On the classification of hypersurfaces in Euclidean spaces satisfying $L_r \overrightarrow{H}_{r+1} = \lambda \overrightarrow{H}_{r+1}$

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Abstract

In this paper, we study isometrically immersed hypersurfaces of the Euclidean space $E^{n+1}$ satisfying the condition $L_r \overrightarrow{H}_{r+1} = \lambda \overrightarrow{H}_{r+1}$ for an integer $r$ ($0 \leq r \leq n - 1$), where $\overrightarrow{H}_{r+1}$ is the $(r+1)$th mean curvature vector field on the hypersurface, $L_r$ is the linearized operator of the first variation of the $(r+1)$th mean curvature of hypersurface arising from its normal variations. Having assumed that on a hypersurface $x : M^n \to E^{n+1}$, the vector field $\overrightarrow{H}_{r+1}$ be an eigenvector of the operator $L_r$ with a constant real eigenvalue $\lambda$, we show that, $M^n$ has to be an $L_r$-biharmonic, $L_r$-1-type, or $L_r$-null-2-type hypersurface. Furthermore, we study the above condition on a well-known family of hypersurfaces, named the weakly convex hypersurfaces (i.e. on which principal curvatures are nonnegative). We prove that, any weakly convex Euclidean hypersurface satisfying the condition $L_r \overrightarrow{H}_{r+1} = \lambda \overrightarrow{H}_{r+1}$ for an integer $r$ ($0 \leq r \leq n - 1$), has constant mean curvature of order $(r+1)$. As an interesting result, we have that, the $L_r$-biharmonicity condition on the weakly convex Euclidean hypersurfaces implies the $r$-minimality.

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1. Introduction

The biharmonic functions as the solution of some well-known partial differential equations frequently appear in mathematical physics. Especially, when it becomes very difficult to find harmonic maps, sometimes the biharmonic ones are helpful. From geometric points of view, the role of biharmonic surfaces in elasticity and fluid mechanics can be considered as a physical motivation for the theory of biharmonicity. From the differential geometric points of view, B.Y. Chen (in the eighties) has started to investigate the properties of biharmonic submanifolds in the Euclidean spaces (whose position vector filed $x : M^n \to E^{n+k}$ satisfies the condition $\Delta^2 x = 0$, where $\Delta$ is the Laplace operator). He introduced some open problems and conjectures in [5], among them, a longstanding conjecture says that a biharmonic submanifold in a Euclidean space is a minimal one. Chen himself has proved the conjecture for surfaces in $E^3$. Later on, I. Dimitrić has verified Chen’s conjecture in several different cases such as special curves, submanifolds of constant mean curvature and also, hypersurfaces of the Euclidean spaces with at most two distinct principal curvatures. T. Hasanis and T. Vlachos in [10] proved the conjecture for hypersurfaces in $E^4$. Having assumed the completeness, Akutagawa and Maeta ([1]) gave an affirmative answer to the global version of Chen’s conjecture for biharmonic submanifolds in Euclidean spaces. Recently, in [8], it is proved that only biharmonic hypersurfaces in space forms with three distinct principal curvatures are minimal ones. An equivalent condition for the biharmonicity of an Euclidean hypersurfaces can be expressed as $\Delta \overrightarrow{H} = 0$, where $\overrightarrow{H}$ is the mean curvature vector field on the hypersurface. In 1988, Chen has started the study of a natural extension of this condition by assuming $\overrightarrow{H}$ to be an eigenvector of $\Delta$ associated to an arbitrary constant real eigenvalue. In [6], Defever has proved that the hypersurfaces of $E^4$ satisfying the condition $\Delta \overrightarrow{H} = \lambda \overrightarrow{H}$ have constant mean curvature.

On the other hand, the Laplacian operator $\Delta$ can be seen as the first one of a sequence of $n$ operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, where $L_r$ stands for the linearized operator of the first variation of the $(r+1)$th mean curvature arising from normal variations of the hypersurface (see, for instance, [2]). These operators are given by $L_r(f) = \text{tr}(P_r \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where $P_r$ denotes the $r$th Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the hessian of $f$. In this paper we consider the Euclidean hypersurfaces satisfying $L_r \overrightarrow{H}_{r+1} = \lambda \overrightarrow{H}_{r+1}$, where
$\overrightarrow{H}_{r+1}$ is the $(r+1)$th mean curvature vector field the Euclidean hypersurface $M$. From this point of view, as an extension of finite type theory, S.M.B. Kashani ([11]) has introduced the notion of $L_r$-finite type hypersurface in the Euclidean space, which can be found in the second edition of Chen’s book [4]. Furthermore, In [3], it is proved that every $L_r$-biharmonic hypersurface in $E^m$ (for arbitrary integer $m > 2$) with at most two distinct principal curvatures is $r$-minimal, $0 < r < m$.

