Strongly \((V^\lambda, A, \Delta^n_{(vm)}, p, q)\)-summable sequence spaces defined by modulus function and statistical convergence

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Received : January 2015. Accepted : April 2015

Abstract

In this paper we introduce strongly \((V^\lambda, A, \Delta^n_{(vm)}, p, q)\)-summable sequences and give the relation between the spaces of strongly \((V^\lambda, A, \Delta^n_{(vm)}, p, q)\)-summable sequences and strongly \((V^\lambda, A, \Delta^n_{(vm)}, p, q)\)-summable sequences with respect to a modulus function when \(A = (a_{ik})\) is an infinite matrix of complex numbers, \((\Delta^n_{(vm)})\) is generalized difference operator, \(p = (p_i)\) is a sequence of positive real numbers and \(q\) is a seminorm. Also we give the relationship between strongly \((V^\lambda, A, \Delta^n_{(vm)}, p, q)\)-convergence with respect to a modulus function and strongly \(S^\lambda(A, \Delta^n_{(vm)})\)-statistical convergence.

AMS Subject Classification (2000) : 40A05, 46A45.

Keywords and Phrases : De la Vallee-Poussin mean, Difference operator, modulus function, statistical convergence.
1. Introduction and Preliminaries

The idea of difference sequence spaces was introduced by Kizmaz [9]. In 1981, Kizmaz [9] defined the sequence spaces:

\[ Z(\Delta) = \left\{ x = (x_k) : \Delta x \in Z \right\}, \]

for \( Z = \ell_\infty, c \) and \( c_0 \), where \( \Delta x = (x_k - x_{k+1}) \).

The notion was further generalized by Et and Çolak [5] by introducing the space \( c_\infty(\Delta^n), c(\Delta^n) \) and \( c_0(\Delta^n) \). Another type of generalization of difference sequence spaces is due to Tripathy and Esi [23]. Who studied the space \( c_\infty(\Delta_m), c(\Delta_m) \) and \( c_0(\Delta_m) \). Tripathy et al. [24] generalized the above notion and define these spaces as follow:

Let \( m, n \) be non negative integers, then for \( Z \) a given sequence space we have.

\[ Z(\Delta^m_n) = \left\{ x = (x_k) : (\Delta^m_n x_k) \in Z \right\} \]

where \( \Delta^m_n x = (\Delta^m_n x_k) = (\Delta^{n-1}_m x_k - \Delta^{n-1}_m x_{k+1}) \) and \( \Delta^0_m x_k = x_k \) for all \( k \in \mathbb{N} \)

Which is equivalent to the following binomial representation.

\[ \Delta^m_n x_k = \sum_{i=0}^{n} (-1)^i (n) i x_{k+i n} \]

Let \( m, n \) be non-negative integers and \( v = (v_k) \) be a sequence of non-zero scalars. Then for \( Z \), a given sequence space, recently Dutta [4] introduced the following sequence spaces:

\[ Z(\Delta^m_n(v_m)) = \left\{ x = (x_k) : (\Delta^m_n(v_m) x_k) \in Z \right\}, \text{ for } Z = \ell_\infty, c \text{ and } c_0. \]

Where \( (\Delta^m_n(v_m) x_k) = (\Delta^{n-1}_m x_k - \Delta^{n-1}_m x_{k-m}) \) and \( \Delta^0_m v_k x_k = v_k x_k \) for all \( k \in \mathbb{N} \) which is equivalent to the following binomial representation:

\[ \Delta^m_n(v_m) x_k = \sum_{i=0}^{n} (-1)^i (n) i v_{k-i} x_{k+i n}. \]
We take $v_{k-mi} = 0$ and $x_{k-mi} = 0$ for non-positive value of $k - mi$. Dutta [4] showed that these spaces can be made $BK$ spaces under the norm

$$\|x\| = \sup_k \left| \Delta^n_{(vm)} x_k \right|.$$ 

For $n = 1$ and $v_k = 1$ for all $k \in \mathbb{N}$. We get the spaces $\ell_\infty(\Delta_m), c(\Delta_m)$ and $c_0(\Delta_m)$. For $m = 1$ and $v_k = 1$ for all $k \in \mathbb{N}$, we get the spaces $\ell_\infty(\Delta^n), c(\Delta^n)$ and $c_0(\Delta^n)$. For $m = n = 1$ and $v_k = 1$ for all $k \in \mathbb{N}$, we get the spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$.

