Holomorphically projective Killing fields with vectorial fields associated in kahlerian manifolds

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Abstract

Taking into account the harmonic and scalar curvatures in the study of Killing transformations between spacial complex (Einsteinian, Peterson-Codazzi, Recurrent) and kaehlerian M spaces with almost complex J structure, we prove that there exists an holomorphically projective transformation between M spaces and complex spaces.

Keywords: *Holomorphically projective, Killing fields, kaehlerian manifolds.*

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1. Introduction and preliminaries

By the end of the 20th century researchers started to link the concept of projectivity with the phenomenon of complex manifolds specially in terms of their holomorphic properties. Then, kaehlerian and hermithian manifolds as well as complex hyper surfaces and other manifolds were considered embedded into special transformations. At this point a vast number of publications arose in relation with the concepts of compact K manifolds, projective infinitesimal transformations in Riemannian manifolds with additive curvature properties and holomorphic projective equivalences and others. Based on [1] [2] and [3], this research studies Kaehler holomorphically projective manifolds with almost complex structures by using the geometric properties of the harmonic and scalar curvatures evaluated over Killing vectorial fields. Two important applications result from this, the Einsteinian and the constant curvature spaces.

Considering (M, g, J) as a kaehlerian manifold of $2n \geq 4$ dimension with a $g = (g_{ij})$ Riemannian metric and an almost-complex structure $J = J_{ij}$ where $J_{ij} = -J_{ji}$ and with a Riemannian curvature tensor of $R_{kji}^h = \partial_k \Gamma_{ji}^h - \partial_j \Gamma_{ki}^h + \Gamma_{ka}^h \Gamma_{ji}^a - \Gamma_{ja}^h \Gamma_{ki}^a$ the $S_{ji} = R_{aji}^a$ then the Ricci tensor and the $r = g^{ba} S_{ba}$ scalar curvature satisfy the following properties:

$$\begin{aligned} i) S_{ji} &= J_i^b J_i^a S_{ba} & ii) H_{ji} &= J_j^a S_{ai} & iii) H_{ji} + H_{ij} &= 0 \\ iv) H_{ji} &= J_i^b J_i^a H_{ba} & v) J_i^a J_a^h &= -\delta_i^h & vi) J_{\underline{j}}^a &= -i\delta_j^a \\ vii) g_{ij} &= J_i^a J_j^b g_{ab} & viii) \nabla_j \tilde{F}_i &= -\nabla_i \tilde{F}_j & ix) \tilde{F}_i &:= J_i^a F_a, \end{aligned}$$

where

$J^j_i = g^{ja} J_a^i$, $H_{ji} = J_a^i S_{ai}$. The Lie operator derivative in the vectorial field direction X for R_{kji}^h and h_{ji} is represented respectively by,

$$L_X R_{kji}^h = \nabla_k L_X \Gamma_{ji}^h - \nabla_j L_X \Gamma_{ki}^h \quad \text{y} \quad L_X \Gamma_{ji}^h = \nabla_j \nabla_i X^h + R_{aji}^h X^a.$$

If X is a vectorial field then

i) X is a Killing field if satisfies

$$(1.1) \quad L_X g_{ji} = 0, \quad \forall \quad i, j = \overline{1, n}.$$

ii) X is an holomorphically projective transformation when

$$(1.2) \quad L_X \Gamma_{ji}^h = \delta_i^h F_j + \delta_j^h F_i - J_j^h J_i^a F_a - J_i^h J_j^a F_a,$$

where $F = (F^i)$ is a particular vector associated to X .

Two metric $g = (g_{ij})$ y $\bar{g} = (\bar{g}_{ij})$ defined on M , they are hollomorphic projective equivalences if

$$\bar{\Gamma}_{ki}^k = \Gamma_{ji}^k + F_i \delta_i^k + F_j \delta_i^k - J_j^k \tilde{F}_i - J_i^k \tilde{F}_j,$$

where $\tilde{F}_i = J_i^a F_a$.

Tensors for harmonic and scalar curvatures are defined on the manifold M by means of the following relations:

$$\nabla_a R_{kji}{}^a = \nabla_k S_{ji} - \nabla_j S_{ki},$$

$$R = g^{ba} S_{ba},$$

respectively where $S_{ji} = R_{aji}{}^a$ is the tensor Ricci. The Laplacian of f is defined by

$$\Delta f = \nabla^a \nabla_a f = \Delta f,$$

where $f = \frac{1}{n+2} \nabla_a X^a$ with $f \in C^\infty(M)$ and $F_j = \nabla_j f$.

