Holomorphically proyective Killing fields with vectorial fields associated in kahlerian manifolds

Richard Malavé Guzmán
U. Politécnica Territorial del oeste de Sucre C. R., Venezuela
Franmary López
U. Politécnica Territorial del oeste de Sucre C. R., Venezuela
and
Rodrigo Martínez
Universidad de Oriente, Venezuela

Received : October 2013. Accepted : April 2015

Abstract

Taking into account the harmonic and scalar curvatures in the study of Killing transformations between spacial complex (Einstenian, Peterson-Codazzi, Recurrent) and kahlerian $M$ spaces with almost complex $J$ structure, we prove that there exists an holomorphically proyective transformation between $M$ spaces and complex spaces.

Keywords: Holomorphically proyective, Killing fields, kahlerian manifolds.

1. Introduction and preliminaries

By the end of the 20th century researchers started to link the concept of proyectivity with the phenomenon of complex manifolds specially in terms of their holomorphic properties. Then, kaehlerian and hermitian manifolds as well as complex hyper surfaces and other manifolds were considered embedded into special transformations. At this point a vast number of publications arose in relation with the concepts of compact K manifolds, proyective in infinitesimal transformations in Riemmanian manifolds with additive curvature properties and holomorphic proyective equivalences and others. Based on [1] [2] and [3], this research studies Kaehler holomorphically proyective manifolds with almost complex structures by using the geometric properties of the harmonic and scalar curvatures evaluated over Killing vectorial fields. Two important applications result from this, the Einsteian and the constant curvature spaces.

Considering \((M,g,J)\) as a kaehlerian manifold of \(2n \geq 4\) dimension with a \(g = (g_{ij})\) Riemannian metric and an almost-complex structure \(J = J_{ij}\) where \(J_{ij} = -J_{ji}\) and with a Riemannian curvature tensor of \(R_{hji} = \partial_k \Gamma_{hji} - \partial_j \Gamma_{hki} + \Gamma_{ka} J_{ji} - \Gamma_{ja} J_{ki}\) the \(S_{ji} = R_{aj}^a\) then the Ricci tensor and the \(r = g^{ba} S_{ba}\) scalar curvature satisfy the following proprieties:

\[
\begin{align*}
\text{i}) & \quad S_{ji} = J_i^b J_s^a S_{ba} \\
\text{ii}) & \quad H_{ji} = J_j^a S_{ai} \\
\text{iii}) & \quad H_{ji} + H_{ij} = 0 \\
\text{iv}) & \quad H_{ji} = J_i^b J_s^a H_{ba} \\
\text{v}) & \quad J_i^b J_j^a = -\delta_i^h \\
\text{vi}) & \quad J_i^b J_j^a = -i\delta_i^h \\
\text{vii}) & \quad g_{ij} = J_i^a J_j^b g_{ab} \\
\text{viii}) & \quad \nabla_j \tilde{F}_i = -\nabla_i \tilde{F}_j \\
\text{ix}) & \quad \tilde{F}_i := J_i^a F_a 
\end{align*}
\]

where

\[
\begin{align*}
J_{ij} &= g^a J_a^i, \quad H_{ji} = J_a^i S_{ai}. \quad \text{The Lie operator derivative in the vectorial field direction } X \text{ for } R_{hji} \text{ and } h_{ji} \text{ is represented respectively by,}
\end{align*}
\]

\[
L_X R_{hji} = \nabla_k L_X \Gamma_{hji} - \nabla_j L_X \Gamma_{hki} \quad \text{and} \quad L_X \Gamma_{hji} = \nabla_j \nabla_i X^h + R_{aj}^a X^a.
\]

If \(X\) is a vectorial field then

i) \(X\) is a Killing field if satisfies

\[
L_X g_{ji} = 0, \quad \forall \quad i, j = \overline{1, n}.
\]

(1.1)

ii) \(X\) is an holomorphically projective transformation when

\[
L_X \Gamma_{hji} = \delta_i^h F_i + \delta_j^h F_i - J_j^h J_i^a F_a - J_i^h J_j^a F_a,
\]

(1.2)
where \( F = (F^i) \) is a particular vector associated to \( X \).