Let $\lambda = (\lambda_r)$ be a non-decreasing sequence of positive numbers tending to $\infty$ such that

$$\lambda_{r+1} \leq \lambda_r + 1, \quad \lambda_1 = 1.$$ 

The generalized de la Vallé-Poussin mean is defined by

$$t_r(x) = \frac{1}{\lambda_r} \sum_{i \in I_r} x_i,$$

where $I_r = [r - \lambda_r + 1, r]$ for $r = 1, 2, \ldots$.

A sequence $x = (x_i)$ is said to be $(V, \lambda)$-summable to a number $s$, if $t_r(x) \to s$ as $r \to \infty$[11].

If $\lambda_r = r$, then $(V, \lambda)$-summability is reduced to $(C, 1)$-summability. We write

$$[V, \lambda] = \left\{ x = (x_i) : \lim_{r \to \infty} \lambda_r^{-1} \sum |x_i - s| = 0 \text{ for some } s \right\}$$

the set of sequences $x = (x_i)$ which are strongly $(V, \lambda)$-summable to $s$ that is $x_i \to s[V, \lambda]$. The strongly $(V, \lambda)$-summable as well as generalized this kind of summable sequence spaces have been studied by various authors(Bilgin[2], Gunor et al[8], Savas[19] and others). The idea of modulus function was introduced by Nakano[15].

We recall that a modulus $f$ is a function from $[0, \infty) \to [0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0, \quad y \geq 0$,

(iii) $f$ is increasing,
(iv) $f$ is continuous from right at 0.

It follows that $f$ must be a continuous everywhere on $[0, \infty)$. The Belgin [2], Kolack [10] Maddox[12,13], Öztürk and Bilgin [2], Ruckle [17] and others used a modulus function for defining some new sequence spaces.

Let $A = (a_{ik})$ be an infinite matrix of complex numbers. We write $Ax = (A_i(x))$ if $A_i(x) = \sum_{k=1}^{\infty} a_{ik}x_k$ converges for each $i$.

Recently, the concept of strong $(V, \lambda)$-summability was generalized by Bilgin [1] as follow:

$$V^\lambda[A, f] = \left\{ x = (x_i) : \lim_{r \to \infty} \lambda_r^{-1} \sum_{i \in I_r} f(|A_i(x) - s|) = 0 \text{ for some } s \right\}.$$

In this paper we introduce the strongly $(V^\lambda, A, \Delta_n^{(vm)}, p, q)$-summable sequences and give the relation between the spaces of strongly $(V^\lambda, A, \Delta_n^{(vm)}, p, q)$-summable sequences and strongly and strongly $(V^\lambda, A, \Delta_n^{(vm)}, p, q)$-summable sequences with respect to a modulus function when $A = (a_{ik})$ be an infinite matrix of real or complex number, $(\Delta_n^{(mv)})$ is generalized difference operator, $p = (p_i)$ is a sequence of positive real numbers and $q$ is a seminorm. Also we give the natural relationship between strongly $(V^\lambda, A, \Delta_n^{(vm)}, p, q)$-convergence with respect to a modulus function and strongly $S^\lambda(A, \Delta_n^{(vm)})$-statistical convergence.

The following inequality will be used throughout the paper:

$$|a_i + b_i|^p_i \leq T \left( |a_i|^{p_i} + |b_i|^{p_i} \right)$$

where $a_i$ and $b_i$ are complex numbers, $T = \max \left( 1, 2^{H-1} \right)$ and $H = \sup p_i < \infty$.

2. Main Results

2. Strongly $(V^\lambda, A, \Delta_n^{(vm)}, p, q)$-summable sequences
Let $A = (a_{ik})$ be an infinite matrix of complex numbers, $p = (p_i)$ be bounded sequence of positive real numbers ($0 < h = \inf p_i \leq p_i \leq \sup p_i = H < \infty$), and $F = (f_k)$ be a sequence of modulus function. We define

$$V^\lambda[A, \Delta^n(\phi_m), F, p, q]$$

$$= \left\{ x : \lim_{r \to \infty} \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta^n(\phi_m)A_i(x) - s | \right) \right) \right\}^p_i = 0 \text{ for some } s \right\}.$$

$$V^\lambda_0[A, \Delta^n(\phi_m), F, p, q] = \left\{ x : \lim_{r \to \infty} \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta^n(\phi_m)A_i(x) | \right) \right) \right\}^p_i = 0 \right\}.$$

$$V^\lambda_{\infty}[A, \Delta^n(\phi_m), F, p, q] = \left\{ x : \sup_{r} \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta^n(\phi_m)A_i(x) | \right) \right) \right\}^p_i < \infty \right\}.$$

