On Zweier I-convergent sequence spaces

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Abstract

In this article we introduce the Zweier I-convergent sequence spaces $Z^I$, $Z^0_I$, and $Z^\infty_I$. We prove the decomposition theorem and study topological, algebraic properties and have established some inclusion relations of these spaces.

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1. Introduction

Let \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \) be the sets of all natural, real and complex numbers respectively. We write

\[
\omega = \{ x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C} \},
\]

the space of all real or complex sequences.

Let \( \ell_\infty, c \) and \( c_0 \) denote the Banach spaces of bounded, convergent and null sequences respectively normed by

\[
||x||_\infty = \sup_k |x_k|.
\]

A sequence space \( \lambda \) with linear topology is called a K-space provided each of maps \( p_i \to \mathbb{C} \) defined by \( p_i(x) = x_i \) is continuous for all \( i \in \mathbb{N} \).

A K-space \( \lambda \) is called an FK-space provided \( \lambda \) is a complete linear metric space.

An FK-space whose topology is normable is called a BK-space.

Let \( \lambda \) and \( \mu \) be two sequence spaces and \( A = (a_{nk}) \) be an infinite matrix of real or complex numbers \( (a_{nk}) \), where \( n, k \in \mathbb{N} \). Then we say that \( A \) defines a matrix mapping from \( \lambda \) to \( \mu \), and we denote it by writing \( A : \lambda \to \mu \).

If for every sequence \( x = (x_k) \in \lambda \) the sequence \( Ax = \{(Ax)_n\} \), the \( A \) transform of \( x \) is in \( \mu \), where

\[
(Ax)_n = \sum_k a_{nk}x_k, \quad (n \in \mathbb{N}).
\]

By \( (\lambda : \mu) \), we denote the class of matrices \( A \) such that \( A : \lambda \to \mu \). Thus, \( A \in (\lambda : \mu) \) if and only if series on the right side of (1) converges for each \( n \in \mathbb{N} \) and every \( x \in \lambda \).

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have recently been employed by Altay, Başar and Mursaleen [1], Başar and Altay [2], Malkowsky [13], Ng and Lee [14], and Wang [21].
Şengönül [18] defined the sequence \( y = (y_i) \) which is frequently used as the \( Z^p \) transform of the sequence \( x = (x_i) \) i.e.,

\[
y_i = px_i + (1 - p)x_{i-1}
\]

where \( x_{-1} = 0 \), \( 1 < p < \infty \) and \( Z^p \) denotes the matrix \( Z^p = (z_{ik}) \) defined by

\[
z_{ik} = \begin{cases} 
p, & (i = k), 
1 - p, & (i - 1 = k); (i, k \in \mathbb{N}), 
0, & \text{otherwise.}
\end{cases}
\]

Following Başar and Altay [2], Şengönül [18] introduced the Zweier sequence spaces \( Z \) and \( Z_0 \) as follows:

\[
Z = \{ x = (x_k) \in \omega : Z^p x \in c \}
\]

\[
Z_0 = \{ x = (x_k) \in \omega : Z^p x \in c_0 \}.
\]

Here we list below some of the results of Şengönül [18] which we will need as a reference in order to establish analogously some of the results of this article.

**Theorem 1.1.** The sets \( Z \) and \( Z_0 \) are linear spaces with the co-ordinate wise addition and scalar multiplication which are the BK-spaces with the norm \( ||x||_Z = ||x||_{Z_0} = ||Z^p x||_c \) [See (Theorem 2.1. [18])].

**Theorem 1.2.** The sequence spaces \( Z \) and \( Z_0 \) are linearly isomorphic to the spaces \( c \) and \( c_0 \) respectively, i.e. \( Z \cong c \) and \( Z_0 \cong c_0 \) [See (Theorem 2.2. [18])].

