On some \( I \)-convergent generalized difference sequence spaces associated with multiplier sequence defined by a sequence of moduli

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Abstract

In this article we introduce the sequence spaces \( c^I(F, \Lambda, \Delta_m, p) \), \( c_0^I(F, \Lambda, \Delta_m, p) \) and \( \ell_\infty^I(F, \Lambda, \Delta_m, p) \), associated with the multiplier sequence \( \Lambda = (\lambda_k) \), defined by a sequence of moduli \( F = (f_k) \). We study some basic topological and algebraic properties of these spaces. Also some inclusion relations are studied.

Key words : Ideal, \( I \)-convergence, modulus function, difference sequence.

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1. Introduction and Preliminaries

The notion of $I$-convergence generalizes and unifies several notions of convergence for sequence spaces. The notion of $I$-convergence was studied at the initial stage by Kostyrko, Salat and Wilczynski [23]. Later on it was studied by Tripathy et al. [5-9, 15], B.Sarma [16], Debnath et al. [25, 26], Khan et al. [28] and many others. They used the notion of ideal $I$ of subsets of the set $N$ of natural numbers to define those concepts.

Let $X$ be a non-empty set. Then a family of subsets $I \subset 2^X$ is said to be an ideal if $I$ is additive, i.e., $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e. $A \in I$, $B \subset A \Rightarrow B \in I$. A non-empty family of subsets $F \subset 2^X$ is said to be a filter on $X$ iff

i) $\emptyset \notin F$
ii) for all $A, B \in F \Rightarrow A \cap B \in F$
iii) $A \in F$, $A \subset B \Rightarrow B \in F$.

An ideal $I \subset 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal $I$ is called admissible iff $I \supset \{\{x\} : x \in X\}$. A non-trivial ideal $I$ is maximal if there does not exist any non-trivial ideal $J \neq I$, containing $I$ as a subset. For each ideal $I$ there is a filter $F(I)$ corresponding to $I$ i.e $F(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N - K$.

A sequence $x = (x_n)$ is said to be $I$-convergent to a number $L \in R$ if for each $\varepsilon > 0$, $A(\varepsilon) = \{n \in N : |x_n - L| \geq \varepsilon\} \in I$. The element $L$ is called the $I$-limit of the sequence $x = (x_n)$.

The natural density of a subset $A$ of $N$ is denoted by $d(A)$ and is defined by

$$d(A) = \lim_{n \to \infty} \frac{1}{n} \{k < n : k \in A\}$$

**Example:** Let $I_f = I_f = \{A \subseteq N : A$ is finite$\}$. Then $I_f$ is a nontrivial admissible ideal of $N$ and the corresponding convergence coincides with ordinary convergence. If $I = I_d = \{A \subseteq N : d(A) = 0\}$, where $d(A)$ denotes the asymptotic density of the set $A$, then $I_d$ is a non-trivial admissible ideal of $N$ and the corresponding convergence coincide with statistical convergence.

The scope for the studies on sequence spaces was extended on introducing the notion of an associated multiplier sequence. S. Goes and G. Goes [18] defined the differentiated sequence space $dE$ and the integrated sequence space $fE$, for a sequence space $E$, by using the multiplier sequence $(k^{-1})$.
and (k), respectively. We shall use a general multiplier sequence \( \Lambda = (\lambda_k) \) for our study.

Throughout the article \( w, c, c_0, \ell_{\infty} \) denote the spaces of all, convergent, null, bounded sequences respectively.

The notion of difference sequences was introduced by H.Kizmaz [19] and it was further generalized as follows:

\[
Z(\Delta_m) = \{ (x_k) \in w : (\Delta_m x_k) \in Z \}
\]

for \( Z = c, c_0, \ell_{\infty} \), where \( \Delta_m x_k = x_k - x_{k+m} \), for all \( k \in N \).

Throughout the article, \( p = (p_k) \) denotes the sequence of positive real numbers. The notion of paranormed sequences was studied and investigated by Tripathy et.al.[10,12] and many others.

The notion of modulus function was introduced by Nakano [17]. It was further investigated with applications to sequences by Tripathy and Chandra [12], Khan et. al[28] and many others.

The following well-known inequality will be used throughout the article.

Let \( p = (p_k) \) be any sequence of positive real numbers with \( 0 < p_k \leq \sup p_k = G \) and \( D = \max\{1, 2G^{-1}\} \). Then \( (|a_k + b_k|)^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}) \) for all \( k \in N \) and \( a_k, b_k \in C \).

