Vertex equitable labeling of union of cyclic snake related graphs

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Received : May 2015. Accepted : March 2016

Abstract

Let $G$ be a graph with $p$ vertices and $q$ edges and $A = \{0, 1, 2, \ldots, \lceil \frac{q}{2} \rceil \}$. A vertex labeling $f : V(G) \rightarrow A$ induces an edge labeling $f^*$ defined by $f^*(uv) = f(u) + f(v)$ for all edges $uv$. For $a \in A$, let $v_f(a)$ be the number of vertices $v$ with $f(v) = a$. A graph $G$ is said to be vertex equitable if there exists a vertex labeling $f$ such that for all $a$ and $b$ in $A$, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$. In this paper, we prove that key graph $KY(m,n)$, $P(2,QS_n)$, $P(m,QS_n)$, $C(n,QS_m)$, $NQ(m)$ and $K_{1,n} \times P_2$ are vertex equitable graphs.

Keywords : Vertex equitable labeling, vertex equitable graph, comb graph, key graph, path union graph, quadrilateral snake graph.

AMS Subject Classification : 05C78.
1. Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling and a detailed survey of graph labeling can be found in [2]. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. The concept of vertex equitable labeling was due to Lourdusamy et al. and further studied in [3]-[10]. Let $G$ be a graph with $p$ vertices and $q$ edges and $A = \{0, 1, 2, \ldots, \left\lfloor \frac{q}{2} \right\rfloor\}$. A graph $G$ is said to be vertex equitable if there exists a vertex labeling $f : V(G) \rightarrow A$ induces an edge labeling $f^*$ defined by $f^*(uv) = f(u) + f(v)$ for all edges $uv$ such that for all $a$ and $b$ in $A$, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$, where $v_f(a)$ be the number of vertices $v$ with $f(v) = a$ for $a \in A$. The vertex labeling $f$ is known as vertex equitable labeling. A graph $G$ is said to be a vertex equitable if it admits vertex equitable labeling. In this paper, we extend our study on vertex equitable labeling and prove that key graph $KY(m, n), P(2QS_n), P(mQS_n), C(nQS_m), NQ(m)$ and $K_{1,n} \times P_2$ are vertex equitable graphs. In [3], it is proved that the comb graph $P_n \odot K_1$ is a vertex equitable graph. In the following theorem we give an another vertex equitable labeling for the same graph $P_n \odot K_1$.

**Theorem 1.1.** The comb graph $P_n \odot K_1$ is a vertex equitable graph.

**Proof.** Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(P_n \odot K_1) = \{u_i, v_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n - 1\}$. Here $|V(P_n \odot K_1)| = 2n$ and $E(P_n \odot K_1) = 2n - 1$. Let $A = \{0, 1, 2, \ldots, \left\lfloor \frac{2n-1}{2} \right\rfloor\}$.

Define a vertex labeling $f : V(P_n \odot K_1) \rightarrow A$ as follows:

**Case (i).** When $n$ is even.

$f(u_{2i-1}) = 2(i - 1), f(u_{2i}) = 2i, f(v_{2i-1}) = f(v_{2i}) = 2i - 1$ if $1 \leq i \leq \frac{n}{2}$.

**Case (ii).** When $n$ is odd.

$f(u_{2i-1}) = 2i - 1, f(u_{2i}) = 2(i - 1)$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, f(v_{2i}) = 2i, f(u_{2i}) = 2i - 1$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$. It can be verified that the induced edge labels of $P_n \odot K_1$ are $1, 2, \ldots, 2n - 1$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence $P_n \odot K_1$ is a vertex equitable graph. □

We use the following theorem and definitions in the subsequent section.
Theorem 1.2. [3] The cycle $C_n$ is a vertex equitable graph if and only if $n \equiv 0$ or $3(\text{mod} \ 4)$.

Theorem 1.3. [7] The $kC_4$-snake is a vertex equitable graph.