**Theorem 1.3.** The inclusions \( Z_0 \subset Z \) strictly hold for \( p \neq 1 \). [See (Theorem 2.3. [18])].

**Theorem 1.4.** \( Z_0 \) is solid. [See (Theorem 2.6. [18])].

**Theorem 1.5.** \( Z \) is not a solid sequence space. [See (Theorem 3.6. [18])].

The concept of statistical convergence was first introduced by Fast [7] and also independently by Buck [3] and Schoenberg [17] for real and complex sequences. Further this concept was studied by Connor [4, 5], Connor,
Fridy and Kline [6] and many others. Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. A sequence \( x = (x_k) \) is said to be statistically convergent to \( L \) if for a given \( \varepsilon > 0 \)

\[
\lim_{k} \frac{1}{k} \left| \{ i : |x_i - L| \geq \varepsilon, i \leq k \} \right| = 0.
\]

The notion of I-convergence generalizes and unifies different notions of convergence including the notion of statistical convergence. At the initial stage it was studied by Kostyrko, Šalát, Wilczyński [12]. Later on it was studied by Šalát, Tripathy, Ziman [15, 16]. Recently further it was studied by Tripathy [19, 20, 21, 22, 23, 24, 25, 26, 27], and V. A.Khan and Khalid Ebadullah [9-11].

Here we give some preliminaries about the notion of I-convergence.

Let \( X \) be a non empty set. Then a family of sets \( I \subseteq 2^X \) (denoting the power set of \( X \)) is said to be an ideal if \( I \) is additive i.e \( A,B \in I \Rightarrow A \cup B \in I \) and hereditary i.e \( A \in I, B \subseteq A \Rightarrow B \in I \).

A non-empty family of sets \( \mathcal{L}(I) \subseteq 2^X \) is said to be filter on \( X \) if and only if \( \emptyset \notin \mathcal{L}(I) \), for \( A, B \in \mathcal{L}(I) \) we have \( A \cap B \in \mathcal{L}(I) \) and for each \( A \in \mathcal{L}(I) \) and \( A \subseteq B \) implies \( B \in \mathcal{L}(I) \).

An Ideal \( I \subseteq 2^X \) is called non-trivial if \( I \neq 2^X \).
A non-trivial ideal \( I \subseteq 2^X \) is called admissible if \( \{ \{ x \} : x \in X \} \subseteq I \). A non-trivial ideal \( I \) is maximal if there cannot exist any non-trivial ideal \( J \neq I \) containing \( I \) as a subset.

For each ideal \( I \), there is a filter \( \mathcal{L}(I) \) corresponding to \( I \) i.e

\[
\mathcal{L}(I) = \{ K \subseteq N : K^c \in I \}, \hspace{1cm} \text{where} \hspace{1cm} K^c = N - K.
\]

**Definition 1.6.** A sequence \( (x_k) \in \omega \) is said to be I-convergent to a number \( L \) if for every \( \varepsilon > 0 \)

\[
\{ k \in N : |x_k - L| \geq \varepsilon \} \in I.
\]

In this case we write \( I - \lim x_k = L \). The space \( c^I \) of all I-convergent sequences to \( L \) is given by

\[
c^I = \{ (x_k) \in \omega : \{ k \in N : |x_k - L| \geq \varepsilon \} \in I, \text{ for some } L \in C \}.
\]
Definition 1.7. A sequence \((x_k)\) is said to be I-null if \(L = 0\). In this case we write \(I \lim x_k = 0\).

Definition 1.8. A sequence \((x_k)\) is said to be I-Cauchy if for every \(\varepsilon > 0\) there exists a number \(m = m(\varepsilon)\) such that
\[
\{k \in N : |x_k - x_m| \geq \varepsilon\} \in I.
\]

Definition 1.9. A sequence \((x_k)\) is said to be I-bounded if there exists \(M > 0\) such that
\[
\{k \in N : |x_k| > M\} \in I.
\]

Example 1.10. Take for I the class \(I_f\) of all finite subsets of \(N\). Then \(I_f\) is a non-trivial admissible ideal and \(I_f\) convergence coincides with the usual convergence with respect to the metric in \(X\). (see [12]).