**Definition 1.1:** A modulus function \( f \) is a mapping from \([0, \infty)\) into \([0, \infty)\) such that

(i) \( f(x) = 0 \) if and only if \( x = 0 \)

(ii) \( f(x + y) \leq f(x) + f(y) \)

(iii) \( f \) is increasing

(iv) \( f \) is continuous from the right at 0

Hence \( f \) is continuous everywhere in \([0, \infty)\)

Let \( X \) be a sequence space. Then the sequence space \( X(f) \) is defined as

\[
X(f) = \{ x = (x_k) : f(x_k) \in X \},
\]
for a modulus function.

**Definition 1.2:** A sequence space \( E \) is said to be solid (or normal) if \((y_k) \in E\) whenever \((x_k) \in E\) and \(|y_k| \leq |x_k|\) for all \(k \in \mathbb{N}\).

**Definition 1.3:** A sequence space \( E \) is said to be monotone if it contains the canonical preimages of all its step spaces.

**Lemma 1.1:** A sequence space \( E \) is normal implies that it is monotone.

**Definition 1.4:** A sequence space \( E \) is said to be symmetric if \((x_{\pi(n)}) \in E\), whenever \((x_n) \in E\), where \(\pi\) is a permutation of \(\mathbb{N}\).

**Definition 1.5:** A sequence space \( E \) is said to be convergence free if \((y_n) \in E\), whenever \((x_n) \in E\) and \(x_n = 0\) implies \(y_n = 0\).

### 2. Main result

**Definition 2.1:** Let \( F = (f_k) \) be a sequence of moduli, then for a given multiplier sequence \( \Lambda = (\lambda_k) \), we introduce the following sequence spaces:

\[
  c^I(F, \Lambda, \Delta_m, p) = \{ (x_k) \in w : \{ n \in N : (f_k(\lambda_k(\Delta_m x_k - L)))^{p_k} \geq \varepsilon \} \in I, \text{ for some } L \in \mathbb{R} \}
\]

\[
  c_0^I(F, \Lambda, \Delta_m, p) = \{ (x_k) \in w : \{ n \in N : (f_k(\lambda_k(\Delta_m x_k)))^{p_k} \geq \varepsilon \} \in I \}
\]

\[
  c_\infty^I(F, \Lambda, \Delta_m, p) = \{ (x_k) \in w : \text{there exist } M > 0 \text{ such that } \{ n \in N : (f_k(\lambda_k(\Delta_m x_k - L)))^{p_k} \geq M \} \in I \}
\]

When \( f_k(x) = f(x) \), for all \(k \in N\), then the above spaces are denoted by \( c^I(f, \Lambda, \Delta_m, p), c_0^I(f, \Lambda, \Delta_m, p), c_\infty^I(f, \Lambda, \Delta_m, p) \).

When \( I = I_f \) and \( f_k(x) = f(x) \), for all \(k \in N\), then the above spaces become \( c(f, \Lambda, \Delta_m, p), c_0(f, \Lambda, \Delta_m, p), c_\infty(f, \Lambda, \Delta_m, p) \), which was studied by Tripathy and Chandra [12].
When \( \lambda_k = p_k = 1 \), for all \( k \in N \) and \( m = 1 \), then the above spaces are denoted by \( c^I(F, \Delta), c_0^I(F, \Delta), \ell_\infty^I(F, \Delta) \), studied by Khan et al[28].

When \( I = I_f \), \( \lambda_k = p_k = 1 \), for all \( k \in N \) and \( f_k(x) = x \), for all \( k \in N \) then the above spaces are denoted by \( c(\Delta_m), c_0(\Delta_m), \ell_\infty(\Delta_m) \), studied by Tripathy et.al.

When \( I = I_f \), \( \lambda_k = p_k = 1 \), for all \( k \in N \) and \( f_k(x) = x \), for all \( k \in N \) and \( m=1 \), then the above spaces reduce to \( c(\Delta), c_0(\Delta), \ell_\infty(\Delta) \), studied by Kizmaz [19].

**Theorem 2.2:** The classes of sequences \( c^I(F, \Lambda, \Delta_m, p), c_0^I(F, \Lambda, \Delta_m, p) \), and \( \ell_\infty^I(F, \Lambda, \Delta_m, p) \) are linear spaces.