Two metric \( g = (g_{ij}) \) and \( \mathcal{F} = (\mathcal{F}_{ij}) \) defined on \( M \), they are holomorphic projective equivalences if

\[
\Gamma^k_{ki} = \Gamma^k_{ji} + F_i \delta^k_j + F_j \delta^k_i - J^k_i F_i - J^k_j F_j,
\]

where \( F_i = J^a_i F_a \).

Tensors for harmonic and scalar curvatures are defined on the manifold \( M \) by means of the following relations:

\[
\nabla_a R_{kji}^a = \nabla^k S_{ji} - \nabla^j S_{ki},
\]

\[
R = g^{ba} S_{ba},
\]

respectively where \( S_{ji} = R_{aji}^a \) is the tensor Ricci. The Laplacian of \( f \) is defined by

\[
\Delta f = \nabla^a \nabla_a f = \Delta f,
\]

where \( f = \frac{1}{n+2} \nabla X^a \) with \( f \in C^\infty(M) \) and \( F_j = \nabla f \).

The classic commutative relationship of \( L_X \) and \( \nabla \) for a tensor \( Y \) of \((1,2)\) type is given by

\[
L_X \nabla_k Y_{ji}^h - \nabla_k L_X Y_{ji}^h = \left( L_X \Gamma_{ka}^h \right) Y_{ji}^h - \left( L_X \Gamma_{kj}^a \right) Y_{ai}^h - \left( L_X \Gamma_{ki}^a \right) Y_{ai}^h - \left( L_X \Gamma_{ki}^a \right) Y_{ai}^h.
\]

Being \( X \) an holomorphically projective transformation with an \( F \) associated vector then the following identities are satisfied, watch [1]

\[ i \quad 2S_{ij}^a F_a = -\nabla_i (\Delta f) \]

\[ ii \quad \nabla_j F_i = J^a_i J^b_j \nabla^c F_a \]

\[ iii \quad \nabla_k \nabla_j F_i = -J^b_k J^a_j R_{ij}^c F_c. \]

In [3] proof

\[ (1.3) \quad S_{ij} = \mathcal{S}_{ij} + \tau (F_{ij} - \mathcal{F}_{ij}), \]

where \( F_{ij} = \nabla_i \nabla_j f \), \( \tau \) – parameter.

An \( A_n = (M, \nabla) \) space is a Peterson Codazzi one if \( \nabla_k S_{ji} = \nabla_j S_{ki} \). If \( \nabla_i R_{ijkl}^h = F_i R_{ijkl}^h \) it is a recurrent space where \( F_i \neq 0 \) or it is an Einstein space if \( S = \lambda g \) taking \( S \) as the Ricci and \( g \) an the metric tensors and \( \lambda \) as a parameter.
Lemma 1.1. If $M$ is a compact Kaehlerian manifold of dimension $n$ with a scalar curvature $R$ and it admits an holomorphically projective transformation then the following equations are fulfilled,

1. $\Delta f = -\frac{2R}{n}f$
2. $S_i^a F_a = \frac{R}{n} F_i$.

Proof. i) Since $A_n$ is a recurrent space and $M$ admits an holomorphically projective transformation then we obtain

\[
(1.4) g^{hi} \nabla_h \nabla_j X_i + g^{hi} R_{abji} X^a - F_b \delta^h_j - F_j \delta^b_h + F_a J^h_b J^j_i + F_a J^a_b J^b_i = 0,
\]

multiplying (1.4) by $g^{hk}$ and applying $\nabla^b$ it results that

\[
\nabla^b \left( \nabla_h \nabla_j X_i + R_{abji} X^a - F_b g_{ji} - F_j g_{bi} + F_a J^a_b J^j_i + F_a R^a_j J^b_i \right) = 0.
\]