A sequence $x = (x_i)$ is said to be strongly $(V^\lambda[A, \Delta^n(\phi_m), F, p, q])$-convergent to a number $s$ with respect to a modulus if there is a complex number $s$ such that $x \in (V^\lambda[A, \Delta^n(\phi_m), F, p, q])$. If $x$ is strongly $(V^\lambda[A, \Delta^n(\phi_m), F, p, q])$-convergent to $s$ with respect to a modulus $F = (f_k)$, then we write $x_i \to s(V^\lambda[A, \Delta^n(\phi_m), F, p, q])$.

Throughout this paper $\phi$ will denote one of the notation 0, 1 or $\infty$.

When $F(x) = x$ then we write the spaces $V^\lambda[A, \Delta^n(\phi_m), F, p, q]$ in place of $V^\lambda_\phi[A, \Delta^n(\phi_m), F, p, q]$. If $p_i = 1$ for all $i$, then $V^\lambda[A, \Delta^n(\phi_m), F, p, q]$ reduces to $V^\lambda[A, \Delta^n(\phi_m), F, q]$ if $q = x$ then $V^\lambda[A, \Delta^n(\phi_m), F, p, q]$ reduces to $V^\lambda[A, \Delta^n(\phi_m), F, p]$. 

In this section we examine some topological properties of $V^\lambda[A, \Delta^n(\phi_m), F, p, q]$ spaces and investigate some inclusion relations between these spaces.

**Theorem 2.1.** Let $F = (f_k)$ be a sequence of moduli, $q$ be a seminorm, $p = (p_i)$ be a sequence of positive real numbers and $X$ denotes the anyone of the spaces $V^\lambda[A, \Delta^n(\phi_m), F, p, q], V^\lambda_0[A, \Delta^n(\phi_m), F, p, q]$ or $V^\lambda_{\infty}[A, \Delta^n(\phi_m), F, p, q]$. Then $X$ is linear space over the complex field $\mathbb{C}$.

**Proof.** Since the proof is analogous for the space $V^\lambda[A, \Delta^n(\phi_m), F, p, q]$, and $V^\lambda_{\infty}[A, \Delta^n(\phi_m), F, p, q]$. So we give the proof of $V^\lambda_0[A, \Delta^n(\phi_m), F, p, q]$. Let
$x, y \in V^\lambda_0[A, \Delta^n_{(vm)}, F, p, q]$ and $a, b \in C$. Then there exist integers $T_a$ and $T_b$ such that $|a| \leq T_a$ and $|b| \leq |T_b$. We have

$$\lambda_r^{-1} \sum_{i \in I_r} \left[ f_k \left( q \left( | \Delta^n_{(vm)} A_i (ax + by) | \right) \right) \right]^{p_i}$$

$$\leq \lambda_r^{-1} \sum_{i \in I_r} \left[ f_k \left( q \left( | \Delta^n_{(vm)} A_i ax + \Delta^n_{(vm)} A_i by | \right) \right) \right]^{p_i}$$

$$\leq T \left\{ \lambda_r^{-1} \sum_{k \in I_r} \left[ T_a f_k \left( q \left( | \Delta^n_{(vm)} A_i x | \right) \right) \right]^{p_i} + \lambda_r^{-1} \sum_{i \in I_r} \left[ T_b f_k \left( q \left( | \Delta^n_{(vm)} A_i y | \right) \right) \right]^{p_i} \right\}$$

$$\leq T \left\{ \left[ T_a \right]^H \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta^n_{(vm)} A_i x | \right) \right)^{p_i} + \left[ T_b \right]^H \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta^n_{(vm)} A_i y | \right) \right)^{p_i} \right\}$$

as $r \to \infty$. This proves that $V^\lambda_0[A, \Delta^n_{(vm)}, F, p, q]$ is linear.