**Proof:** We prove the theorem for the class of sequences \( c_0^I(F, \Lambda, \Delta_m, p) \). The other cases can be proved similarly.

Let \( (x_k), (y_k) \in c_0^I(F, \Lambda, \Delta_m, p) \), then

\[
A = \{ k \in N : (f_k(\lambda_k(\Delta_m x_k)))^{p_k} \geq \frac{e}{2D(\|\alpha\|+1)} \} \subset I
\]

and \( B = \{ k \in N : (f_k(\lambda_k(\Delta_m y_k)))^{p_k} \geq \frac{e}{2D(\|\beta\|+1)} \} \subset I \)

Our aim is to show that \( (\alpha x_k + \beta y_k) \in c_0^I(F, \Lambda, \Delta_m, p) \), for scalars \( \alpha, \beta \).

We have

\[
(f_k(\lambda_k(\Delta_m(\alpha x_k + \beta y_k))))^{p_k} \\
\leq (f_k(\alpha \lambda_k(\Delta_m x_k)) + f_k(\beta \lambda_k(\Delta_m x_k)))^{p_k} \\
\leq D(\|\alpha\| + 1)(f_k(\lambda_k(\Delta_m x_k)))^{p_k} + D(\|\beta\| + 1)(f_k(\lambda_k(\Delta_m y_k)))^{p_k}
\]

Now, \( C = \{ k \in N : (f_k(\lambda_k(\Delta_m(\alpha x_k + \beta y_k)))^{p_k} \geq e \} \)
\[
\subset \{ k \in N : D(\|\alpha\| + 1)(f_k(\lambda_k(\Delta_m x_k)))^{p_k} \geq \frac{e}{2} \} \cup \{ k \in N : D(\|\beta\| + 1)(f_k(\lambda_k(\Delta_m y_k)))^{p_k} \geq \frac{e}{2} \}
\]
\[
= \{ k \in N : (f_k(\lambda_k(\Delta_m x_k)))^{p_k} \geq \frac{e}{2D(\|\alpha\|+1)} \} \cup \{ k \in N : (f_k(\lambda_k(\Delta_m y_k)))^{p_k} \geq \frac{e}{2D(\|\beta\|+1)} \}
\]
\[
= A \cup B
\]
i.e, \( C \subset A \cup B \)
But \( A, B \in I \), hence \( A \cup B \in I \), therefore \( C \in I \).

**Theorem 2.3:** The classes of sequences \( \ell^I(F, \Lambda, \Delta_m, p) \), \( \ell^0_0(F, \Lambda, \Delta_m, p) \), and \( \ell^\infty(F, \Lambda, \Delta_m, p) \) are paranormed spaces paranormed by \( g \),

\[
g(x) = \sup_k (f_k(\lambda_k(x_k)))^{\frac{p_k}{\lambda_k}},
\]
where \( M = \max(1, \sup_k p_k) \).

**Proof:** Clearly \( g(x) \geq 0, g(-x) = g(x), g(x + y) \leq g(x) + g(y) \).

Next we show the continuity of the product. Let \( \alpha \) be fixed and \( g(x) \to 0 \). Then it is obvious that \( g(\alpha x) \to 0 \).

Next let \( \alpha \to 0 \) and \( x \) be fixed. Since \( f_k \) are continuous, we have \( f_k(\lambda_k(\Delta_m x_k)) \to 0 \), as \( \alpha \to 0 \).

Thus we have

\[
\sup_k (f_k(\lambda_k(\Delta_m x_k)))^{\frac{p_k}{\lambda_k}} \to 0, \text{ as } \alpha \to 0.
\]

Hence \( g(\alpha x) \to 0 \), as \( \alpha \to 0 \).
Therefore \( g \) is a paranorm.

**Proposition 2.1:** \( \ell^0_0(F, \Lambda, \Delta_m, p) \subset \ell^I(F, \Lambda, \Delta_m, p) \subset \ell^\infty(F, \Lambda, \Delta_m, p) \) and the inclusion is proper.

**Example:** Let \( I = I_f \), \( f_k(x_k) = x_k = (-1)^k \), \( \lambda_k = p_k = 1 \), \( m = 1 \), then \((x_k) \in \ell^\infty(F, \Lambda, \Delta_m, p) \) but \((x_k) \notin \ell^0_0(F, \Lambda, \Delta_m, p) \) or \( \ell^I(F, \Lambda, \Delta_m, p) \).

