f(z))'}}{(1 - \alpha)(J^p_\lambda)^\mu f(z)) + \alpha z^2(J^p_\lambda)^\mu f(z))''} \right\} > \beta
\]

for \( z \in U \) and \( 0 \leq \alpha < 1, 0 \leq \beta < p \) and \( \lambda > -1 \).

The class \( \Omega_p(\alpha, \beta, \lambda) \) contains many well-known classes of analytic functions such as:

- For \( \alpha = \lambda = 0 \), \( \Omega_p(\alpha, \beta, \lambda) \) reduces to the class \( S^*(\beta) \) of starlike functions of order \( \beta \).

- For \( \alpha = \lambda = -1 \), \( \Omega_p(\alpha, \beta, \lambda) \) reduces to the class \( K(\beta) \) of convex functions of order \( \beta \).

2. Main Results

The coefficient bounds for the functions \( f \in \Omega_p(\alpha, \beta, \lambda) \) are found in the following theorem:

**Theorem 2.1.** Let \( f \in A_p, \ z \in U \) be of the form (1.1). Then \( f \in \Omega_p(\alpha, \beta, \lambda) \) iff
\[
\sum_{k=n+p}^{\infty} \frac{\alpha \beta(1 + k - k^2) + k - \beta}{p - \beta + \alpha \beta(1 + p - p^2)} B_p^\lambda(1 - \beta) a_k < 1
\]

where \( 0 \leq \alpha < 1, 0 \leq \beta < p \) and \( \lambda > -1 \).
Proof. Since \( f \in \Omega_p(\alpha, \beta, \lambda) \)

\[
(2.2) \quad \text{Re} \left\{ \frac{z (J_p^{\lambda \mu} f(z))'}{(1 - \alpha) (J_p^{\lambda \mu} f(z)) + \alpha z (J_p^{\lambda \mu} f(z))''} \right\} > \beta
\]

Making use of equation (1.9) in the above inequality, we obtain

\[
(2.3) \quad \text{Re} \left\{ \frac{pz^{p-1} - \sum_{k=n+p}^{\infty} ka_k B_p^{\lambda \mu}(k) z^{k-1}}{(1 - \alpha + \alpha p(p - 1)) z^p - \sum_{k=n+p}^{\infty} [(1 - \alpha) + \alpha (k - 1)]a_k B_p^{\lambda \mu}(k) z^k} \right\} > \beta
\]

Therefore, we obtain

\[
(2.4) \quad \sum_{k=n+p}^{\infty} \alpha \beta (1 + k - k^2) + k - \beta \frac{p - \beta + \alpha \beta (1 + p - p^2)}{B_p^{\lambda \mu}(k)} a_k < 1.
\]

In this theorem, we will show that this class is closed under linear combination.

**Theorem 2.2.** Let for \( j \in \{1, 2, 3, ..., m\} \)

\[
f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \in \Omega_p(\alpha, \beta, \lambda)
\]

Then for \( 0 < P_j < 1, \sum_{j=1}^{m} P_j = 1 \), the function \( F(z) \) defined by

\[
(2.5) \quad F(z) = \sum_{j=1}^{m} P_j f_j(z)
\]

is also in \( \Omega_p(\alpha, \beta, \lambda) \).

**Proof.** For every \( j \in \{1, 2, 3, ..., m\} \), we obtain

\[
(2.6) \quad \sum_{k=n+p}^{\infty} \frac{\alpha \beta (1 + k - k^2) + k - \beta}{p - \beta + \alpha \beta (1 + p - p^2)} B_p^{\lambda \mu}(k) a_{k,j} < 1.
\]

Since

\[
(2.7) \quad F(z) = \sum_{j=1}^{m} P_j \left( z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \right) = z^p - \sum_{k=n+p}^{\infty} \left( \sum_{j=1}^{m} P_j a_{k,j} \right) z^k
\]
Therefore,
\[
(2.8) \quad \sum_{k=n+p}^{\infty} \frac{\alpha\beta(1 + k - k^2) + k - \beta}{p - \beta + \alpha\beta(1 + p - p^2)} B^{\lambda,\mu}_p(k) \left( \sum_{j=1}^{m} P_j a_{k,j} \right) < \sum_{j=1}^{m} P_j = 1
\]
which proves the Theorem.