Now using Ricci’s and Bianchi’s identities we obtain

\[
(R_{abji} - 2R_{bjia}) \nabla^b X^a - R_{ai} \nabla_j X^a + R^a_i \nabla_a X_i - (\nabla_a R_{ji}) X^a = 0,
\]

finally by applying $\nabla^j$ the result is

\[
-2 \nabla^i R_{ba} \nabla^b X^a = 0 \Rightarrow \nabla_i R_{ba} L_X g^{ba} = 0
\]

\[
\Rightarrow -2RF_i = n \nabla_i (\Delta f)
\]

\[
\Rightarrow \nabla_i (\Delta f) = -\frac{2R}{n}f = \nabla_i \left(-\frac{2R}{n}f\right)
\]

\[
\Rightarrow \Delta f = -\frac{2R}{n}f,
\]

due to $(n\Delta f + 2Rf)$ is constant for being

\[
\int_M \Delta f d\sigma = \int_M f d\sigma = 0,
\]

a compact $M$ and $d\sigma$ is a volumetric element of $M$. Finally we conclude that

\[
\Delta f = -\frac{2R}{n}f.
\]

ii) The demonstration is obtained by using I and part (i) from this lemma.

Lemma 1.2. Let $X$ be an holomorphically projective transformation with an $F$ associated vector then

\[
L_X S_{ji} = -(n+2)\nabla_j F_i.
\]
Holomorphically projective Killing fields with vectorial fields ...

**Proof.** Using the definition of $L_X R_{kji}^h$ we have

$$L_X S_{ji} = L_X R_{hji}^h = \nabla_h L_X \Gamma_{ji}^h - \nabla_j L_X \Gamma_{hi}^h,$$

since $X$ is an holomorphically projective transformation then

$$L_X S_{ji} = \nabla_j F_i + \nabla_i F_j - J_i^h J^a_j F_a - J_j^h J^a_i F_a - n \nabla_j F_i - \nabla_j F_i +$$

$$+ in J^a_i \nabla_j F_a - \nabla_j F_i.$$

By considering the real part we obtain the desired result

$$L_X S_{ji} = - n \nabla_j F_i - 2 \nabla_j F_i = -(n + 2) \nabla_j F_i.$$

2. Results

The following theorem allows a Kaehlerian space to become into a Peterson-Codazzi space under the hypothesis that the former is holomorphically projective.

**Theorem 2.1.** Let $M$ be a Kaehlerian manifold and $X$ be an holomorphically projective killing field with an associated vectorial field $F$ then

$$L_X (\nabla_j S_{ki} - \nabla_k S_{ji})$$

$$= \left\{ (n + 2) R^a_{jki} - S_{ki} \delta^a_j + S_{ji} \delta^a_k - J_i^a H_{ki} - J_j^a H_{ji} + 2 J^a_i H_{jk} \right\} F_a.$$

(2.1)

**Proof.** Using the classic relation of commutation for a (0,2) type tensor we obtain that

$$(L_X \nabla_j S_{ki} - L_X \nabla_k S_{ji}) - (\nabla_j L_X S_{ki} - \nabla_k L_X S_{ji}) = (L_X \Gamma_{ki}^a) S_{ja} - \left( L_X \Gamma_{ji}^a \right) S_{ka}. $$

(2.2)

If by hypothesis we consider $X$ as an holomorphically projective transformation by using (1.2) then we have that

(2.3) $$L_X \Gamma_{ji}^a = \delta^a_j F_i + \delta^a_i F_j - J^a_i J^h_j F_h - J^a_j J^h_i F_h.$$

Furthermore according to lemma (1.2),

(2.4) $$L_X S_{ji} = - (n + 2) \nabla_j F_i$$
and analogically we obtain $L_X \Gamma^a_{ki}$ and $L_X S_{ki}$.