In this paper, we try to classify the Euclidean hypersurfaces satisfying $L_r \overrightarrow{H}_{r+1} = \lambda \overrightarrow{H}_{r+1}$. Also, we study this condition together with the weak convexity. Here are our main results:

**Theorem 1.1.** If $x : M^n \to E^{n+1}$ is an isometric immersion of a hypersurface into Euclidean space, then the $(r+1)$th mean curvature vector field $\overrightarrow{H}_{r+1}$ is an eigenvector of $L_r$ if and only if it satisfies one of the following families:
(a) $L_r$-biharmonic hypersurfaces,
(b) $L_r$-1-type hypersurfaces,
(c) $L_r$-null-2-type hypersurfaces.

**Theorem 1.2.** Let $x : M^n \to E^{n+1}$ be an isometrically immersed Euclidean hypersurface satisfying $L_{n-1} \overrightarrow{H}_{n} = \lambda \overrightarrow{H}_{n}$, then $H_n$ is constant. Moreover, if $\lambda = 0$ then $M^n$ is $n$-minimal or ordinary minimal.

**Theorem 1.3.** Let $x : M^n \to E^{n+1}$ be a weakly convex hypersurface satisfying $L_r \overrightarrow{H}_{r+1} = \lambda \overrightarrow{H}_{r+1}$. Then the $(r+1)$th mean curvature is constant.

**Theorem 1.4.** Assume that $x : M^n \to E^{n+1}$ is a weakly convex $L_r$-biharmonic hypersurface in $E^{n+1}$, i.e. $L_r^2 x = 0$. Then $H_{r+1} = 0$

2. Preliminaries

In this section we recall some prerequisites about Newton transformations $P_r$ and their associated second order differential operators $L_r$ from [2].

Let $x : M^n \to E^{n+1}$ be an isometrically immersed hypersurface in the Euclidean space, with the Gauss map $N$. We denote by $\nabla^0$ and $\nabla$ the Levi-Civita connections on $E^{n+1}$ and $M$, respectively, then, the basic Gauss and Weingarten formulae of the hypersurface are written as

$$\nabla^0_X Y = \nabla_X Y + <SX,Y>N$$
and

\[ SX = -\nabla_X^0 N \]

for all tangent vector fields \( X, Y \in \chi(M) \), where \( S : \chi(M) \to \chi(M) \) is the shape operator (or Weingarten endomorphism) of \( M \) with respect to the Gauss map \( N \). As is well known, \( S \) defines a self-adjoint linear operator on each tangent plane \( T_pM \), and its eigenvalues \( \lambda_1(p), \ldots, \lambda_n(p) \) are the principal curvatures of the hypersurface. Associated to the shape operator there are \( n \) algebraic invariants given by

\[ s_r(p) = \sigma_r(\lambda_1(p), \ldots, \lambda_n(p)), \quad 1 \leq r \leq n, \]

where \( \sigma_r : \mathbb{R}^n \to \mathbb{R} \) is the elementary symmetric function in \( \mathbb{R}^n \) given by

\[ \sigma_r(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}. \]

Observe that the characteristic polynomial of \( S \) can be written in terms of the \( s_r \) as

\[ Q_s(t) = det(tI - S) = \sum_{r=0}^{n} (-1)^r s_r t^{n-r}, \quad (2.1) \]

where \( s_0 = 1 \) by definition. Then for any integer \( r \in \{0, 1, \ldots, n-1\} \), we introduce \( r \)th mean curvature function \( H_r \) and \((r + 1)\)th mean curvature vector field \( \overrightarrow{H}_{r+1} \) as follows:

\[ \left( \begin{array}{c} n \\ r \end{array} \right) H_r = s_r, \quad \overrightarrow{H}_{r+1} = H_{r+1} N. \]

In particular, when \( r = 1 \)

\[ H_1 = \frac{1}{n} \sum_{i=1}^{n} \lambda_i = \frac{1}{n} tr(S) = H \]

is nothing but the mean curvature of \( M \), which is the main extrinsic curvature of the hypersurface. On the other hand, \( H_n = \lambda_1 \cdots \lambda_n \) is called the Gauss-Kronecker curvature of \( M \). A hypersurface with zero \((r + 1)\)th mean curvature in \( \mathbb{E}^{n+1} \) is called \( r \)-minimal (see [14]).