**Theorem 2.3.** Let
\[
f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k
\]
and
\[
g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k
\]
belong to \( \Omega_p(\alpha, \beta, \lambda) \). Then the function
\[
G(z) = z^p - \sum_{k=n+p}^{\infty} (a_k^2 + b_k^2) z^k
\]
is in \( \in \Omega_p(\alpha, \beta, \lambda_1) \), where
\[
(2.9) \quad \lambda_1 \leq \inf_k \left[ \frac{(k-p)[\alpha\beta(1 + k - k^2) + k - \beta]}{2[p - \beta + \alpha\beta(1 + p - p^2)]} (B^{\lambda,\mu}_p(k))^2 - 1 \right].
\]

**Proof.**
Since \( f, g \in \Omega_p(\alpha, \beta, \lambda) \),
\[
\sum_{k=n+p}^{\infty} \left[ \frac{\alpha\beta(1 + k - k^2) + k - \beta}{p - \beta + \alpha\beta(1 + p - p^2)} B^{\lambda,\mu}_p(k) \right]^2 a_k^2
\leq \left[ \sum_{k=n+p}^{\infty} \frac{\alpha\beta(1 + k - k^2) + k - \beta}{p - \beta + \alpha\beta(1 + p - p^2)} B^{\lambda,\mu}_p(k) a_k \right]^2 < 1.
(2.10)

Similarly,
\[
\sum_{k=n+p}^{\infty} \left[ \frac{\alpha\beta(1 + k - k^2) + k - \beta}{p - \beta + \alpha\beta(1 + p - p^2)} B^{\lambda,\mu}_p(k) \right]^2 b_k^2
\]
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\[ \sum_{k=n+p}^{\infty} \frac{\alpha \beta (1+k-k^2)+k-\beta}{p-\beta + \alpha \beta (1+p-p^2)} B_{p}^{\lambda,\mu}(k) b_k^2 < 1. \] (2.11)

Therefore

\[ \sum_{k=n+p}^{\infty} \frac{1}{2} \left[ \frac{\alpha \beta (1+k-k^2)+k-\beta}{p-\beta + \alpha \beta (1+p-p^2)} B_{p}^{\lambda,\mu}(k) \right]^2 (a_k^2 + b_k^2) < 1 \] (2.12)

Now, we must show that

\[ \sum_{k=n+p}^{\infty} \frac{\alpha \beta (1+k-k^2)+k-\beta}{p-\beta + \alpha \beta (1+p-p^2)} B_{p}^{\lambda,\mu}(k) (a_k^2 + b_k^2) < 1 \] (2.13)

This inequality holds if

\[ \frac{\alpha \beta (1+k-k^2)+k-\beta}{p-\beta + \alpha \beta (1+p-p^2)} B_{p}^{\lambda,\mu}(k) \leq \frac{1}{2} \left[ \frac{\alpha \beta (1+k-k^2)+k-\beta}{p-\beta + \alpha \beta (1+p-p^2)} B_{p}^{\lambda,\mu}(k) \right]^2 \] (2.14)

which is equivalent to

\[ B_{p}^{\lambda,\mu}(k) = \frac{1}{2} \frac{\alpha \beta (1+k-k^2)+k-\beta}{p-\beta + \alpha \beta (1+p-p^2)} B_{p}^{\lambda,\mu}(k)^2 \] (2.15)

Since \( \frac{\lambda_1 + 1}{k-p} \leq B_{p}^{\lambda,\mu}(k) \), we obtain

\[ \frac{\lambda_1 + 1}{k-p} \leq \frac{1}{2} \frac{\alpha \beta (1+k-k^2)+k-\beta}{p-\beta + \alpha \beta (1+p-p^2)} B_{p}^{\lambda,\mu}(k)^2 \] (2.16)

and this gives the required result.