The classic commutative relationship of L_X and de ∇ for a tensor Y of (1,2) type is given by

$$L_X \nabla_k Y_{ji}{}^h - \nabla_k L_X Y_{ji}{}^h = (L_X \Gamma_{ka}^h) Y_{ji}{}^h - (L_X \Gamma_{kj}^h) Y_{ai}{}^h - (L_X \Gamma_{kj}^a) Y_{ai}{}^h - (L_X \Gamma_{ki}^a) Y_{aj}{}^h.$$

Being X an holomorphically projective transformation with an F associated vector then the following identities are satisfied, watch [1]

$$\text{i} \quad 2S_i^a F_a = -\nabla_i(\Delta f)$$

$$\text{ii} \quad \nabla_j F_i = J_i^a J_j^b \nabla_b F_a$$

$$\text{iii} \quad \nabla_k \nabla_j F_i = -J_k{}^b J_j^a R_{iab}^c F_c.$$

In [3] proof

$$(1.3) \quad S_{ij} = \bar{S}_{ij} + \tau(F_{ij} - \bar{F}_{ji}),$$

where $F_{ij} = \nabla_i \nabla_j f$, τ - parameter.

An $A_n = (M, \nabla)$ space is a Peterson Codazzi one if $\nabla_k S_{ji} = \nabla_j S_{ki}$. If $\nabla_l R_{ijk}^h = F_l R_{ijk}^h$ it is a recurrent space where $F_l \neq 0$ or it is an Einstein space if $S = \lambda g$ taking S as the Ricci and g as the metric tensors and λ as a parameter.

Lemma 1.1. *If M is a compact Kaehlerian manifold of dimension n with a scalar curvature R and it admits an holomorphically projective transformation then the following equations are fulfilled,*

- i) $\Delta f = -\frac{2R}{n}f$
ii) $S_i^a F_a = \frac{R}{n}F_i$.

Proof. i) Since A_n is a recurrent space and M admits an holomorphically projective transformation then we obtain

$$(1.4) g^{hi} \nabla_b \nabla_j X_i + g^{hi} R_{abji} X^a - F_b \delta_j^h - F_j \delta_b^h + F_a J_b^a J_j^h + F_a J_j^a J_b^h = 0,$$

multiplying (1.4) by g_{hk} and applying ∇^b it results that

$$\nabla^b \left(\nabla_b \nabla_j X_i + R_{abji} X^a - F_b g_{ji} - F_j g_{bi} + F_a J_b^a J_{ji} + F_a J_j^a J_{bi} \right) = 0.$$

Now using Ricci's and Bianchi's identities we obtain

$$(R_{abji} - 2R_{bjia}) \nabla^b X^a - R_{ai} \nabla_j X^a + R_j^a \nabla_a X_i - (\nabla_a R_{ji}) X^a = 0,$$

finally by applying ∇^j the result is

$$\begin{aligned} -2\nabla_i R_{ba} \nabla^b X^a &= 0 \Rightarrow \nabla_i R_{ba} L_X g^{ba} = 0 \\ &\Rightarrow -2RF_i = n\nabla_i (\Delta f) \\ &\Rightarrow \nabla_i (\Delta f) = -\frac{2R}{n}f = \nabla_i \left(-\frac{2R}{n}f \right) \\ &\Rightarrow \Delta f = -\frac{2R}{n}f, \end{aligned}$$

due to $(n\Delta f + 2Rf)$ is constant for being

$$\int_M \Delta f d\sigma = \int_M f d\sigma = 0,$$

a compact M and $d\sigma$ is a volumetric element of M . Finally we conclude that

$$\Delta f = -\frac{2R}{n}f.$$

ii) The demonstration is obtained by using I and part (i) from this lema.

Lemma 1.2. *Let X be an holomorphically projective transformation with an F associated vector then*

$$L_X S_{ji} = -(n+2) \nabla_j F_i.$$

Proof. Using the definition of $L_X R_{kji}^h$ we have

$$L_X S_{ji} = L_X R_{hji}^h = \nabla_h L_X \Gamma_{ji}^h - \nabla_j L_X \Gamma_{hi}^h,$$

since X is an holomorphically projective transformation then

$$\begin{aligned} L_X S_{ji} &= \nabla_j F_i + \nabla_i F_j - J_j^h J_i^a F_a - J_i^h J_j^a F_a - n \nabla_j F_i - \nabla_j F_i + \\ &\quad + i n J_i^a \nabla_j F_a - \nabla_j F_i. \end{aligned}$$

By considering the real part we obtain the desired result

$$L_X S_{ji} = -n \nabla_j F_i - 2 \nabla_j F_i = -(n+2) \nabla_j F_i.$$

2. Results

The following theorem allows a Kaehlerian space to become into a Peterson-Codazzi space under the hypothesis that the former is holomorphically projective.