Theorem 1.4. [4] Let $G_1(p_1,2n+1)$ and $G_2(p_2,q_2)$ be any two vertex equitable graphs with equitable labeling $f$ and $g$ respectively. Let $u$ and $v$ be the vertices of $G_1$ and $G_2$ respectively such that $f(u) = n+1$ and $g(v) = 0$. Then the graph $G$ obtained by joining $u$ and $v$ by an edge is a vertex equitable graph.

Theorem 1.5. [9] Let $G_1(p_1,q), G_2(p_2,q), \ldots, G_m(p_m,q)$ be the vertex equitable graphs with $q$ is odd and $u_i,v_i$ be the vertices of $G_i$ for $1 \leq i \leq m$ labeled by $0$ and $\left\lfloor \frac{q}{2} \right\rfloor$. Then the graph $G$ obtained by joining $v_1$ with $u_2$ and $v_2$ with $u_3$ and $v_3$ with $u_4$ and so on until we join $v_{m-1}$ with $u_m$ by an edge is also a vertex equitable graph.

Definition 1.6. Let $NQ(m)$ be the $n$th quadrilateral snake obtained from the path $u_1,u_2,\ldots,u_m$ by joining $u_i,u_{i+1}$ with $2n$ new vertices $v^i_j$ and $w^i_j$, $1 \leq i \leq m-1, 1 \leq j \leq n$.

Definition 1.7. The key graph is a graph obtained from $K_2$ by appending one vertex of $C_m$ to one end point and comb graph $P_n \odot K_1$ to the other end of $K_2$. It is denoted as $KY(m,n)$.

Definition 1.8. [11] Let $G_1,G_2,\ldots,G_n$, $n \geq 2$ be $n$ graphs and $u_i$ be a vertex of $G_i$ for $1 \leq i \leq n$. The graph obtained by adding an edge between $u_i$ and $u_{i+1}$ for $1 \leq i \leq n-1$ is called a path union of $G_1,G_2,\ldots,G_n$ and is denoted by $P(G_1,G_2,\ldots,G_n)$. When all the $n$ graphs are isomorphic to a graph $G$, it is denoted by $P(n.G)$.

Definition 1.9. Let $G_1,G_2,\ldots,G_n$, be $n$ graphs and $u_i$ be a vertex of $G_i$ for $1 \leq i \leq n$. The graph obtained by adding an edge between $u_i$ and $u_{i+1}(1 \leq i \leq n-1)$, $u_n$ and $u_1$ is called a cycle union of $G_1,G_2,\ldots,G_n$ and is denoted by $C(G_1,G_2,\ldots,G_n)$. When all the $n$ graphs are isomorphic to a graph $G$, it is denoted by $C(n.G)$.

2. Main Results

Theorem 2.1. The key graph $KY(m,n)$ is a vertex equitable graph if $m \equiv 0$ or $3(\text{mod} \ 4)$. 
Proof. 
Case(i). \( m \equiv 3(\text{mod} \ 4) \).
Let \( G_1 = C_m, G_2 = P_n \odot K_1 \). Since \( G_1 \) has \( m \) edges and \( G_2 \) has \( 2n - 1 \) edges, by Theorem 1.1 and Theorem 1.2, \( P_n \odot K_1, C_m \) are vertex equitable graphs. Hence, by Theorem 1.4, \( KY(m, n) \) is a vertex equitable graph.

Case(ii). \( m \equiv 0(\text{mod} \ 4) \).
Let \( G_1 = P_n \odot K_1, G_2 = C_m \). Since \( G_1 \) has \( 2n - 1 \) edges and \( G_2 \) has \( m \) edges, by Theorem 1.1 and Theorem 1.2, \( P_n \odot K_1 \) and \( C_n \) are vertex equitable graphs. Hence by Theorem 1.4, \( KY(m, n) \) is a vertex equitable graph. \( \square \)

An example for the vertex equitable labeling of \( KY(7, 5) \) is shown in Figure 1.