The classical Newton transformations \( P_r : \chi(M) \to \chi(M) \) are defined inductively by

\[ P_0 = I \quad \text{and} \quad P_r = s_r I - S \circ P_{r-1} = \left( \begin{array}{c} n \\ r \end{array} \right) H_r I - S \circ P_{r-1} \]
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for every $r = 1, \ldots, n$ where $I$ denotes the identity in $\chi(M)$.

Equivalently,

\begin{equation}
P_r = \sum_{j=0}^{r} (-1)^j s_{r-j} S^j = \sum_{j=0}^{r} (-1)^j \binom{n}{r-j} H_{r-j} S^j.
\end{equation}

Note that by the Cayley-Hamilton theorem stating that any operator $T$ is annihilated by its characteristic polynomial, we have $P_n = 0$ from (2.1).

Each $P_r(p)$ is also a self-adjoint linear operator on the tangent space $T_p M$ which commutes with $S(p)$. Indeed, $S(p)$ and $P_r(p)$ can be simultaneously diagonalized: if $\{E_1, \ldots, E_n\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\lambda_1(p), \ldots, \lambda_n(p)$, respectively, then they are also the eigenvectors of $P_r(p)$ with corresponding eigenvalues given by

\begin{equation}
\mu_{i,r}(p) = \sum_{i_1 < \cdots < i_r, i_r \neq i} \lambda_{i_1}(p) \cdots \lambda_{i_r}(p),
\end{equation}

for every $1 \leq i \leq n$.

Associated to each Newton transformation $P_r$, we consider the second-order linear differential operator $L_r : C^\infty(M) \to C^\infty(M)$ given by

$L_r(f) = tr(P_r \circ \nabla^2 f)$.

Here, $\nabla^2 f : \chi(M) \to \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of $f$ and is given by

$\langle \nabla^2 f(X), Y \rangle = \langle \nabla X(\nabla f), Y \rangle$, $X, Y \in \chi(M)$.

3. Hypersurfaces in Euclidean spaces satisfying $L_r H_{r+1} = \lambda H_{r+1}$

First, we recall the definition of an $L_r$-finite type hypersurface from [11], which is the basic notion of the paper.

**Definition 3.1.** An isometrically immersed hypersurface $x : M^n \to \mathbb{E}^{n+1}$ is said to be of $L_r$-finite type if $x$ has a finite decomposition $x = \sum_{i=0}^{m} x_i$, for some positive integer $m$ satisfying the condition that $L_r x_i = \kappa_i x_i$, $\kappa_i \in \mathbb{R}$, $1 \leq i \leq m$, where $x_i : M^n \to \mathbb{E}^{n+1}$ are smooth maps, $1 \leq i \leq m$, and $x_0$ is constant. If all $\kappa_i$’s are mutually different, $M^n$ is said to be of $L_r$-m-type. An $L_r$-m-type hypersurface is said to be null if some $\kappa_i ; 1 \leq i \leq m$, is zero.
Let \( x : M^n \to E^{n+1} \) be a connected orientable hypersurface immersed into Euclidean space, with Gauss map \( N \). Then, as is well known (see [2]),

\[
L_r x = c_r \overrightarrow{H}_{r+1},
\]

where \( c_r = (n-r)(n) \). This shows, in particular, that \( M^n \) is an \( r \)-minimal hypersurface of \( E^{n+1} \) if and only if its coordinate functions are \( L_r \)-harmonic (i.e., if they are eigenfunctions with eigenvalue 0):

\[
\overrightarrow{H}_{r+1} = 0 \iff L_r x = 0.
\]

Condition (3.2) can be generalized in several directions. In [13] and inspired by Takahashi theorem, the first author jointly with Kashani studied and classified hypersurfaces in Euclidean spaces for which

\[
L_r x = \lambda x; \quad \lambda \in \mathbb{R},
\]

that is, hypersurfaces for which all coordinate functions are eigenfunctions of \( L_r \) with the same eigenvalue \( \lambda \). In terms of \( L_r \)-finite type theory, condition (3.3) characterizes the \( L_r \)-1-type hypersurfaces of \( E^{n+1} \). In [13], the authors showed that \( r \)-minimal hypersurfaces and open parts of hyperspheres are the only \( L_r \)-1-type Euclidean hypersurfaces.