Definition 1.11. For \(I = I_\delta\) and \(A \subset N\) with \(\delta(A) = 0\) respectively. \(I_\delta\) is a non-trivial admissible ideal, \(I_\delta\)-convergence is said to be logarithmic statistical convergence(see[12]).

Definition 1.12. A map \(h\) defined on a domain \(D \subset X\) i.e \(h : D \subset X \rightarrow R\) is said to satisfy Lipschitz condition if
\[
|h(x) - h(y)| \leq K|x - y|,
\]
where \(K\) is known as the Lipschitz constant. The class of \(K\)-Lipschitz functions defined on \(D\) is denoted by \(h \in (D, K)(\text{see}[15,16])\).

Definition 1.13. A convergence field of I-covergence is a set
\[
F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I \lim x \in R\}.
\]

The convergence field \(F(I)\) is a closed linear subspace of \(l_\infty\) with respect to the supremum norm, \(F(I) = l_\infty \cap c^I\) (See [15,16]).

Define a function \(h : F(I) \rightarrow R\) such that \(h(x) = I \lim x\), for all \(x \in F(I)\), then the function \(h : F(I) \rightarrow R\) is a Lipschitz function. (see [15, 16]).

Definition 1.14. Let \((x_k), (y_k)\) be two sequences. We say that \((x_k) = (y_k)\) for almost all \(k\) relative to \(I\) (a.a.k.r.I), if
\[
\{k \in N : x_k \neq y_k\} \in I(\text{see}[19,20]).
\]
The following Lemmas will be used for establishing some results of this article:

Lemma 1.15. Let $E$ be a sequence space. If $E$ is solid then $E$ is monotone. (see [8], page 53).

Lemma 1.16. If $I \subset 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$. (see [19, 20]).

2. Main Results

In this section we introduce the following classes of sequence spaces:

\[ Z^I = \{ x = (x_k) \in \omega : \{ k \in N : I - \lim Z^p x = L \}, \text{ for some } L \in C \} \]

\[ Z^I_0 = \{ x = (x_k) \in \omega : \{ k \in N : I - \lim Z^p x = 0 \} \} \]

\[ Z^I_\infty = \{ x = (x_k) \in \omega : \{ k \in N : \sup_k |Z^p x| < \infty \} \} \]

We also denote by

\[ m^I_Z = Z_\infty \cap Z^I \]

and

\[ m^I_{Z_0} = Z_\infty \cap Z^I_0. \]

Throughout the article, for the sake of convenience now we will denote by

\[ Z^p(x_k) = x', Z^p(y_k) = y', Z^p(z_k) = z' \text{ for } x, y, z \in \omega. \]

Theorem 2.1. The classes of sequences $Z^I, Z^I_0, m^I_Z$ and $m^I_{Z_0}$ are linear spaces.

Proof. We shall prove the result for the space $Z^I$.

The proof for the other spaces will follow similarly.

Let $(x_k), (y_k) \in Z^I$ and let $\alpha, \beta$ be scalars. Then

\[ I - \lim |x'_k - L_1| = 0, \text{ for some } L_1 \in C; \]
\[ I - \lim \|y'_k - L_2\| = 0, \text{ for some } L_2 \in C; \]

That is for a given \( \epsilon > 0 \), we have
\[
A_1 = \{ k \in N : |x'_k - L_1| > \frac{\epsilon}{2} \} \in I,
\]
\[
A_2 = \{ k \in N : |y'_k - L_2| > \frac{\epsilon}{2} \} \in I.
\]
(2.1)

we have
\[
|(\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2)| \leq |\alpha||x'_k - L_1| + |\beta||y'_k - L_2|
\]
\[
\leq |x'_k - L_1| + |y'_k - L_2|
\]

Now, by (1) and (2), \{ \{ k \in N : |(\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2)| > \epsilon \} \} \subset A_1 \cup A_2.

Therefore \((\alpha x_k + \beta y_k) \in Z^I\)

Hence \(Z^I\) is a linear space.