![Figure 1](attachment:image.png)

Theorem 2.2. The path union graph \( P(2.QS_n) \) is a vertex equitable graph.

Proof. Let \( V(P(2.QS_n)) = \{u_i, v_{ij}, w_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\} \) and \( E(P(2.QS_n)) = \{u_1u_2, u_iv_{i1}, u_iw_{i1} : 1 \leq i \leq 2\} \cup \{u_{ij}v_{ij}, u_{ij}w_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\} \cup \{u_{ij}v_{ij+1}, u_{ij}w_{ij+1} : 1 \leq i \leq 2, 1 \leq j \leq n-1\} \). Here \( |V(P(2.QS_n))| = 6n + 2 \) and \( |E(P(2.QS_n))| = 8n + 1 \). Let \( A = \{0, 1, 2, \ldots, \lceil \frac{8n+1}{2} \rceil \} \).

Define a vertex labeling \( f : V((P(2.QS_n))) \rightarrow A \) as follows:
\[
f(u_1) = 0, f(u_2) = \lfloor \frac{8n+1}{2} \rfloor.
\]
For $1 \leq j \leq n$, $f(u_{1j}) = f(w_{1j}) = 2j$, $f(v_{1j}) = 2j - 1$, $f(v_{2j}) = f(w_{2j}) = \left\lceil \frac{8n+1}{2} \right\rceil - 2j$, $f(v_{2j}) = \left\lceil \frac{8n+1}{2} \right\rceil - 2j + 1$.

It can be verified that the induced edge labels of $P(2.QS_n)$ are $1, 2, \ldots, 8n+1$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence $P(2.QS_n)$ is a vertex equitable graph.

**Theorem 2.3.** The path union graph $P(m.QS_n)$ is a vertex equitable graph if $m > 2$.

**Proof.** Here $V(P(m.QS_n)) = m(3n + 1)$ and $E(P(m.QS_n)) = 4mn + m - 1$.

**Case(i).** $m$ is even.

Let $G_i = P(2.QS_n)$ for $1 \leq i \leq \frac{m}{2}$. By Theorem 2.2, $P(2.QS_n)$ is a vertex equitable graph. Since each $G_i$ has $8n + 1$ edges, by Theorem 1.5, $P(m.QS_n)$ admits vertex equitable labeling if $m$ is even.

**Case(ii).** $m$ is odd and take $m = 2k + 1$.

By Case (i) $P(2k.QS_n)$ is a vertex equitable graph. By Theorem 1.3, $nC_4$ snake is a vertex equitable graph. Let $G_1 = P(2k.QS_n)$ and $G_2 = nC_4$. Since $G_1$ has $8mn + 2m - 1$ edges, by Theorem 1.4, $P(2m + 1.QS_n)$ admits vertex equitable labeling.

An example for the vertex equitable labeling of the graph obtained by the path union of 4 copies of $3C_4$-snake is shown in Figure 2.

![Figure 2](image_url)
Theorem 2.4. The graph obtained by the cycle union of \( n \) copies of \( mC_4 \)-snake, \( C(n.QS_m) \) is a vertex equitable graph if \( n \equiv 0,3(\text{mod } 4) \).

Proof. Let \( V(C(n.QS_m)) = \{ u_i, u_{ij}, v_{ij}, w_{ij} : 1 \leq i \leq n, 1 \leq j \leq m \} \) and \( E(C(n.QS_m)) = \{ u_iu_{i+1} : 1 \leq i \leq n-1 \} \cup \{ u_nu_1 \} \cup \{ u_iv_{i1}, uw_{i1} : 1 \leq i \leq n \} \cup \{ u_{ij}v_{ij+1}, u_{ij}w_{ij+1} : 1 \leq i \leq n, 1 \leq j \leq m-1 \} \). Here \( |V(C(n.QS_m))| = 3mn + n \) and \( |E(C(n.QS_m))| = 4mn + n \). Let \( A = \{ 0, 1, 2, \ldots, \left[ \frac{4mn+n}{2} \right] \} \).