Most recently, condition (3.2) generalized in another direction by Aminian and Kashani([3]), they studied the hypersurfaces of \( E^{n+1} \) satisfying

\[
L_r \overrightarrow{H}_{r+1} = 0 \iff L_r^2 x = 0.
\]

Hypersurfaces of \( E^{n+1} \) satisfying (3.4) called \( L_r \)-biharmonic hypersurfaces. Conditions (3.3) and (3.4) may be generalized and combined into the

\[
L_r \overrightarrow{H}_{r+1} = \lambda \overrightarrow{H}_{r+1}, \quad \lambda \in \mathbb{R}.
\]

Theorem 1.1 determines hypersurfaces of \( E^{n+1} \) which satisfy \( L_r \overrightarrow{H}_{r+1} = \lambda \overrightarrow{H}_{r+1} \) for some \( \lambda \in \mathbb{R} \).

Proof of Theorem 1.1. Under the hypothesis, assume that \( L_r \overrightarrow{H}_{r+1} = \lambda \overrightarrow{H}_{r+1} \) holds for some real number \( \lambda \). If \( \lambda = 0 \), then \( M^n \) is a \( L_r \)-biharmonic hypersurface, which gives (a). Now, assume that \( L_r \overrightarrow{H}_{r+1} = \lambda \overrightarrow{H}_{r+1} \) with \( \lambda \neq 0 \). Taking

\[
x_p = \frac{1}{\lambda} L_r x \quad \text{and} \quad x_0 = x - x_p,
\]

where \( x_p \) is a point of intersection of \( \overrightarrow{H}_{r+1} = 0 \) and the hypersurface \( M^n \). Then, for some positive integer \( k \), we have

\[
L_r x_p = \frac{1}{\lambda} L_r x + k \overrightarrow{H}_{r+1} = 0.
\]

This contradicts the hypothesis, and hence \( L_r x = \lambda x \) for some \( \lambda \in \mathbb{R} \).
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we find

\[ L_r x_p = \frac{1}{\lambda} L_r^2 x = \frac{c_r}{\lambda} L_r \overrightarrow{H}_{r+1} = c_r \overrightarrow{H}_{r+1} = L_r x. \]

Hence, \( M \) is either of \( L_r \)-1-type or of \( L_r \)-null-2-type, depending on \( x_0 \) is a constant or non-constant. Conversely, if \( M \) is \( L_r \)-biharmonic or \( L_r \)-null-2-type hypersurface, then \( L_r^2 x = \lambda x; \lambda \in R \), so formula (3.1) gives the result. If \( M \) is \( L_r \)-1-type hypersurface, then \( L_r x = \lambda x; \lambda \in R \), so by taking \( L_r \) of this equation and using (3.1) we get the result. \( \square \)

By formulae in [2] page 122, we have

\[ L_r^2 x = -c_r \left( \frac{n}{r+1} \right) H_{r+1} \nabla H_{r+1} - 2(S \circ P_r)(\nabla H_{r+1}) \]

\[ -c_r \left( \frac{n}{r+1} \right) H_{r+1} \left( n H_1 H_{r+1} - (n - r - 1) H_{r+2} - L_r H_{r+1} \right) \n. \]

(3.6)

By identifying normal and tangent parts of (3.6), one obtains necessary and sufficient conditions for the \((r+1)\)th mean curvature vector field \( \overrightarrow{H}_{r+1} \) be an eigenvector of \( L_r \), namely

\[ L_r H_{r+1} = \left( \frac{n}{r+1} \right) H_{r+1} (n H_1 H_{r+1} - (n - r - 1) H_{r+2}) = \lambda H_{r+1} \]

(3.7)

and

\[ (S \circ P_r)(\nabla H_{r+1}) = -\frac{1}{2} \left( \frac{n}{r+1} \right) H_{r+1} \nabla H_{r+1}. \]

(3.8)

Since \( P_n = 0, \) \( n = \dim M; \) \( S \circ P_{n-1} = H_n I \), by equations (3.7) and (3.8), hence one leads to consider the case \( r = n - 1 \), at first. Here we prove Theorem 1.2.