**Theorem 2.2.** The spaces \(m^I_Z\) and \(m^I_{Z_0}\) are normed linear spaces, normed by
\[
\|x'_k\|_* = \sup_k |Z^p(x)|,
\]
(2.2)

where \(x'_k = Z^p(x)\).

**Proof:** It is clear from Theorem 2.1 that \(m^I_Z\) and \(m^I_{Z_0}\) are linear spaces.

It is easy to verify that (3) defines a norm on the spaces \(m^I_Z\) and \(m^I_{Z_0}\).

**Theorem 2.3.** A sequence \(x = (x_k) \in m^I_Z\) I-converges if and only if for every \( \epsilon > 0 \) there exists \( N_\epsilon \in N \) such that
\[
\{ k \in N : |x'_k - x'_{N_\epsilon}| < \epsilon \} \in m^I_Z
\]
(2.3)

**Proof.** Suppose that \( L = I - \lim x' \). Then
\[
B_\epsilon = \{ k \in N : |x'_k - L| < \frac{\epsilon}{2} \} \in m^I_Z \text{ for all } \epsilon > 0
\]
Fix an $N_\varepsilon \in B_\varepsilon$. Then we have
\[ |x^{'}_{N_\varepsilon} - x^{'}_k| \leq |x^{'}_{N_\varepsilon} - L| + |L - x^{'}_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]
which holds for all $k \in B_\varepsilon$.

Hence $\{ k \in \mathbb{N} : |x^{'}_k - x^{'}_{N_\varepsilon}| < \varepsilon \} \in m^I_\mathbb{Z}$.

Conversely, suppose that $\{ k \in \mathbb{N} : |x^{'}_k - x^{'}_{N_\varepsilon}| < \varepsilon \} \in m^I_\mathbb{Z}$.
That is $\{ k \in \mathbb{N} : |x^{'}_k - x^{'}_{N_\varepsilon}| < \varepsilon \} \in m^I_\mathbb{Z}$ for all $\varepsilon > 0$. Then the set
\[ C_\varepsilon = \{ k \in \mathbb{N} : x^{'}_k \in [x^{'}_{N_\varepsilon} - \varepsilon, x^{'}_{N_\varepsilon} + \varepsilon] \} \in m^I_\mathbb{Z} \] for all $\varepsilon > 0$.

Let $J_\varepsilon = [x^{'}_{N_\varepsilon} - \varepsilon, x^{'}_{N_\varepsilon} + \varepsilon]$. If we fix an $\varepsilon > 0$ then we have $C_\varepsilon \in m^I_\mathbb{Z}$ as well as $C_{\varepsilon/2} \in m^I_\mathbb{Z}$. Hence $C_\varepsilon \cap C_{\varepsilon/2} \in m^I_\mathbb{Z}$. This implies that
\[ J = J_\varepsilon \cap J_{\varepsilon/2} \neq \emptyset \]
that is
\[ \{ k \in \mathbb{N} : x^{'}_k \in J \} \in m^I_\mathbb{Z} \]
that is
\[ \text{diam} J \leq \text{diam} J_\varepsilon \]
where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals
\[ J_\varepsilon = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_k \supseteq \ldots \]
with the property that $\text{diam} I_k \leq \frac{1}{2} \text{diam} I_{k-1}$ for (k=2,3,4,...) and
\[ \{ k \in \mathbb{N} : x^{'}_k \in I_k \} \in m^I_\mathbb{Z} \] for (k=1,2,3,4,...).

Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi^{'} = I - \lim x^{'}$, that is $L = I - \lim x^{'}$.

**Theorem 2.4.** Let $I$ be an admissible ideal. Then the following are equivalent.

(a) $(x_k) \in \mathcal{Z}^I$;

(b) there exists $(y_k) \in \mathcal{Z}$ such that $x_k = y_k$, for a.a.k.r. $I$;
(c) there exists \((y_k) \in Z\) and \((z_k) \in Z_0^I\) such that \(x_k = y_k + z_k\) for all \(k \in \mathbb{N}\) and \(\{k \in \mathbb{N} : |y_k - L| \geq \epsilon\} \in I\);

(d) there exists a subset \(K = \{k_1 < k_2 \ldots\}\) of \(\mathbb{N}\) such that \(K \in \mathcal{L}(I)\) and \(\lim_{n \to \infty} |x_{k_n} - L| = 0\).