**Theorem 2.2.** Let $F = (f_k)$ be a sequence of moduli, $q$ be a seminorm and $p = (p_i)$ be a sequence of positive real numbers, then the inclusions $V^\lambda_0[A, \Delta^n_{(vm)}, F, p, q] \subset V^\lambda[A, \Delta^n_{(vm)}, F, p, q] \subset V^\lambda_\infty[A, \Delta^n_{(vm)}, F, p, q]$ hold.

**Proof.** The inclusion $V^\lambda_0[A, \Delta^n_{(vm)}, F, p, q] \subset V^\lambda[A, \Delta^n_{(vm)}, F, p, q]$ is obvious. Now let $x \in V^\lambda[\Delta^n_{(vm)}, A, F, p, q]$ such that $x_i \to s \left( V^\lambda[\Delta^n_{(vm)}, A, F, p, q] \right)$. By using (1), we have

$$\sup_r \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta^n_{(vm)} A_i | x \right) \right)^{p_i}$$

$$= \sup_r \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta^n_{(vm)} A_i | x - s \right) \right)^{p_i}$$

$$\leq T \left\{ \sup_r \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta^n_{(vm)} A_i | x - s \right) \right)^{p_i} \right\}$$
\[ + \sup_{r} \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( |s| \right) \right) \]
\[ \leq T \left\{ \sup_{r} \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( |\Delta_n^{(v)} A_i(x) - s| \right) \right) \right. \]
\[ + \max \left\{ f_k \left( q \left( |s| \right) \right)^h, f_k q \left( |s| \right)^H \right\} < \infty. \]

Hence \( x \in V^\lambda \left[ A, \Delta_n^{(v)} \Delta, p, q \right] \). This proves that inclusion \( V^\lambda \left[ A, \Delta_n^{(v)} \Delta, F, p, q \right] \subseteq V^\lambda \left[ A, \Delta_n^{(v)} \Delta, F, p, q \right] \) holds, which completes the proof.

**Corollary 1.** \( V^\lambda_0 \left[ A, \Delta_n^{(v)} \Delta, F, p, q \right] \) and \( V^\lambda_\infty \left[ A, \Delta_n^{(v)} \Delta, F, p, q \right] \) are nowhere dense subsets of \( V^\lambda_\infty \left[ A, \Delta_n^{(v)} \Delta, F, p, q \right] \). Let \( X \) be a sequence space.

(i) \( X \) is called solid(normal) if \( (\alpha_i x_i) \in X \), whenever \( (x_i) \in X \) for all sequences \( (\alpha_i) \) of scalars with \( |\alpha_i| \leq 1 \), for all \( i \in \mathbb{N} \).

(ii) Monotone provided \( X \) contains the canonical pre-images of all its step spaces. If \( X \) is normal, then \( X \) is monotone.

**Theorem 2.3.** The sequence spaces \( V^\lambda_0 \left[ A, \Delta_n^{(v)} \Delta, F, p, q \right] \) and \( V^\lambda_\infty \left[ A, \Delta_n^{(v)} \Delta, F, p, q \right] \) are solid and hence monotone.

**Proof.** Let \( \alpha = (\alpha_i) \) be a sequence of scalars such that \( |\alpha_i| \leq 1 \), for all \( i \in \mathbb{N} \). Since \( F = (f_k) \) is monotone, we get

\[ \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( |\Delta_n^{(v)} A_i(\alpha x) | \right) \right) \]
\[ \leq \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( \sup |\alpha_i| |\Delta_n^{(v)} A_i(x) | \right) \right) \]
\[ \leq \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( |\Delta_n^{(v)} A_i(x) | \right) \right)^{p_i} \]

Which leads to the proof.

**Theorem 2.4.** Let \( F = (f_k) \) be any modulus. Then \( V^\lambda_\varphi \left[ A, \Delta_n^{(v)} \Delta, p, q \right] \subset V^\lambda_\varphi \left[ A, \Delta_n^{(v)} \Delta, F, p, q \right] \).