Proof of Theorem 1.2. By (3.8) we have

\[ (S \circ P_{n-1})(\nabla H_n) = -\frac{1}{2} H_n \nabla H_n. \]
We know that \( P_n = 0 \), hence \( S \circ P_{n-1} = H_n I \). So \( \frac{3}{2} \nabla H^2_n = 0 \). Therefore \( H_n \) is constant.
If \( \lambda = 0 \) and \( H_n \neq 0 \), by using (3.7) we obtain that \( H = 0 \). □

4. Weakly convex hypersurfaces in Euclidean spaces

Recently, in [12], the ordinary biharmonicity condition is verified on the hypersurfaces of space forms of nonpositive sectional curvature with an additional condition named weak convexity. A hypersurfaces of a space form is said to be weakly convex if all of its principal curvatures be non-negative. Here, we study the \( L_r \)-biharmonicity condition and in general \( L_r H_{r+1} = \lambda H_{r+1} \) on weakly convex Euclidean hypersurfaces. We prove the Theorem 1.3.

Proof of Theorem 1.3.

Define
\[
B := \{ p \in M : \nabla H^2_{r+1}(p) \neq 0 \}.
\]

We will prove that \( B \) is an empty set by a contradiction argument, and so \((r+1)\)th mean curvature is constant and we are done. We choose a local orthonormal frame \( \{ E_1, \ldots, E_n \} \) such that \( S(E_i) = \lambda_i E_i \) and \( P_r(E_i) = \mu_{i,r} E_i \), where \( \lambda_i, s \) and \( \mu_{i,r}, s \) are eigenvalues of \( S \) and \( P_r \), respectively, \( 1 \leq i \leq n \), which are nonnegative by the assumption that \( M^n \) is weakly convex.

We have \( \nabla H_{r+1} = \sum_{i=1}^{n} < \nabla H_{r+1}, E_i > E_i \), so (3.8) is equivalent to

\[
< \nabla H_{r+1}, E_i > \left( \lambda_i \mu_{i,r} + \frac{1}{2} \left( \begin{array}{c} n \\ r + 1 \end{array} \right) H_{r+1} \right) = 0, \quad \text{on } B,
\]

for every \( i = 1, \ldots, n \). Therefore, for every \( i \) such that \( < \nabla H_{r+1}, E_i > \neq 0 \) on \( B \) we get

\[
(\lambda_i \mu_{i,r} + \frac{1}{2} \left( \begin{array}{c} n \\ r + 1 \end{array} \right) H_{r+1}) = 0, \quad \text{on } B.
\]

So by the assumption that \( M^n \) is weakly convex, we obtain that \( H_{r+1} = 0 \) locally on \( B \), which is a contradiction with the definition of \( B \). This finishes the proof. □

Using the idea of the last proof, we prove Theorem 1.4 as follows.

Proof of Theorem 1.4. By Theorem 1.3, the \((r + 1)\)th mean curvature \( H_{r+1} \) is constant. It is always true that

\[
H_{i-1} H_{i+1} \leq H_i^2
\]
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and

$$H_1 \geq H_2^{1/2} \geq H_3^{1/3} \geq \cdots \geq H_i^{1/i} \quad (1 \leq i < n),$$

provided $H_1, H_2, \ldots, H_i$ are nonnegative, [page 52 of [9]].

Then, from these above inequalities, we obtain

$$HH_{r+1} - H_{r+2} \geq \frac{H_{r+1}}{H_r}(HH_r - H_{r+1}) \geq \frac{H_{r'+1}}{H_r}(HH_r - H_{r'+1}) \geq H_{r+1}(H - H_r^2) \geq 0.$$

(4.2)

And the other hand, since $H_{r+1}$ is a constant and $M^n$ is $L_r$-biharmonic, by using formula (3.7) we get

$$nHH_{r+1} = (n - r - 1)H_{r+2},$$

so, when $r = n - 1$, we have $H = 0$ therefore from the above inequalities, we get $H_n = 0$. When $r < n - 1$, formula (4.2) and this above equation we get $H_{r+1} = 0$. \qed

References


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