**Proof.** (a) implies (b). Let \((x_k) \in Z^I\). Then there exists \(L \in C\) such that

\[\{k \in \mathbb{N} : |x_k' - L| \geq \epsilon\} \in I\]

Let \((m_t)\) be an increasing sequence with \(m_t \in \mathbb{N}\) such that

\[\{k \leq m_t : |x_k' - L| \geq \frac{1}{t}\} \in I\].

Define a sequence \((y_k)\) by

\[y_k = x_k, \text{ for all } k \leq m_1.\]

For \(m_t < k \leq m_{t+1}, t \in \mathbb{N}\),

\[y_k = \begin{cases} x_k, & \text{if } |x_k' - L| < \epsilon^{-1}, \\ L, & \text{otherwise.} \end{cases}\]

Then \((y_k) \in Z\) and form the following inclusion

\[\{k \leq m_t : x_k \neq y_k\} \subseteq \{k \leq m_t : |x_k' - L| \geq \epsilon\} \in I.\]

We get \(x_k = y_k\), for a.a.k.r.I.

(b) implies (c). For \((x_k) \in Z^I\).

Then there exists \((y_k) \in Z\) such that \(x_k = y_k\), for a.a.k.r.I.

Let \(K = \{k \in \mathbb{N} : x_k \neq y_k\}\), then \(K \in I\).

Define a sequence \((z_k)\) by

\[z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}\]

Then \(z_k \in Z_0^I\) and \(y_k \in Z\).
(c) implies (d). Let \( P_1 = \{ k \in \mathbb{N} : |z_k| \geq \varepsilon \} \in \mathcal{L}(I) \) and \( K = P_1^c = \{ k_1 < k_2 < k_3 < \ldots \} \in \mathcal{L}(I) \). Then we have \( \lim_{n \to \infty} |x_{k_n} - L| = 0 \).

(d) implies (a). Let \( K = \{ k_1 < k_2 < k_3 < \ldots \} \in \mathcal{L}(I) \) and \( \lim_{n \to \infty} |x_{k_n} - L| = 0 \).

Then for any \( \varepsilon > 0 \), and by Lemma, we have
\[
\{ k \in \mathbb{N} : |x'_k - L| \geq \varepsilon \} \subseteq K^c \cup \{ k \in K : |x'_k - L| \geq \varepsilon \}.
\]

Thus \( (x_k) \in \mathcal{Z}^I \).

**Theorem 2.5.** The inclusions \( \mathcal{Z}^I_0 \subset \mathcal{Z}^I \subset \mathcal{Z}^I_\infty \) are proper.

**Proof:** Let \( (x_k) \in \mathcal{Z}^I \). Then there exists \( L \in \mathbb{C} \) such that
\[
I - \lim |x'_k - L| = 0
\]
We have
\[
|x'_k| \leq \frac{1}{2}|x'_k - L| + \frac{1}{2}|L|
\]
Taking the supremum over \( k \) on both sides we get \( (x_k) \in \mathcal{Z}^I_\infty \).

The inclusion \( \mathcal{Z}^I_0 \subset \mathcal{Z}^I \) is obvious.

**Theorem 2.6.** The function \( \bar{h} : m^I_Z \to \mathbb{R} \) is the Lipschitz function, where \( m^I_Z = \mathcal{Z}^I \cap \mathcal{Z}_\infty \), and hence uniformly continuous.

