Some geometric properties of lacunary Zweier Sequence Spaces of order $\alpha$

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Received : July 2016. Accepted : September 2016

Abstract

In this paper we introduce a new sequence space using Zweier matrix operator and lacunary sequence of order $\alpha$. Also we study some geometrical properties such as order continuous, the Fatou property and the Banach-Saks property of the new space.

Keywords and phrases : Lacunary sequence; Zweier operator; order continuous; Fatou property; Banach-Saks property

1. Introduction

Throughout the article, \( w, c, c_0 \) and \( \ell_\infty \) denotes the spaces of all, convergent, null and bounded sequences, respectively. Also, by \( \ell_1 \) and \( \ell_p \), we denote the spaces of all absolutely summable and \( p \)-absolutely summable series, respectively. Recall that a sequence \( (x(i))_{i=1}^\infty \) in a Banach space \( X \) is called Schauder (or basis) of \( X \) if for each \( x \in X \) there exists a unique sequence \( (a(i))_{i=1}^\infty \) of scalars such that \( x = \sum_{i=1}^\infty a(i)x(i) \), i.e. \( \lim_{n \to \infty} \sum_{i=1}^n a(i)x(i) = x \). A sequence space \( X \) with a linear topology is called a K-space if each of the projection maps \( P_i : X \to \mathbb{C} \) defined by \( P_i(x) = x(i) \) for \( x = (x(i))_{i=1}^\infty \in X \) is continuous for each natural \( i \). A Fréchet space is a complete metric linear space and the metric is generated by a F-norm and a Fréchet space which is a K-space is called an FK-space. In other words, \( X \) is an FK-space if \( X \) is a Fréchet space with continuous coordinatewise projections. All the sequence spaces mentioned above are FK-space except the space \( c_{00} \) which is the space of real sequences which have only a finite number of non-zero coordinates. An FK-space \( X \) which contains the space \( c_{00} \) is said to have the property AK if for every sequence \( (x(i))_{i=1}^\infty \in X, x = \sum_{i=1}^\infty x(i)e(i) \) where \( e(i) = (0,0,...,1^{i^{th}}\text{place},0,0,...) \).

A Banach space \( X \) is said to be a Köthe sequence space if \( X \) is a subspace of \( w \) such that

(a) if \( x \in w, y \in X \) and \( |x(i)| \leq |y(i)| \) for all \( i \in \mathbb{N}, \) then \( x \in X \) and \( ||x|| \leq ||y|| \)

(b) there exists an element \( x \in X \) such that \( x(i) > 0 \) for all \( i \in \mathbb{N} \).

We say that \( x \in X \) is order continuous if for any sequence \( (x_n) \in X \) such that \( x_n(i) \leq |x(i)| \) for all \( i \in \mathbb{N} \) and \( x_n(i) \to 0 \) as \( n \to \infty \) we have \( ||x_n|| \to 0 \) holds.

A Köthe sequence space \( X \) is said to be order continuous if all sequences in \( X \) are order continuous. It is easy to see that \( x \in X \) order continuous if and only if \( ||(0,0,\ldots,0,x(n+1),x(n+2),\ldots)|| \to 0 \) as \( n \to \infty \).

A Köthe sequence space \( X \) is said to be the Fatou property if for any real sequence \( x \) and \( (x_n) \) in \( X \) such that \( x_n \uparrow x \) coordinatewisely and \( \sup_n ||x_n|| < \infty \), we have that \( x \in X \) and \( ||x_n|| \to ||x|| \).
A Banach space $X$ is said to have the Banach-Saks property if every bounded sequence $(x_n)$ in $X$ admits a subsequence $(z_n)$ such that the sequence $(t_k(z))$ is convergent in $X$ with respect to the norm, where

$$t_k(z) = \frac{z_1 + z_2 + ... + z_k}{k} \text{ for all } k \in \mathbb{N}.$$  

Some of works on geometric properties of sequence space can be found in [3, 4, 8, 9, 13, 16, 17, 18, 19, 20, 22, 23].