Theorem 2.1. *Let M be a Kaehlerian manifold and X be an holomorphically projective killing field with an associated vectorial field F then*

$$\begin{aligned} &L_X (\nabla_j S_{ki} - \nabla_k S_{ji}) \\ &= \left\{ (n+2) R_{jki}^a - S_{ki} \delta_j^a + S_{ji} \delta_k^a - J_i^a H_{ki} + J_k^a H_{ji} + 2 J_i^a H_{jk} \right\} F_a. \end{aligned} \quad (2.1)$$

Proof. Using the classic relation of commutation for a (0,2) type tensor we obtain that

$$(L_X \nabla_j S_{ki} - L_X \nabla_k S_{ji}) - (\nabla_j L_X S_{ki} - \nabla_k L_X S_{ji}) = (L_X \Gamma_{ki}^a) S_{ja} - (L_X \Gamma_{ji}^a) S_{ka}. \quad (2.2)$$

If by hypothesis we consider X as an holomorphically projective transformation by using (1.2) then we have that

$$(2.3) \quad L_X \Gamma_{ji}^a = \delta_j^a F_i + \delta_i^a F_j - J_j^a J_i^h F_h - J_i^a J_j^h F_h.$$

Furthermore according to lemma (1.2),

$$(2.4) \quad L_X S_{ji} = -(n+2) \nabla_j F_i$$

and analogically we obtain $L_X \Gamma_{ki}^a$ and $L_X S_{ki}$.

By Substituting (2.3) and (2.4) in (2.2) it results

$$\begin{aligned} & (L_X \nabla_j S_{ki} - L_X \nabla_k S_{ji}) - (\nabla_j [-(n+2) \nabla_k F_i] - \nabla_k [-(n+2) \nabla_j F_i]) \\ & = (\delta_k^a F_i + \delta_i^a F_k - J_k^a J_i^h F_h - J_i^a J_k^h F_h) S_{ja} \\ & \quad - (\delta_j^a F_i + \delta_i^a F_j - J_j^a J_i^h F_h - J_i^a J_j^h F_h) S_{ka}. \end{aligned}$$

By doing certain manipulations and using simplification we conclude that

$$\begin{aligned} & \left\{ (n+2) R_{jki}{}^a - S_{ki} \delta_j^a + S_{ji} \delta_k^a - J_i^a H_{ki} + J_k^a H_{ji} + 2J_i^a H_{jk} \right\} F_a \\ & = L_X (\nabla_j S_{ki} - \nabla_k S_{ji}). \end{aligned}$$

From here on some applications of the previous results will be given.

1) If $\nabla_j S_{ki} = \nabla_k S_{ji}$ then M is Kaehler-Peterson-Codazzi space and

$$\begin{aligned} & \left\{ (n+2) R_{jki}{}^a - S_{ki} \delta_j^a + S_{ji} \delta_k^a - J_i^a H_{ki} + J_k^a H_{ji} + 2J_i^a H_{jk} \right\} F_a = 0. \\ (2.5) \end{aligned}$$

Consequence i

A Kaehler-Peterson-Codazzi space has an harmonic curvature since,

$$\nabla_j S_{ki} = \nabla_k S_{ji} \Leftrightarrow \nabla_a R_{jki}^a = 0.$$

Consequence ii

A Kaehler-Peterson-Codazzi space is an Einstenian space if the former has a constant scalar curvature. Factually by applying g^{ki} into (2.5) results in

$$\left\{ (n+2) g^{ki} R_{jki}{}^a - R \delta_j^a + g^{ai} S_{ji} - J_i^a g^{ki} H_{ki} + J_k^a g^{ki} H_{ji} + 2J_i^a g^{ki} H_{jk} \right\} F_a = 0.$$

Since $F_a \neq 0$ and developing the three last terms we have,

$$(n+2) g^{ki} R_{jki}{}^a - R \delta_j^a + g^{ai} S_{ji} - J_j^a J_k^b g^{ki} S_{bi} + 3J_k^a J_j^b g^{ki} S_{bi} = 0,$$

by making the contraction $a = j$ and adding up from 1 to n we obtain

$$g_{ki} (nR + 2R - nR + R) = 3S_{ki},$$

In this way we conclude that $S_{ki} = \frac{R}{n}g_{ki}$. In other words the Kaehler-Peterson-Codazzi space is an Einstenian space.

2) If M is a recurrent space then

$$(n+2)R_{jki}^a F_a - L_X(R_{jki}^a F_a) = S_{ki}\delta_j^a - S_{ji}\delta_k^a + J_j^a H_{ki} - J_k^a H_{ji} - 2J_i^a H_{jk} F_a.$$

Consequence

If M is an harmonic curvature and $W = \{F = (F^i) : F \neq 0\}$ with

$$F_j F^k = \begin{cases} \|F\|^2 & si \quad j = k \\ 0 & si \quad k \neq j \end{cases},$$

then M has a null scalar curvature.