By Substituting (2.3) and (2.4) in (2.2) it results

$$(L_X \nabla_j S_{ki} - L_X \nabla_k S_{ji}) - (\nabla_j [- (n+2) \nabla_k F_i] - \nabla_k [- (n+2) \nabla_j F_i])$$

$$= (\delta^a_k F_i + \delta^a_k F_k - J^a_i J^b_k F_h - J^a_i J^b_k F_h) S_{ja}$$

$$= \left( \delta^a_j F_i + \delta^a_i F_j - J^a_i J^b_i F_h - J^a_i J^b_i F_h \right) S_{ka}.$$  

By doing certain manipulations and using simplification we conclude that

$$(n+2) \{ (n+2) R_{jki}^a - S_{ki} \delta^a_j + S_{ji} \delta^a_k - J^a_i H_{ki} + J^a_k H_{ji} + 2 J^a_i H_{jk} \} F_a$$

$$= L_X (\nabla_j S_{ki} - \nabla_k S_{ji}).$$

From here on some applications of the previous results will be given.

1) If $\nabla_j S_{ki} = \nabla_k S_{ji}$ then $M$ is Kaehler-Peterson-Codazzi space and

$$(n+2) \{ (n+2) R_{jki}^a - S_{ki} \delta^a_j + S_{ji} \delta^a_k - J^a_i H_{ki} + J^a_k H_{ji} + 2 J^a_i H_{jk} \} F_a = 0.$$

(2.5)

**Consequence i**

A Kaehler-Peterson-Codazzi space has an harmonic curvature since,

$$\nabla_j S_{ki} = \nabla_k S_{ji} \iff \nabla_a R^a_{jki} = 0.$$

**Consequence ii**

A Kaehler-Peterson-Codazzi space is an Einstenian space if the former has a constant scalar curvature. Factually by applying $g^{ki}$ into (2.5) results in

$$\{ (n+2) g^{ki} R_{jki}^a - R \delta^a_j + g^{ai} S_{ji} - J^a_i g^{ki} H_{ki} + J^a_k g^{ki} H_{ji} + 2 J^a_i g^{ki} H_{jk} \} F_a = 0.$$  

Since $F_a \neq 0$ and developing the three last terms we have,

$$(n+2) g^{ki} R_{jki}^a - R \delta^a_j + g^{ai} S_{ji} - J^a_i J^b_i g^{k} S_{bi} + 3 J^a_k J^b_j g^{ki} S_{bi} = 0,$$

by making the contraction $a = j$ and adding up from 1 to $n$ we obtain

$$g_{ki} (nR + 2R - nR + R) = 3S_{ki},$$
In this way we conclude that $S_{ki} = \frac{R}{n} g_{ki}$. In other words the Kaehler-Peterson-Codazzi space is an Einstenian space.

2) If $M$ is a recurrent space then

$$(n + 2) R^a_{jki} F_a - L_X \left( R^a_{jki} F_a \right) = S_{ki} \delta^a_j - S_{ji} \delta^a_k + J^a_j H_{ki} - J^a_k H_{ji} - 2J^a_i H_{jk} F_a.$$

**Consequence**

If $M$ is an harmonic curvature and $W = \{ F = (F^i) : F \neq 0 \}$ with

$$F_j F^k = \begin{cases} \|F\|^2 & \text{si } j = k \\ 0 & \text{si } k \neq j \end{cases},$$

then $M$ has a null scalar curvature.

As a matter of fact if $M$ admits an harmonic curvature then making the contraction $l = a$ and summing up from 1 to $n$ in the relation

$$\nabla_l R^a_{jki} = R^a_{jki} F_l,$$

we obtain,

$$\nabla_l R^a_{jki} = R^a_{jki} F_l \Rightarrow R^a_{jki} = 0.$$

from (2.1) we obtain

$$S_{ki} F_j - S_{ji} F_k + H_{ki} F_j - H_{ji} F_k - 2H_{jk} F_i = 0$$

and multiplying the previous relation by $g^{ki}$ it results that

$$g^{ki} S_{ki} F_j - g^{ki} S_{ji} F_k + g^{ki} H_{ki} F_j - g^{ki} H_{ji} F_k + 2g^{ki} H_{kj} F_i = 0.$$

Therefore

$$S^k_j F_k - r F_j = H^k_j F_k,$$

wherein by applying $F^j$ it results that $r = 0$. This way we conclude that the manifold is plain.