S¸engönül [24] defined the sequence $y = (y_k)$ which is frequently used as the $Z^i$-transformation of the sequence $x = (x_k)$ i.e.

$$y_k = ix_k + (1 - i)x_{k-1}$$

where $x_{-1} = 0, k \neq 0, 1 < k < \infty$ and $Z^i$ denotes the matrix $Z^i = (z_{nk})$ defined by

$$z_{nk} = \begin{cases} 
  i, & \text{if } n = k; \\
  1 - i, & \text{if } n - 1 = k; \\
  0, & \text{otherwise.}
\end{cases}$$

S¸engönül [24] introduced the Zweier sequence spaces $Z$ and $Z_0$ as follows

$$Z = \{ x = (x_k) \in w : Z^i x \in c \}$$

and

$$Z_0 = \{ x = (x_k) \in w : Z^i x \in c_0 \}.$$  

For details on Zweier sequence spaces we refer to [5, 10, 11, 12, 14, 15].

2. Lacunary Zweier sequence spaces of order $\alpha$

by lacunary sequence we mean an increasing sequence $\theta = (k_r)$ of positive integers satisfying $k_0 = 0$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. We denote the intervals, by $I_r = (k_{r-1}, k_r]$, which determines $\theta$. Let $\alpha \in (0,1]$ be any real number and let $p$ be a positive real number such that $1 \leq p < \infty$. Now we define the following sequence space.

$$[Z^0_{\theta}]_\infty(p) = \left\{ x \in w : \sup_r \frac{1}{h_r^p} \sum_{k \in I_r} |(Z^i x)_k|^p < \infty \right\}.$$  

Special cases:
(a) For $p = 1$ we have $[Z_\theta^\alpha]_\infty(p) = [Z_\phi^\alpha]_\infty$.

(b) For $\alpha = 1$ and $p = 1$ we have $[Z_\theta^\alpha]_\infty(p) = [Z_\theta]_\infty$.

For details on sequence spaces of order $\alpha$ we refer to [1, 2, 6, 7].

**Theorem 2.1.** Let $\alpha \in (0, 1]$ and $p$ be a positive real number such that $1 \leq p < \infty$. Then the sequence space $[Z_\theta^\alpha]_\infty(p)$ is a BK-space normed by

\[
\|x\|_\alpha = \sup_r \frac{1}{h_r^\alpha} \left( \sum_{k \in I_r} |(Z^i x)_k|^p \right)^\frac{1}{p}.
\]

**Proof.** Since the matrix $Z^i$ is a triangle, we have the result by norm (2.1) and the Theorem 4.3.12 of Wilansky [25], p. 63. $\square$ $[Z_\theta^\alpha]_\infty \subset [Z_\theta^\beta]_\infty(p)$.

**Theorem 2.2.** Let $\alpha$ and $\beta$ be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $p$ be a positive real number such that $1 \leq p < \infty$. Then $[Z_\theta^\alpha]_\infty(p) \subset [Z_\phi^\beta]_\infty(p)$.

**Proof.** The proof of theorem follows from the following inequality. For all $r \in \mathbb{N}$ we have is straightforward, so omitted.

\[
\frac{1}{h_r^\alpha} \sum_{k \in I_r} |(Z^i x)_k|^p \leq \frac{1}{h_r^\beta} \sum_{k \in I_r} |(Z^i x)_k|^p.
\]

$\square$

**Theorem 2.3.** Let $\alpha$ and $\beta$ be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $p$ be a positive real number such that $1 \leq p < \infty$. For two lacunary sequences $\theta = (h_r)$ and $\phi = (l_r)$ for all $r$, then $[Z_\theta^\alpha]_\infty(p) \subset [Z_\phi^\beta]_\infty(p)$ if and only if $\sup_r \left( \frac{h_r^\alpha}{l_r^\beta} \right) < \infty$.