**Proof:** Let \( x, y \in m^I_Z, x \neq y \). Then the sets
\[
A_x = \{ k \in \mathbb{N} : |x'_k - \bar{h}(x')| \geq ||x' - y'||_s \} \in I,
A_y = \{ k \in \mathbb{N} : |y'_k - \bar{h}(y')| \geq ||x' - y'||_s \} \in I.
\]
Thus the sets,
\[
B_x = \{ k \in \mathbb{N} : |x'_k - \bar{h}(x')| < ||x' - y'||_s \} \in m^I_Z,
B_y = \{ k \in \mathbb{N} : |y'_k - \bar{h}(y')| < ||x' - y'||_s \} \in m^I_Z.
\]
Hence also \( B = B_x \cap B_y \in m^I_Z \), so that \( B \neq \phi \).
Now taking $k$ in $B$,

$$|\bar{h}(x') - h(y')| \leq |\bar{h}(x') - x'| + |x' - y'| + |y' - h(y')| \leq 3|x' - y'|_\star.$$  

Thus $\bar{h}$ is a Lipschitz function.

For $m^I_{Z_0}$ the result can be proved similarly.

**Theorem 2.7.** If $x,y \in m^I_Z$, then $(x.y) \in m^I_Z$ and $\bar{h}(x.y) = \bar{h}(x)\bar{h}(y)$.

**Proof:** For $\varepsilon > 0$

$$B_x = \{k \in N : |x' - \bar{h}(x')| < \varepsilon\} \in m^I_Z,$$

$$B_y = \{k \in N : |y' - \bar{h}(y')| < \varepsilon\} \in m^I_Z.$$

Now,

$$|x'.y' - \bar{h}(x')h(y')| = |x'.y' - x'\bar{h}(y') + x'\bar{h}(y') - \bar{h}(x')\bar{h}(y')|$$

$$\leq |x'| |y' - \bar{h}(y')| + |\bar{h}(y')| |x' - \bar{h}(x')|$$  

(2.4)

As $m^I_Z \subseteq Z_\infty$, there exists an $M \in R$ such that $|x'| < M$ and $|\bar{h}(y')| < M$.

Using eqn(5) we get

$$|x'.y' - \bar{h}(x')h(y')| \leq M\varepsilon + M\varepsilon = 2M\varepsilon$$

For all $k \in B_x \cap B_y \in m^I_Z$.

Hence $(x,y) \in m^I_Z$ and $\bar{h}(x.y) = \bar{h}(x)\bar{h}(y)$.

For $m^I_{Z_0}$ the result can be proved similarly.

**Theorem 2.8.** The spaces $Z^I_0$ and $m^I_{Z_0}$ are solid and monotone.

**Proof:** We prove the result for the case $Z^I_0$.

Let $(x_k) \in Z^I_0$. Then

$$I - \lim_k |x'_k| = 0$$  

(2.5)
Let \((\alpha_k)\) be a sequence of scalars with \(|\alpha_k| \leq 1\) for all \(k \in \mathbb{N}\). Then the result follows from (6) and the following inequality

\[|\alpha_k x_k' \leq |\alpha_k||x_k' \leq |x_k'|\text{ for all } k \in \mathbb{N}.
\]

That the space \(Z^I_0\) is monotone follows from the Lemma 1.15.

For \(m^I_{Z_0}\) the result can be proved similarly.

**Theorem 2.9.** The spaces \(Z^I\) and \(m^I_Z\) are neither monotone nor solid, if \(I\) is neither maximal nor \(I = I_f\) in general.

**Proof:** Here we give a counter example.

Let \(I = I_\delta\). Consider the K-step space \(X_K\) of \(X\) defined as follows,

Let \((x_k) \in X\) and let \((y_k) \in X_K\) be such that

\[(y_k)' = \begin{cases} (x_k'), & \text{if } k \text{ is odd,} \\ 1, & \text{otherwise.} \end{cases}
\]

Consider the sequence \((x_k')\) defined by \((x_k') = k^{-1}\) for all \(k \in \mathbb{N}\).

Then \((x_k) \in Z^I\) but its K-step space preimage does not belong to \(Z^I\). Thus \(Z^I\) is not monotone. Hence \(Z^I\) is not solid.