A modified Komatu operator \( K_{n,p}^\gamma : A \rightarrow A \) is defined for \( \gamma \geq 0 \) and \( c > -p \) as

\[ K_{c,p}^\gamma f(z) = \frac{(c+p)^\gamma}{1(\gamma)z^c} \int_0^1 t^c (\log \frac{1}{t})^{\gamma-1} f(tz) dt \] (2.17)

It can be easily verified that for \( f \in A_p \)

\[ K_{c,p}^\gamma f(z) = z^p - \sum_{k=p+1}^{\infty} \left( \frac{c+p}{c+k} \right)^\gamma a_k z^k \] (2.18)
Theorem 2.4. If \( f \in \Omega_p(\alpha, \beta, \lambda) \), then \( K^\gamma_{c,p}f \in \Omega_p(\alpha, \beta, \lambda) \).

Proof. Since \( f \in \Omega_p(\alpha, \beta, \lambda) \) and \( \left\{ \frac{c+p}{c+k} \right\}^\gamma < 1 \), we have

\[
\sum_{k=n+p}^{\infty} \frac{\alpha \beta(1+k-k^2) + k - \beta}{p - \beta + \alpha \beta(1+p-p^2)} \left( \frac{c+p}{c+k} \right)^\gamma B^\lambda_{c,p}(k)a_k < 1
\]

This completes the proof.

3. Radius of starlikeness and convexity

Now we obtain the radii of starlikeness and convexity for the functions \( K^\gamma_{c,p}f \).

Theorem 3.1. The function \( K^\gamma_{c,p}f \) is starlike of order \( \eta \) in

\[
|z| < r_1(\alpha, \beta, \lambda, c, \gamma, \eta), \text{ where }
\]

\[
r_1(\alpha, \beta, \lambda, c, \gamma, \eta) = \inf_k \left[ \frac{1-\eta}{k-\eta-p+1} \frac{\alpha \beta(1+k-k^2) + k - \beta}{p - \beta + \alpha \beta(1+p-p^2)} \left( \frac{c+p}{c+k} \right)^\gamma B^\lambda_{c,p}(k)a_k \right]^{\frac{1}{1-p}}
\]

Proof. We must show that

\[
\left| \frac{z(K^\gamma_{c,p}f(z))'}{K^\gamma_{c,p}f(z)} - p \right| < 1 - \eta.
\]

i.e.

\[
\left| \frac{z(K^\gamma_{c,p}f(z))'}{K^\gamma_{c,p}f(z)} - p \right| \leq \frac{\sum_{k=n+p}^{\infty} \gamma(k - p)a_k |z|^{k-p}}{1 - \sum_{k=n+p}^{\infty} \gamma a_k |z|^{k-p}} < 1 - \eta
\]

or to show that

\[
\sum_{k=n+p}^{\infty} \left( \frac{c+p}{c+k} \right)^\gamma (k - p)a_k |z|^{k-p} + (1 - \eta) \sum_{k=n+p}^{\infty} \left( \frac{c+p}{c+k} \right)^\gamma a_k |z|^{k-p} < 1 - \eta
\]
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(3.4)

Now, in view of (2.1), the theorem holds if

\[
|z|^{k-p} < \frac{1 - \eta}{k - p + 1 - \eta p - \beta + \alpha \beta (1 + p - p^2)} \left( \frac{c + p}{c + k} \right)^\gamma B_p^{\lambda, \mu}(k)
\]

This proves the result.

**Theorem 3.2.** The function \( K_{c,p}^\gamma f \) is convex of order \( \eta \) in

\[
|z| < r_2(\alpha, \beta, \lambda, c, \gamma, \eta), \text{ where}
\]

\[
r_2(\alpha, \beta, \lambda, c, \gamma, \eta) = \inf_k \left[ \frac{p(1+k^2) + k - \beta}{k(1+\alpha \beta (1+p-k^2))} \left( \frac{c+p}{c+k} \right)^\gamma B_p^{\lambda, \mu}(k) \right]^{\frac{1}{k-p}}
\]

(3.6)