As a matter of fact if M admits an harmonic curvature then making the contraction $l = a$ and summing up from 1 to n in the relation

$$\nabla_l R_{jki}^a = R_{jki}^a F_l,$$

we obtain,

$$\nabla_l R_{jki}^a = R_{jki}^a F_l \Rightarrow R_{jki}^a = 0.$$

from (2.1) we obtain

$$S_{ki}F_j - S_{ji}F_k + H_{ki}\tilde{F}_j - H_{ji}\tilde{F}_k - 2H_{jk}\tilde{F}_i = 0$$

and multiplying the previous relation by g^{ki} it results that

$$g^{ki}S_{ji}F_k - g^{ki}S_{ki}F_j + g^{ki}H_{ji}\tilde{F}_k - g^{ki}H_{ki}\tilde{F}_j + 2g^{ki}H_{kj}\tilde{F}_i = 0.$$

Therefore

$$S_j^k F_k - r F_j = H_j^k \tilde{F}_k,$$

wherein by applying F^j it results that $r = 0$. This way we conclude that the manifold is plain.

Example

Be Einstein compact Kaehlerian spaces $A_n = (M, \nabla)$ and $\overline{A}_n = (M, \overline{\nabla})$, with metric $g = (g_{ij})$ y $\overline{g} = (\overline{g}_{ij})$ hollomorphic projective equivalences, get an expression that relates the scalar curvature R and $\overline{R}$.

Solution Using [4],

$$S_{ij} = \overline{S}_{ij} + \tau(F_{ij} - \overline{F}_{ji}),$$

and as spaces of Einstein:

$$S_{ij} = c_1 g_{ij}, \quad \bar{S}_{ij} = c_2 \bar{g}_{ij}$$

o

$$S_{ij} = \frac{R}{n} g_{ij}, \quad \bar{S}_{ij} = \frac{\bar{R}}{n} \bar{g}_{ij},$$

it must be

$$\frac{R}{n} g^{ij} = \frac{\bar{R}}{n} \bar{g}_{ij} + \tau(F_{ij} - \bar{F}_{ji}), \quad \tau \in C,$$

Applying g^{ij} result

$$R = \frac{\bar{R}}{n} g^{ij} \bar{g}_{ij} + \tau(g^{ij} F_{ij} - g^{ij} \bar{F}_{ji}),$$

$$R = \frac{\bar{R}}{n} g^{ij} \bar{g}_{ij} + \tau(\|F\| - g^{ij} \bar{F}_{ji})$$

o

$$Rg_{ij} = \frac{\bar{R}}{n} \bar{g}_{ij} + \tau(\|F\|g_{ij} - \bar{F}_{ij})$$

Applying now $\bar{g}^{ij}$,

$$Rg_{ij} = \bar{R}\bar{g}_{ij} + \tau(\|F\|g_{ij} - \|\bar{F}\|\bar{g}_{ij}),$$

from here

$$(R - \tau\|F\|)g_{ij} = (\bar{R} - \tau\|\bar{F}\|)\bar{g}_{ij},$$

then

$$\frac{(R - \tau\|F\|)}{(\bar{R} - \tau\|\bar{F}\|)} = \frac{\det(\bar{g}_{ij})}{\det(g_{ij})}$$

In [3] proof

$$\ln \sqrt{\frac{\det(\bar{g}_{ij})}{\det(g_{ij})}} = (n+2)h, \quad h \in C^\infty(M).$$

Then he concludes

$$(R - \tau\|F\|) = (\bar{R} - \tau\|\bar{F}\|) \exp[2h(n+2)].$$

Example

Get an expression that compute the tensor Ricci in a compact kahlerian manifolds admitting proyective hollomorphic transformations with associated vector F , if A_n this is recurrent.

Solution In this case

$$\nabla_k R_{lji}^h = R_{lji}^h F_k,$$

by making the contraction $a = j$ and adding up from 1 to n we obtain

$$(2.6) \quad \nabla_k S_{ji} = S_{ji} F_k.$$

But

$$\nabla_k S_{ji} = \partial_k(S_{ji}) - \Gamma_{kj}^a S_{ai} - \Gamma_{ki}^a S_{ja}.$$

Applying g^{ij} result

$$(2.7) \quad \nabla_k S_{ji} = \partial_k(S_{ji}).$$

From (2.6) and (2.7) results

$$\partial_k(S_{ji}) - S_{ji} \partial_k f = 0.$$

The solution of this partial differential equation is the tensor de Ricci, hence is obtained scalar curvature.

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