**Theorem 2.10.** The spaces \(Z^I\) and \(Z^I_0\) are sequence algebras.

**Proof:** We prove that \(Z^I_0\) is a sequence algebra.

Let \((x_k), (y_k) \in Z^I_0\. Then

\[I - \lim |x_k'| = 0
\]

and

\[I - \lim |y_k'| = 0
\]

Then we have

\[I - \lim |(x_k', y_k')| = 0
\]

Thus \((x_k, y_k) \in Z^I_0\)
Hence $\mathcal{Z}_0^I$ is a sequence algebra.

For the space $\mathcal{Z}^I$, the result can be proved similarly.

**Theorem 2.11.** The spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not convergence free in general.

**Proof:** Here we give a counter example.

Let $I = I_f$. Consider the sequence $(x_k^I)$ and $(y_k^I)$ defined by

$$x_k^I = \frac{1}{k} \quad \text{and} \quad y_k^I = k \quad \text{for all } k \in \mathbb{N}$$

Then $(x_k) \in \mathcal{Z}^I$ and $\mathcal{Z}_0^I$, but $(y_k) \notin \mathcal{Z}^I$ and $\mathcal{Z}_0^I$.

Hence the spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not convergence free.

**Theorem 2.12.** If $I$ is not maximal and $I \neq I_f$, then the spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not symmetric.

**Proof:** Let $A \in I$ be infinite.

If

$$x_k^I = \begin{cases} 1, & \text{for } k \in A, \\ 0, & \text{otherwise}. \end{cases}$$

Then by lemma 1.16. $x_k \in \mathcal{Z}_0^I \subset \mathcal{Z}^I$. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & \text{otherwise}. \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(k)}^I \notin \mathcal{Z}^I$ and $x_{\pi(k)} \notin \mathcal{Z}_0^I$.

Hence $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not symmetric.

**Theorem 2.13.** The sequence spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are linearly isomorphic to the spaces $c^I$ and $c_0^I$ respectively, i.e $\mathcal{Z}^I \cong c^I$ and $\mathcal{Z}_0^I \cong c_0^I$. 
Proof. We shall prove the result for the space $\mathcal{Z}$ and $c^I$.

The proof for the other spaces will follow similarly.

We need to show that there exists a linear bijection between the spaces $\mathcal{Z}$ and $c^I$. Define a map $T : \mathcal{Z} \rightarrow c^I$ such that $x \rightarrow x' = Tx$

$$T(x_k) = px_k + (1 - p)x_{k-1} = x'_k$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$.

Clearly $T$ is linear.

Further, it is trivial that $x = 0 = (0, 0, 0, \ldots)$, whenever $Tx = 0$ and hence injective.

Let $x'_k \in c^I$ and define the sequence $x = x_k$ by

$$x_k = M \sum_{i=0}^{k} (-1)^{k-i}N^{k-i}x'_i \quad (i \in \mathbb{N}),$$

where $M = \frac{1}{p}$ and $N = \frac{1-p}{p}$.

Then we have

$$\lim_{k \rightarrow \infty} px_k + (1 - p)x_{k-1}$$

$$= p \lim_{k \rightarrow \infty} M \sum_{i=0}^{k} (-1)^{k-i}N^{k-i}x'_i + (1 - p) \lim_{k \rightarrow \infty} M \sum_{i=0}^{k-1} (-1)^{k-i}N^{k-i}x'_i$$

$$= \lim_{k \rightarrow \infty} x'_k$$

which shows that $x \in \mathcal{Z}$.

Hence $T$ is a linear bijection.

Also we have $||x||_* = ||Z^px||_c$.

Therefore,

$$||x||_* = \sup_{k \in \mathbb{N}} |px_k + (1 - p)x_{k-1}|,$$
\[ \sup_{k \in \mathbb{N}} \left| pM \sum_{i=0}^{k} (-1)^{k-i} N^{k-i} x_i' + (1 - p)M \sum_{i=0}^{k-1} (-1)^{k-i} N^{k-i} x_i' \right| \]
\[ = \sup_{k \in \mathbb{N}} \left| x_k' \right| = \left\| x' \right\|_{c'} \]

Hence \( Z^I \cong c' \).

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References


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