**Proof.** Let $x = (x_k) \in [Z_\theta^\alpha]_\infty(p)$ and $\sup_r \left( \frac{h_r^\alpha}{l_r^\beta} \right) < \infty$. Then

\[
\sup_r \frac{1}{h_r^\alpha} \sum_{k \in I_r} |(Z^i x)_k|^p < \infty
\]
and there exists a positive number $K$ such that $h^\alpha_r \leq K l^\beta_r$ and so that \( \frac{1}{l^\alpha_r} \leq \frac{K}{h^\beta_r} \) for all $r$. Therefore, we have

\[
\frac{1}{l^\alpha_r} \sum_{k \in I_r} |(Z^i x)_k|^p \leq \frac{K}{h^\beta_r} \sum_{k \in I_r} |(Z^i x)_k|^p.
\]

Now taking supremum over $r$, we get

\[
\sup_r \frac{1}{l^\alpha_r} \sum_{k \in I_r} |(Z^i x)_k|^p \leq \sup_r \frac{K}{h^\beta_r} \sum_{k \in I_r} |(Z^i x)_k|^p
\]

and hence $x \in [Z^\beta_\phi]_\infty(p)$.

Next suppose that $[Z^\beta_\theta]_\infty(p) \subset [Z^\beta_\phi]_\infty(p)$ and $\sup_r \left( \frac{h^\alpha_r}{l^\beta_r} \right) = \infty$. Then there exists an increasing sequence $(r_i)$ of natural numbers such that $\lim_i \left( \frac{h^\alpha_{r_i}}{l^\beta_{r_i}} \right) = \infty$. Let $L$ be a positive real number, then there exists $i_0 \in \mathbb{N}$ such that $\frac{h^\alpha_{r_i}}{l^\beta_{r_i}} > L$ for all $r_i \geq i_0$. Then $h^\alpha_{r_i} > L l^\beta_{r_i}$ and so $\frac{1}{l^\alpha_{r_i}} > \frac{L}{h^\beta_{r_i}}$. Therefore we can write

\[
\frac{1}{l^\alpha_{r_i}} \sum_{k \in I_{r_i}} |(Z^i x)_k|^p > \frac{L}{h^\beta_{r_i}} \sum_{k \in I_{r_i}} |(Z^i x)_k|^p \text{ for all } r_i \geq i_0.
\]

Now taking supremum over $r_i \geq i_0$ then we get

\[
\sup_{r_i \geq i_0} \frac{1}{l^\alpha_{r_i}} \sum_{k \in I_{r_i}} |(Z^i x)_k|^p > \sup_{r_i \geq i_0} \frac{L}{h^\beta_{r_i}} \sum_{k \in I_{r_i}} |(Z^i x)_k|^p.
\]

Since the relation (2.2) holds for all $L \in \mathbb{R}^+$ (we may take the number $L$ sufficiently large), we have

\[
\sup_{r_i \geq i_0} \frac{1}{l^\alpha_{r_i}} \sum_{k \in I_{r_i}} |(Z^i x)_k|^p = \infty
\]

but $x = (x_k) \in [Z^\beta_\phi]_\infty(p)$ with

\[
\sup_r \left( \frac{h^\alpha_r}{l^\beta_r} \right) < \infty.
\]

Therefore $x \notin [Z^\beta_\phi]_\infty(p)$ which contradicts that $[Z^\beta_\theta]_\infty(p) \subset [Z^\beta_\phi]_\infty(p)$. Hence $\sup_{r \geq 1} \left( \frac{h^\alpha_r}{l^\beta_r} \right) < \infty$. \( \square \)
Corollary 2.4. Let $\alpha$ and $\beta$ be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $p$ be a positive real number such that $1 \leq p < \infty$. For any two lacunary sequences $\theta = (h_r)$ and $\phi = (l_r)$ for all $r \geq 1$, then

(a) $[Z^\alpha_\theta]_\infty(p) = [Z^\beta_\phi]_\infty(p)$ if and only if $0 < \inf_r \left( \frac{h_r}{l_r} \right) < \sup_r \left( \frac{h_r}{l_r} \right) < \infty$.

(b) $[Z^\alpha_\theta]_\infty(p) = [Z^\alpha_\phi]_\infty(p)$ if and only if $0 < \inf_r \left( \frac{h_r}{l_r} \right) < \sup_r \left( \frac{h_r}{l_r} \right) < \infty$.

(c) $[Z^\alpha_\phi]_\infty(p) = [Z^\beta_\theta]_\infty(p)$ if and only if $0 < \inf_r \left( \frac{h_r}{l_r} \right) < \sup_r \left( \frac{h_r}{l_r} \right) < \infty$.