Define a vertex labeling \( f : V(C(n.QS_m)) \rightarrow A \) as follows:

Case (i). \( n \equiv 0(\text{mod } 4) \).
\[
f(u_{2i}) = (4m + 1)i \quad \text{if } 1 \leq i \leq \frac{n}{4},
\]
\[
f(u_{2i-1}) = \begin{cases} 
(4m + 1)(i - 1) & \text{if } 1 \leq i \leq \frac{n}{4}, \\
(4m + 1)(i - 1) + 1 & \text{if } \frac{n}{4} + 1 \leq i \leq \frac{n}{2}, 
\end{cases}
\]
For \( 1 \leq j \leq m \),
\[
f(v_{(2i-1)j}) = \begin{cases} 
(4m + 1)(i - 1) + 2j & \text{if } 1 \leq i \leq \frac{n}{4}, \\
(4m + 1)(i - 1) + (2j - 1) & \text{if } \frac{n}{4} + 1 \leq i \leq \frac{n}{2}, 
\end{cases}
\]
\[
f(v_{2ij}) = (4m + 1)i - 2j + 1 \quad \text{if } 1 \leq i \leq \frac{n}{4},
\]
\[
f(w_{(2i-1)j}) = \begin{cases} 
(4m + 1)(i - 1) + 2j - 1 & \text{if } 1 \leq i \leq \frac{n}{4}, \\
(4m + 1)(i - 1) + 2j & \text{if } \frac{n}{4} + 1 \leq i \leq \frac{n}{2}, 
\end{cases}
\]
\[
f(w_{2ij}) = (4m + 1)i - 2j \quad \text{if } 1 \leq i \leq \frac{n}{4},
\]
\[
f(u_{2i-1}) = (4m + 1)(i - 2j) \quad \text{if } 1 \leq i \leq \frac{n}{4},
\]
\[
f(u_{2i}) = (4m + 1)(i - 1) + 2j \quad \text{if } 1 \leq i \leq \frac{n}{4},
\]
\[
\]
Case (ii). \( n \equiv 3(\text{mod } 4) \).
\[
f(u_{2i}) = (4m + 1)(i - 1) + (2m + 1) \quad \text{if } 1 \leq i \leq \left[ \frac{n}{4} \right],
\]
\[
f(u_{2i-1}) = \begin{cases} 
(4m + 1)(i - 1) + 2m & \text{if } 1 \leq i \leq \left[ \frac{n}{4} \right], \\
(4m + 1)(i - 1) + (2m + 1) & \text{if } \left[ \frac{n}{4} \right] + 1 \leq i \leq \left[ \frac{n}{2} \right], 
\end{cases}
\]
For \( 1 \leq j \leq m \),
\[
f(u_{2ij}) = (4m + 1)(i - 1) + (2m + 1) + 2j \quad \text{if } 1 \leq i \leq \left[ \frac{n}{4} \right],
\]
\[
f(u_{2i-1}) = \begin{cases} 
(4m + 1)(i - 1) + 2m - 2j & \text{if } 1 \leq i \leq \left[ \frac{n}{4} \right], \\
(4m + 1)(i - 1) + 2m - 2j + 1 & \text{if } \left[ \frac{n}{4} \right] + 1 \leq i \leq \left[ \frac{n}{2} \right], 
\end{cases}
\]
\[
f(v_{(2i-1)j}) = \begin{cases} 
(4m + 1)(i - 1) + 2m - 2(j - 1) & \text{if } 1 \leq i \leq \left[ \frac{n}{4} \right], \\
(4m + 1)(i - 1) + 2m + 1 - 2j & \text{if } \left[ \frac{n}{4} \right] + 1 \leq i \leq \left[ \frac{n}{2} \right], 
\end{cases}
\]
\[
f(v_{2ij}) = (4m + 1)(i - 1) + 2m + 2j - 1 \quad \text{if } 1 \leq i \leq \left[ \frac{n}{4} \right],
\]
\[
f(v_{2i}) = (4m + 1)(i - 1) + 2m + 2j \quad \text{if } 1 \leq i \leq \left[ \frac{n}{4} \right],
\]
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\[
f(w_{2i-1}j) = \begin{cases} 
(4m + 1)(i - 1) + 2m - 2j + 1 & \text{if } 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil \\
(4m + 1)(i - 1) + 2m - 2(j - 1) & \text{if } \left\lceil \frac{n}{4} \right\rceil + 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil ,
\end{cases}
\]