**Proof.** We consider the case \( V^\lambda_0 \left[ A, \Delta_n^{(v)} \Delta, p, q \right] \subset V^\lambda_0 \left[ A, \Delta_n^{(v)} \Delta, F, p, q \right] \). Let \( x \in V^\lambda_0 \left[ A, \Delta_n^{(v)} \Delta, p, q \right] \) and \( \epsilon > 0 \). We choose \( 0 < \delta < 1 \) such that \( f_k(u) < \epsilon \) for every \( u \) with \( 0 \leq u \leq \delta \).
we can write

\[ \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta_{(vm)}^n A_i(x) - s | \right) \right)^{p_i} \]

\[ = \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta_{(vm)}^n A_i(x) - s | \right) \right)^{p_i} + \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta_{(vm)}^n A_i(x) - s | \right) \right)^{p_i} \]

\[ \leq \max \left( \epsilon^h, \epsilon \right) + \max \left( 1, (2f_k(1)\delta^{-1})^H \right) \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta_{(vm)}^n A_i(x) - s | \right) \right)^{p_i} \]

where

\[ \sum_i f_k \left( q \left( | \Delta_{(vm)}^n A_i(x) - s | \right) \right)^{p_i} \leq \delta \quad \text{and} \quad \sum_i f_k \left( q \left( | \Delta_{(vm)}^n A_i(x) - s | \right) \right)^{p_i} > \delta. \]

Hence

\[ \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta_{(vm)}^n A_i(x) - s | \right) \right)^{p_i} \]

\[ \leq \max \left( \epsilon^h, \epsilon \right) + \max \left( 1, (2f_k(1)\delta^{-1})^H \right) \lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta_{(vm)}^n A_i(x) - s | \right) \right)^{p_i} \]

therefore, \( x \in V_0^\lambda[A, F, p, q] \)

**Theorem 2.5.** Let \( F = (f_k) \) be any modulus. If \( \lim_{t \to \infty} \frac{f(t)}{t} = \beta > 0 \), then \( V^\lambda[A, F, p, q] = V^\lambda[A, F, p, q] \).

**Proof.** The existence of positive limit for any modulus function given with \( \beta \) was introduced by Maddox[13]

Let \( \beta > 0 \) and \( x \in V^\lambda[A, F, p, q] \). Since \( \beta > 0 \), we have \( f_k(t) \geq \beta t \) for all \( t > 0 \) It is easy to see that \( x \in V^\lambda[A, F, p, q] \), by using Theorem 2.4 the proof is completed.

we consider that \( (p_i) \) and \( p'_i \) are any bounded sequences of positive real numbers. We can prove \( V^\lambda[A, F, p', q] \subset V^\lambda[A, F, p, q] \) only under addition condition

**Theorem 2.6.** Let \( 0 < p_i \leq p'_i \), for all \( i \) and let \( \frac{p'_i}{p_i} \) be bounded. Then \( V^\lambda[A, F, p', q] \subset V^\lambda[A, F, p, q] \)

**Proof.** If we take \( t_i = f_k(|A_i(x)|)^{p'_i} \) for all \( i \), then using the same technique in proof of Theorem 2.2 of Öztürk and Bilgin [16], it is easy to prove
Corollary 2.

if $0 < \inf p_i \leq 1$ for all $i$, $V_{\varphi}[A, \Delta^n_{(vm)} F, q] \subset V_{\varphi}^\lambda[A, \Delta^n_{(vm)}], F, p, q]$ if

$1 \leq p_i \leq H < \infty$, then $V_{\varphi}^\lambda[A, \Delta^n_{(vm)}, F, p, q] \subset V_{\varphi}[A, \Delta^n_{(vm)}), F, q]$.

3. $S_{\lambda}(A, \Delta^n_{(vm)})$-Statistical Convergence

In this section, we introduce natural relationship between strongly $V_{\lambda}^\lambda[A, \Delta^n_{(vm)}, p, q]$-convergence with respect to modulus function and strongly $S_{\lambda}(A, \Delta^n_{(vm)})$-statistical convergence. In [6], Fast introduce the idea of statistical convergence. These idea was later studied by Connor [3], Freedman and Sember [7], Salat[19], Savas[20], Schoenberg [21], Rath and Tripathy [18], Tripathy [22], Tripathy and Sen [25, 26] and the other authors independently.

A complex number sequence $x = (x_i)$ is said to be statistically convergent to the number $\ell$ if for every $\epsilon > 0$, $\lim_{n \to \infty} |K(\epsilon)| = 0$, where $|K(\epsilon)|$ denotes the number of elements in $K(\epsilon) = \{i \in \mathbb{N} : |x_i - \ell| \geq \epsilon\}$. The set of statistically convergent sequences is denoted by $S$.