Theorem 2.5. $\ell_p \subset [Z^\alpha_\theta]_\infty(p) \subset \ell_\infty$.

Proof. The proof of the result is straightforward, so omitted. □

Theorem 2.6. If $0 < p < q$, then $[Z^\alpha_\theta]_\infty(p) \subset [Z^\alpha_\theta]_\infty(q)$.

Proof. The proof of the result is straightforward, so omitted. □

3. Some geometric properties

In this section we study some of the geometric properties like order continuous, the Fatou property and the Banach-Saks property in this new sequence space.

Theorem 3.1. The space $[Z^\alpha_\theta]_\infty(p)$ is order continuous.

Proof. We have to show that the space $[Z^\alpha_\theta]_\infty(p)$ is an AK-space. It is easy to see that $[Z^\alpha_\theta]_\infty(p)$ contains $c_0$ which is the space of real sequences which have only a finite number of non-zero coordinates. By using the definition of AK-properties, we have that $x = (x(i)) \in [Z^\alpha_\theta]_\infty(p)$ has a unique representation $x = \sum_{i=1}^\infty x(i)e(i)$ i.e. $\|x - x[j]\|_\alpha = \|(0, 0, \ldots, x(j), x(j + 1), \ldots)\|_\alpha \to 0$ as $j \to \infty$, which means that $[Z^\alpha_\theta]_\infty(p)$ has AK. Therefore BK-space $[Z^\alpha_\theta]_\infty(p)$ containing $c_0$ has AK-property, hence the space $[Z^\alpha_\theta]_\infty(p)$ is order continuous. □

Theorem 3.2. The space $[Z^\alpha_\theta]_\infty(p)$ has the Fatou property.
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Proof. Let $x$ be a real sequence and $(x_j)$ be any nondecreasing sequence of non-negative elements form $[Z^\alpha_\theta]_\infty(p)$ such that $x_j(i) \to x(i)$ as $j \to \infty$ coordinatewisely and $\sup_j ||x_j||_\alpha < \infty$.

Let us denote $T = \sup_j ||x_j||_\alpha$. Since the supremum is homogeneous, then we have

$$
1 \over T \sup_r \frac{1}{h_r^\alpha} \left( \sum_{k \in I_r} \left| (Z^i x_j(i))_k \right|^p \right)^{1 \over p} 
$$

$$
\leq \sup_r \frac{1}{h_r^\alpha} \left( \sum_{k \in I_r} \left| (Z^i x_j(i))_k \right|^p \left| \frac{||x_n||_\alpha}{||x_n||_\alpha} \right| \right)^{1 \over p} 
$$

$$
= \frac{1}{||x_n||_\alpha} ||x_n||_\alpha = 1.
$$

Also by the assumptions that $(x_j)$ is non-decreasing and convergent to $x$ coordinatewisely and by the Beppo-Levi theorem, we have

$$
1 \over T \lim_{j \to \infty} \sup_r \frac{1}{h_r^\alpha} \left( \sum_{k \in I_r} \left| (Z^i x_j(i))_k \right|^p \right)^{1 \over p} 
$$

$$
= \sup_r \frac{1}{h_r^\alpha} \left( \sum_{k \in I_r} \left| (Z^i x(i))_k \right|^p \left| \frac{T}{T} \right| \right)^{1 \over p} \leq 1,
$$

whence

$$
||x||_\alpha \leq T = \sup_j ||x_j||_\alpha = \lim_{j \to \infty} ||x_j||_\alpha < \infty.
$$

Therefore $x \in [Z^\alpha_\theta]_\infty(p)$. On the other hand, since $0 \leq x$ for any natural number $j$ and the sequence $(x_j)$ is non-decreasing, we obtain that the sequence $(||x_j||_\alpha)$ is bounded form above by $||x||_\alpha$. Therefore $\lim_{j \to \infty} ||x_j||_\alpha \leq ||x||_\alpha$ which contradicts the above inequality proved already, yields that $||x||_\alpha = \lim_{j \to \infty} ||x_j||_\alpha$. □

Theorem 3.3. The space $[Z^\alpha_\theta]_\infty(p)$ has the Banach-Saks property.

Proof. The proof of the result follows from the standard technique. □
References


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