Proof. Noting the fact that \( K_{c,p}^\gamma f \) is convex if \( z(K_{c,p}^\gamma f)' \) is starlike. Therefore, we must show that,

\[
\left| \frac{z(K_{c,p}^\gamma f(z))'}{(K_{c,p}^\gamma f(z))'} \right| < 1 - \eta.
\]

i.e.

\[
p(1-p)z^{p-1} - \sum_{k=n+p}^{\infty} \left( \frac{c+p}{c+k} \right)^\gamma (k - p) a_k |z|^{k-1} < 1 - \eta
\]

(3.7)

or to show that

\[
\sum_{k=n+p}^{\infty} \left( \frac{c+p}{c+k} \right)^\gamma [k(1-\eta)] a_k |z|^{k-p} < p(p-\eta)
\]

(3.8)

Now, by (2.1), the last inequality holds if

\[
|z|^{k-p} < \frac{p(p-\eta) \alpha \beta (1+k-k^2) + k - \beta}{k(1+\alpha \beta (1+p-k^2))} \left( \frac{c+p}{c+k} \right)^\gamma B_p^{\lambda, \mu}(k)
\]

(3.9)

This complete the proof.
**Theorem 3.3.** If \( f \in \Omega_p(\alpha, \beta, \lambda) \), then the function \( F^\mu_p(z) \), \( z \in U \) defined by

\[
F^\mu_p(z) = (1 - \mu)z^p + p\mu \int_0^z \frac{f(t)}{t} \, dt, \quad 0 \leq \mu < \frac{2}{p}
\]

is in \( \Omega_p(\alpha, \beta, \lambda) \).

Proof. We have

\[
F^\mu_p(z) = (1 - \mu)z^p + p\mu \left[ \int_0^z t^{p-1} \, dt - \sum_{k=n+p}^{\infty} \int_0^z a_k t^{k-1} \, dt \right] = z^p - \sum_{k=n+p}^{\infty} a_k \frac{p\mu}{k} z^k
\]

(3.12)

Now, by (2.1), we obtain

\[
\sum_{k=n+p}^{\infty} \alpha \beta (1 + k - k^2) + k - \beta \frac{p\mu}{k} B^{\lambda,\mu}_p(k) a_k
\]

\[
\leq \sum_{k=n+p}^{\infty} \frac{\alpha \beta (1 + k - k^2) + k - \beta}{p - \beta + \alpha \beta (1 + p - p^2)} \left( \frac{2}{p} \right) B^{\lambda,\mu}_p(k) a_k < 1
\]

(3.13)

and this proves the theorem.

**Remark 3.4.** \( F^\mu_p(z) \) is starlike of order \( \eta \) in \( |z| < r^1_p(\alpha, \beta, \lambda, \eta, \mu) \), where

\[
r^1_p(\alpha, \beta, \lambda, \eta, \mu) = \inf_k \left[ \frac{k(1 - \eta)}{p\mu(p-k+1-\eta)} \frac{\alpha \beta (1 + k - k^2) + k - \beta}{p - \beta + \alpha \beta (1 + p - p^2)} B^{\lambda,\mu}_p(k) a_k \right]^{\frac{1}{p-1}}.
\]

(3.14)

Also, \( F^\mu_p(z) \) is convex of order \( \eta \) in \( |z| < r^2_p(\alpha, \beta, \lambda, \eta, \mu) \), where

\[
r^2_p(\alpha, \beta, \lambda, \eta, \mu) = \inf_k \left[ \frac{(p - \eta)}{p\mu(k - \eta)} \frac{\alpha \beta (1 + k - k^2) + k - \beta}{p - \beta + \alpha \beta (1 + p - p^2)} B^{\lambda,\mu}_p(k) a_k \right]^{\frac{1}{p-1}}.
\]

(3.15)

The proof of the above remark is made by similar arguments of the Theorems 3.1 and 3.2.
4. Fractional Calculus on $\Omega_p(\alpha, \beta, \lambda)$

In this section, we apply the fractional calculus techniques and discuss the properties of the family $\Omega_p(\alpha, \beta, \lambda)$ (see [10]). In this theorem, we find the distortion bounds for $f(z)$.