**Example**

Be Einstein compact Kaehlerian spaces $A_n = (M, \nabla)$ and $\overline{A}_n = (M, \overline{\nabla})$, with metric $g = (g_{ij})$ y $\overline{g} = (\overline{g}_{ij})$ holomorphic projective equivalences, get an expression that relates the scalar curvature $R$ and $\overline{R}$.

**Solution** Using [4],

$$S_{ij} = \overline{S}_{ij} + \tau (F_{ij} - \overline{F}_{ji}),$$
and as spaces of Einstein:

\[ S_{ij} = c_1 g_{ij}, \quad \bar{S}_{ij} = c_2 \bar{g}_{ij} \]

or

\[ S_{ij} = \frac{R}{n} g_{ij}, \quad \bar{S}_{ij} = \frac{R}{n} \bar{g}_{ij}, \]

it must be

\[ \frac{R}{n} g_{ij} = \frac{R}{n} \bar{g}_{ij} + \tau (F_{ij} - F_{ji}), \quad \tau \in \mathbb{C}, \]

Applying \( g^{ij} \) result

\[ R = \frac{R}{n} g^{ij} \bar{g}_{ij} + \tau (g^{ij} F_{ij} - g^{ij} F_{ji}), \]

\[ R = \frac{R}{n} g^{ij} \bar{g}_{ij} + \tau (\|F\| - g^{ij} F_{ji}) \]

or

\[ R g_{ij} = \frac{R}{n} \bar{g}_{ij} + \tau (\|F\| g_{ij} - F_{ij}) \]

Applying now \( \bar{g}^{ij} \),

\[ R g_{ij} = \frac{R}{n} \bar{g}^{ij} + \tau (\|F\| g_{ij} - \|F\| \bar{g}_{ij}), \]

from here

\[ (R - \tau \|F\|) g_{ij} = (\bar{R} - \tau \|F\|) \bar{g}_{ij}, \]

then

\[ \frac{(R - \tau \|F\|)}{(\bar{R} - \tau \|F\|)} = \frac{\text{det}(\bar{g}_{ij})}{\text{det}(g_{ij})} \]

In [3] proof

\[ \ln \sqrt{\frac{\text{det}(\bar{g}_{ij})}{\text{det}(g_{ij})}} = (n + 2)h, \quad h \in C^\infty(M). \]

Then he concludes

\[ (R - \tau \|F\|) = (\bar{R} - \tau \|F\|) \exp[2h(n + 2)]. \]

Example
Get an expression that compute the tensor Ricci in a compact kahlerian manifolds admitting proyective holomorphic transformations with associated vector $F$, if $A_n$ this is recurrent.

**Solution** In this case
\[
\nabla_k R^h_{lji} = R^h_{lji} F_k,
\]
by making the contraction $a = j$ and adding up from 1 to $n$ we obtain
\[
(2.6) \quad \nabla_k S_{ji} = S_{ji} F_k.
\]
But
\[
\nabla_k S_{ji} = \partial_k(S_{ji}) - \Gamma^a_{kj} S_{ai} - \Gamma^a_{ki} S_{ja}.
\]
Applying $g^{ji}$ result
\[
(2.7) \quad \nabla_k S_{ji} = \partial_k(S_{ji}).
\]
From (2.6) and (2.7) results
\[
\partial_k(S_{ji}) - S_{ji}\partial_k f = 0.
\]
The solution of this partial differential equation is the tensor de Ricci, hence is obtained scalar curvature.

**References**


Richard Malavé Guzmán
Departamento de Electricidad,
Universidad Politécnica Territorial del oeste de
Sucre Clodosbaldo Russian,
6101 Cumaná, Edo. Sucre,
Venezuela
e-mail : rmalaveg@gmail.com

Franmary López
Departamento de Electricidad,
Universidad Politécnica Territorial del oeste de
Sucre Clodosbaldo Russian,
6101 Cumaná, Edo. Sucre,
Venezuela
e-mail : franmalopezv@hotmail.com

and

Rodrigo Martínez
Departamento de Matemática,
Universidad de Oriente,
6101 Cumaná, Edo. Sucre,
Venezuela
e-mail : yigo54@cantv.net