\[
f(w_{2i}j) = \begin{cases} 
(4m + 1)(i - 1) + 2m + 2j & \text{if } 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil \\
(4m + 1)(i - 1) + 2m + 2j + 1 & \text{if } \left\lceil \frac{n}{4} \right\rceil + 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil .
\end{cases}
\]

It can be verified that the induced edge labels of $C(n.QS_m)$ are 1, 2, \ldots, $4mn + n$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence $C(n.QS_m)$ is a vertex equitable graph. \Box

An example for the vertex equitable labeling of the graph obtained by the cycle union of 7 copies of $2C_4$-snake is shown in Figure 3.

![Figure 3](image_url)

**Theorem 2.5.** The $n^{th}$ quadrilateral snake $NQ(m)$ is a vertex equitable graph if $n \geq 2$ is even.
Proof. Let \( V(NQ(m)) = \{u_i/1 \leq i \leq m\} \cup \{v_j^i/1 \leq i \leq m, 1 \leq j \leq n\}, E(NQ(m)) = \{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{w_j^i/1 \leq i \leq m-1, 1 \leq j \leq n\} \cup \{u_jw_j^i/2 \leq i \leq m, 1 \leq j \leq n\}. \) Clearly \( NQ(m) \) has \( 2(m-1)n+m \) vertices and \( 3(m-1)n+m+1 \) edges. Let \( A = \{0, 1, 2, \ldots, \left\lfloor \frac{3n(m-1)+m-1}{2} \right\rfloor \} \).

Define a vertex labeling \( f : V(NQ(m)) \rightarrow A \) as follows:

For \( 1 \leq i \leq m \), \( f(u_i) = \left\lfloor \frac{3n+1(i-1)}{2} \right\rfloor \),

For \( 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor, 1 \leq j \leq n \), \( f(v_j^{2i-1}) = (3n+1)(i-1) + j \),
\( f(v_j^{2i}) = (3n+1)(i-1) + \left\lceil \frac{3n+1}{2} \right\rceil + (j-1) \).

For \( 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil, 1 \leq j \leq \frac{n}{2} \), \( f(w_j^{2i-1}) = (3n+1)(i-1) + \left\lceil \frac{3n+1}{2} \right\rceil - 2j \),
\( f(w_j^{2i}) = (3n+1)(i-1) + \left\lceil \frac{3n+1}{2} \right\rceil - (2j-1) \).

For \( 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil, 1 \leq j \leq \frac{n}{2} \), \( f(w_j^{2i}) = (3n+1)i - (2j - 2) \).
\( f(w_j^{2i}) = (3n+1)i - (2j - 2) \).

It can be verified that the induced edge labels of \( NQ(m) \) are \( 1, 2, \ldots, 3(m-1)n+m-1 \) and \( |v_f(a) - v_f(b)| \leq 1 \) for all \( a, b \in A \). Hence \( NQ(m) \) is a vertex equitable graph.

An example for the vertex equitable labeling of \( 4Q(4) \) is shown in Figure 4.

![Figure 4](image-url)
Corollary 2.6. The book graph $K_{1,n} \times P_2$ is a vertex equitable graph.

References


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