A sequence $x = (x_i)$ is said to strongly $S_{\lambda}(A, \Delta^n_{(vm)})$-statistically convergent to $s$ if any $\epsilon > 0$, $\lim_{r \to \infty} \lambda^{-1} r |KA(\epsilon)| = 0$, where $|K(\epsilon)|$ denotes the number of elements in $KA(\epsilon) = \{i \in I_r : \Delta^n_{(vm)} A_i(x) - s \geq \epsilon\}$.

The set of all strongly $S_{\lambda}(A, \Delta^n_{(vm)})$-statistically convergent sequences is denoted by $S_{\lambda}(A, \Delta^n_{(vm)})$.

Now we give the relation between $S_{\lambda}(A, \Delta^n_{(vm)})$-statistically convergence and strongly $V_{\lambda}^\lambda(A, \Delta^n_{(vm)}, p, q)$-convergence with respect to modulus.

Theorem 3.1. Let $F = (f_k)$ be any modulus. Then $V_{\lambda}^\lambda[A, \Delta^n_{(vm)}, F, p, q] \subset S_{\lambda}(A, \Delta^n_{(vm)})$.

Proof. Let $x \in V_{\lambda}^\lambda[A, \Delta^n_{(vm)}, F, p, q]$. Then
\[
\lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta^n_{(vm)} A_i(x) - s | \right) \right)^{p_i} \\
\geq \lambda_r^{-1} \sum_{i} f_k \left( q \left( | \Delta^n_{(vm)} A_i(x) - s | \right) \right)^{p_i} \geq \lambda_r^{-1} \sum_{i} f_k \left( q \right)^{p_i} \\
\geq \lambda_r^{-1} \sum_{i} \min \left( f_k(\epsilon)^h, f_k(\epsilon) \right)^H \\
\geq \lambda_r^{-1} \left\{ i \in I : | \Delta^n_{(vm)} A_i(x) - s \geq \epsilon \right\} \min \left\{ f_k(\epsilon)^h, (\epsilon)^H \right\}.
\]

where the summation \( \sum_1 \) is over \( \left\{ | \Delta^n_{(vm)} A_i(x) - s \right\} \geq \epsilon \). Hence \( S^{\lambda}( A, \Delta^n_{(vm)} A_i(x) ) \).

**Theorem 3.2.** Let \( F = (f_k) \) be any modulus. Then \( V^\lambda[A, \Delta^n_{(vm)}] \subset S^\lambda(A, \Delta^n_{(vm)}). \)

**Proof.** By Theorem 3.1. it is sufficient to show that \( S^\lambda[A, \Delta^n_{(vm)}] \subset S^\lambda(A, \Delta^n_{(vm)}). \)

Let \( x \in S^\lambda(A, \Delta^n_{(vm)}). \) Since \( f_k \) is bounded, so there exists an integer \( K > 0 \) such that \( f_k( | \Delta^n_{(vm)} A_i(x) - s | ) \leq K. \) Then for a given \( \epsilon > 0 \), we have.

\[
\lambda_r^{-1} \sum_{i \in I_r} f_k \left( q \left( | \Delta^n_{(vm)} A_i(x) - s | \right) \right)^{p_i} \\
= \lambda_r^{-1} \sum_{i} f_k \left( q \left( | \Delta^n_{(vm)} A_i(x) - s | \right) \right)^{p_i} + \lambda_r^{-1} \sum_{2} f_k \left( q \left( | \Delta^n_{(vm)} A_i(x) - s | \right) \right)^{p_i} \\
\leq K^H \lambda_r^{-1} \left| \left\{ i \in I : | \Delta^n_{(vm)} A_i(x) - s \geq \epsilon \right\} \right| + \max \left\{ f_k(\epsilon)^h, f_k(\epsilon)^H \right\}.
\]

where the summation \( \sum_1 f_k \left( q \left( | \Delta^n_{(vm)} A_i(x) - s | \right) \right) \geq \epsilon \) and \( \sum_2 f_k \left( q \left( | \Delta^n_{(vm)} A_i(x) - s | \right) \right) < \epsilon. \) Taking \( \epsilon \to 0 \) and \( r \to \infty. \) It follows that \( x \in V^\lambda(A, \Delta^n_{(vm)}). \) This completes the proof.
Strongly \((V^\lambda, A, \Delta^n_{(vm)}; p, q)\)-summable sequence ...

\((ii)\) References


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