**Theorem 4.1.** Let $f \in \Omega_p(\alpha, \beta, \lambda)$, $\lambda \geq 0$. Then

$$|z|^{p+\delta} \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} [1 - M|z|^n] \leq |D_z^{-\delta} f(z)| \leq |z|^{p+\delta} \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} [1 + M|z|^n]$$

(4.1)

where

$$M = \frac{(p+1)n[p - \beta + \alpha \beta(1 + p - p^2)](\nu + \mu - \lambda + 2)n\Gamma(n+1)}{(p+\delta+1)n[p + n + \beta\{\alpha(1 + p + n - (p + n)^2) - 1\}]\Gamma(n+1)\Gamma(n+\nu+2)}$$

and $f(z)$ is analytic function

Proof.

By equation (1.8), we have

$$\frac{\Gamma(\delta + p + 1)}{\Gamma(p+1)} z^{-\delta} D_z^{-\delta} f(z) = z^{p} - \sum_{k=n+p}^{\infty} a_k H_p(k, \delta) z^k$$

(4.2)

where

$$H_p(k, \delta) = \frac{\Gamma(\delta + p + 1)\Gamma(k+1)}{\Gamma(k+\delta+1)\Gamma(p+1)}$$

(4.3)

But $H_p(k, \delta)$ is a decreasing function for $k \geq n + p$ and also $B^{\lambda, \mu}_p(k)$ is increasing function of $k$, thus, we have

$$H_p(k, \delta) \leq \frac{\Gamma(\delta + p + 1)\Gamma(n+p+1)}{\Gamma(n+p+\delta+1)\Gamma(p+1)} = \frac{(p+1)n}{(\delta + p + 1)n}$$

(4.4)

and

$$B^{\lambda, \mu}_p(k) \geq \frac{(\mu + 1)n(\nu + 2)n}{(\mu - \lambda + \nu + 2)n\Gamma(n+1)}$$

(4.5)

So, we conclude that

$$\left| \frac{\Gamma(\delta + p + 1)}{\Gamma(p+1)} z^{-\delta} D_z^{-\delta} f(z) \right|$$

$$\leq |z|^p + \frac{(p+1)n}{(\delta + p + 1)n} |z|^{n+p} \sum_{k=n+p}^{\infty} a_k$$
where $M$ is defined in the theorem statement. Thus, we get

\begin{equation}
|D_z^{-\delta} f(z)| \leq |z|^{p+\delta} \frac{\Gamma(p+1)}{\Gamma(\delta+p+1)} [1 + M|z|^n]
\end{equation}

(4.7)

Also, we have

\begin{equation}
\left| \frac{\Gamma(\delta+p+1)}{\Gamma(p+1)} z^{-\delta} D_z^{-\delta} f(z) \right| \\
\geq |z|^p - \frac{(p+1)\Gamma(p+1)}{\Gamma(p+\delta+1)} |z|^{n+p} \sum_{k=n+p}^{\infty} a_k
\end{equation}

(4.8)

Then

\begin{equation}
|D_z^{-\delta} f(z)| \geq |z|^{p+\delta} \frac{\Gamma(p+1)}{\Gamma(\delta+p+1)} [1 - M|z|^n]
\end{equation}

(4.9)

This completes the proof of the theorem.

**Theorem 4.2.** Let $f \in \Omega_p(\alpha, \beta, \lambda)$, $\lambda \geq 0$. Then

\begin{equation}
|z|^{p-\delta} \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} [1 - N|z|^n] \leq |D_z^\delta f(z)| \leq |z|^{p-\delta} \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} [1 + N|z|^n]
\end{equation}

(4.10)

where

\begin{align*}
N &= \frac{(p+1)_n[p - \beta + \alpha\beta(1+p-p^2)](\nu + \mu - \lambda + 2)n\Gamma(n+1)}{(p-\delta+1)_n[p + n + \beta\{(\alpha(1+p+n-(p+n)^2) - 1)](\mu+1)n(\nu+2)_n}
\end{align*}

and $f(z)$ is analytic function

Proof. By equation (1.6), we have

\begin{equation}
\frac{\Gamma(p-\delta+1)}{\Gamma(p+1)} z^\delta D_z^\delta f(z) = z^p - \sum_{k=n+p}^{\infty} a_k R_p(k, \delta) z^k
\end{equation}