**Theorem 2.4:** The spaces \( \ell^I(F, \Lambda, \Delta_m, p) \), \( \ell^0_0(F, \Lambda, \Delta_m, p) \), and \( \ell^\infty(F, \Lambda, \Delta_m, p) \) are neither solid nor monotone in general, but the spaces \( \ell^0_0(F, \Lambda, p) \), and \( \ell^\infty(F, \Lambda, p) \) are solid and as such are monotone.

**Proof:** Let \((x_k)\) be a given sequence and \((\alpha_k)\) be a sequence of scalars such that \( |\alpha_k| \leq 1 \), for all \( k \in N \).
Then we have

\[(f_k(\lambda_k \alpha_k x_k))^p_k \leq (f_k(\lambda_k x_k))^p_k, \text{ for all } k \in N.\]

The solidness of \(c_0^I(F, \Lambda, p)\), and \(\ell_\infty^I(F, \Lambda, p)\) follows from this inequality. The monotonicity follows by lemma \(2.1\).

The first part of the proof follows from the following example:

**Example:** Let \(I = I_f\), \(f_k(x) = x\), for all \(x \in [0, \infty]\), \(m = 1\), \(\lambda_k = 1\) for all \(k \in N\), \(p_k = 1\) for \(k\) odd, \(p_k = 3\) for \(k\) even, \(x_k = k\), for all \(k \in N\) belongs to \(c^I(\Delta, p)\) and \(\ell_\infty^I(\Delta, p)\). For \(E\), a sequence space, consider its step space \(E_J\) defined by \((y_k) \in E_J\) implies \(y_k = 0\) for all \(k\) odd and \(y_k = x_k\) for \(k\) even. Then \((y_k)\) neither belongs to \((c^I(\Delta, p))_J\) nor to \((\ell_\infty^I(\Delta, p))_J\). Hence the spaces are not monotone. Hence are not solid.

**Theorem 2.5:** The spaces \(c^I(F, \Lambda, \Delta, p)\), \(c_0^I(F, \Lambda, \Delta, p)\) and \(\ell_\infty^I(F, \Lambda, \Delta, p)\) are not symmetric in general.

**Proof:** The result follows from the following example:

**Example:** Let \(I = I_f\), \(f_k(x) = x\), for all \(x \in [0, \infty]\), \(m = 0\), \(\lambda_k = k\) for all \(k \in N\), \(p_k = 1\) for \(k\) odd, \(p_k = 4\) for \(k\) even, \(x_k = k^{-2}\), for all \(k \in N\). Then \((x_k)\) belongs to \(c^I(F, \Lambda, p)\), \(c_0^I(F, \Lambda, p)\). Consider its rearrangement \((y_k)\) defined as follows:

\[(y_n) = (x_1, x_3, x_4, x_2, x_6, x_7, x_8, \ldots, x_{24}, x_5, x_{26}, x_{27}, \ldots, x_{624}, x_{25}, x_{626}, \ldots).\]

Then \((y_n)\) neither belongs to \(c^I(F, \Lambda, p)\) nor to \(c_0^I(F, \Lambda, p)\). Hence the spaces \(c^I(F, \Lambda, \Delta, p)\), \(c_0^I(F, \Lambda, \Delta, p)\) and \(\ell_\infty^I(F, \Lambda, \Delta, p)\) are not symmetric in general.

**Theorem 2.6:** The spaces \(c^I(F, \Lambda, \Delta, p)\), \(c_0^I(F, \Lambda, \Delta, p)\), and \(\ell_\infty^I(F, \Lambda, \Delta, p)\) are not convergence free.
Example: Let $I = I_I$, $f_k(x) = x$, for all $x \in [0, \infty]$, $m = 1, \lambda_k = 1$ for all $k \in N$, $p_k = 1$ for $k$ odd, $p_k = 2$ for $k$ even, consider the sequence $(x_k)$ defined by $x_k = k^{-1}$, for all $k \in N$, then $(x_k)$ belongs to each of $c^I(\Delta, p), c_0^I(\Delta, p), \text{and } \ell_\infty^I(\Delta, p)$. Consider the sequence $(y_k)$ defined by $y_k = k^2$, for all $k \in N$. Then $(y_k)$ neither belongs to $c^I(\Delta, p)$ nor to $c_0^I(\Delta, p)$ nor to $\ell_\infty^I(\Delta, p)$. Hence the spaces are not convergence free.

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