(4.11)
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where

\[ R_p(k, \delta) = \frac{\Gamma(p - \delta + 1)\Gamma(k + 1)}{\Gamma(k - \delta + 1)\Gamma(p + 1)} \]  

But \( R_p(k, \delta) \) is a decreasing function for \( k \geq n + p \) and also \( B_p^{\lambda,\mu}(k) \) is increasing function of \( k \), thus, we have

\[ R_p(k, \delta) \leq \frac{\Gamma(p - \delta + 1)\Gamma(n + p + 1)}{\Gamma(n + p - \delta + 1)\Gamma(p + 1)} = \frac{(p + 1)_n}{(p - \delta + 1)_{n+1}} \]  

and

\[ B_p^{\lambda,\mu}(k) \geq \frac{(\mu + 1)_n(\nu + 2)_n}{(\mu - \lambda + \nu + 2)_{n+1}\Gamma(n + 1)} \]  

So, we conclude that

\[ \left| \frac{\Gamma(p - \delta + 1)}{\Gamma(p + 1)} z^\delta D^\delta_z f(z) \right| \]

\[ \leq |z|^p + \frac{(p+1)_n}{(p-\delta+1)_{n+1}} |z|^{n+p} \sum_{k=n+p}^\infty a_k \]

\[ \leq |z|^p \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} [1 + N|z|^{n+p}] \]  

(4.15)

where \( N \) is defined in the theorem statement. Then, we get

\[ |D^\delta_z f(z)| \leq |z|^{p-\delta} \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} [1 + N|z|^n] \]  

Also, we have

\[ \left| \frac{\Gamma(p - \delta + 1)}{\Gamma(p + 1)} z^\delta D^\delta_z f(z) \right| \]

\[ \geq |z|^p - \frac{(p+1)_n}{(p-\delta+1)_{n+1}} |z|^{n+p} \sum_{k=n+p}^\infty a_k \]

\[ \geq |z|^p \frac{\Gamma(p+1)}{\Gamma(p+\delta+1)} [1 - N|z|^n] \]  

(4.17)

Then

\[ |D^\delta_z f(z)| \geq |z|^{p-\delta} \frac{\Gamma(p+1)}{\Gamma(p-\delta+1)} [1 - N|z|^n] \]  

This completes the proof.

Letting \( \delta = 1 \) in Theorem 4.1, we obtain
Corollary 4.3. Let \( f \in \Omega_p(\alpha, \beta, \lambda), \lambda \geq 0 \). Then

\[
|z|^{p+1} \left[ 1 - M |z|^n \right] \leq \left| \int_0^z f(t) \, dt \right| \leq |z|^{p+1} \left[ 1 + M |z|^n \right]
\]

where

\[
M = \frac{(p + 1)^p[p - \beta + \alpha \beta(1 + p - p^2)](\nu + \mu - \lambda + 2)\Gamma(n + 1)}{(p + n + 1)^n[p + n + \beta(\alpha(1 + p + n - (p + n)^2) - 1)](\mu + 1)n(\nu + 2)}
\]

and \( f(z) \) is analytic function

Letting \( \delta = 0 \) in Theorem 4.2, we obtain

Corollary 4.4. Let \( f \in \Omega_p(\alpha, \beta, \lambda), \lambda \geq 0 \). Then

\[
|z|^{p} \left[ 1 - N |z|^n \right] \leq |f(z)| \leq |z|^{p} \left[ 1 + N |z|^n \right]
\]

where

\[
N = \frac{[p - \beta + \alpha \beta(1 + p - p^2)](\nu + \mu - \lambda + 2)\Gamma(n + 1)}{[p + n + \beta(\alpha(1 + p + n - (p + n)^2) - 1)](\mu + 1)n(\nu + 2)}
\]

and \( f(z) \) is analytic function

References


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