An equivalence in generalized almost-Jordan algebras

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Received : September 2016. Accepted : October 2016

Abstract

In this paper we work with the variety of commutative algebras satisfying the identity $\beta((x^2y)x - ((yx)x)x) + \gamma(x^3y - ((yx)x)x) = 0$, where $\beta, \gamma$ are scalars. They are called generalized almost-Jordan algebras. We prove that this variety is equivalent to the variety of commutative algebras satisfying $(3\beta + \gamma)(G_y(x,z,t) - G_x(y,z,t)) + (\beta + 3\gamma)(J(x,z,t)y - J(y,z,t)x) = 0$, for all $x, y, z, t \in A$, where $J(x,y,z) = (xy)z + (yz)x + (zx)y$ and $G_x(y,z,t) = (yz,x,t) + (yt,x,z) + (zt,x,y)$. Moreover, we prove that if $A$ is a commutative algebra, then $J(x,z,t)y = J(y,z,t)x$, for all $x, y, z, t \in A$, if and only if $A$ is a generalized almost-Jordan algebra for $\beta = 1$ and $\gamma = -3$, that is, $A$ satisfies the identity $(x^2y)x + 2((yx)x)x - 3x^3y = 0$ and we study this identity. We also prove that if $A$ is a commutative algebra, then $G_y(x,z,t) = G_x(y,z,t)$, for all $x, y, z, t \in A$, if and only if $A$ is an almost-Jordan or a Lie Triple algebra.

Subjclass : Primary 17A30; Secondary 17C50.

Keywords : Jordan algebras, generalized almost-Jordan algebras, Lie Triple algebras, baric algebras.

Thanks : Part if this research was done while the first author was visiting the Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, supported by the FONDECYT grant 1120844 and the second author supported by Programa de Estímulo a la Excelencia Institucional-Universidad de Chile, was visiting the University of São Paulo.
1. Introduction

In this work, \( F \) is a field of \( \text{char} F \neq 2 \) and \( A \) be a commutative not necessarily associative algebra over \( F \).

The algebra \( A \) is called Jordan algebra if satisfies \( (y^2, x, y) = 0 \), for all \( y, x \in A \). For properties of these algebras see [10]. It is know, see Osborn [7], that a Jordan algebra satisfies the identity

\[
3(x^2y)x - 2((yx)x)x - x^3y = 0.
\]

Algebras satisfying identity (1.1), called Lie Triple algebras or almost-Jordan algebras have been studied by Hentzel, Peresi, Osborn, Peterson and Sidorov [5, 7, 8, 9, 11].

Identity (1.1) was generalized in 1988 by Carini, Hentzel and Piacentini-Cattaneo, see [3]. After that, Arenas and Labra call them generalized almost-Jordan algebras, see [1].

We say that \( A \) is a generalized almost-Jordan algebra if it satisfies:

\[
\beta((x^2y)x - ((yx)x)x) + \gamma(x^3y - ((yx)x)x) = 0,
\]

for all \( x, y \in A \), where \( \beta, \gamma \in F \) and \( (\beta, \gamma) \neq (0, 0) \).

In the study of degree four identities not implied by commutativity, Osborn [8] classified those that were implied by the fact of possessing a unit element. Carini, Hentzel and Piacentini-Cattaneo [3] extended this work by dropping the restriction on the existence of the unit element. The identity defining a generalized almost-Jordan algebra with \( \beta, \gamma \in F \) appears as one of these identities.

We have:

\[
(x^2, y, x) = (x^2y)x - x^2(yx), (x^2, x, y) = x^3y - x^2(yx), (yx, x, x) = ((yx)x)x - (yx)x^2,
\]

so

\[
(x^2, y, x) - (yx, x, x) = (x^2y)x - ((yx)x)x, (x^2, x, y) - (yx, x, x) = x^3y - ((yx)x)x
\]

and
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\[
0 = \beta \left( (x^2y)x - ((yx)x)x \right) + \gamma \left( x^3y - ((yx)x)x \right) = \\
\beta \left( (x^2, y, x) - (yx, x, x) \right) + \gamma \left( (x^2, x, y) - (yx, x, x) \right)
\]

Therefore, in terms of associators a generalized almost-Jordan algebra satisfies,

\[
\beta(x^2, y, x) + \gamma(x^2, x, y) = (\beta + \gamma)(yx, x, x) \quad (1.3)
\]

If \( \beta = 3 \) and \( \gamma = -1 \), we obtain an almost-Jordan algebra, that is, \( A \) satisfies

\[
3(x^2, y, x) = (x^2, x, y) + 2(yx, x, x).
\]

Generalized almost-Jordan algebras \( A \) have been studied in [3] where the authors proved that for almost all the algebras, simplicity implies associativity, in [1], where the authors proved that these algebras always have a trace form in terms of the trace of right multiplication operators. They also prove that if \( A \) is finite-dimensional and solvable, then it is nilpotent. In [2] the author found the Wedderburn decomposition of \( A \) assuming that for every ideal \( I \) of \( A \) either \( I \) has a non zero idempotent or \( I \subset R, R \) the solvable radical of \( A \) and the quotient \( A/R \) is separable, in [4] the authors give a characterization of representations and irreducibles modules of these algebras, and in [6] where, assuming that \( A \) also satisfies \((xx)x = 0\) the authors proved the existence of an ideal \( I \) of \( A \) such that \( AI = IA = 0 \) and the quotient algebra \( A/I \) is power-associative.

In this paper we prove the equivalence between generalized almost-Jordan algebras, and commutative algebras satisfying the identity \((3\beta + \gamma)(G_y(x, z, t) - G_z(x, y, t)) + (3\gamma)(J(x, z, t)y - J(y, z, t)x) = 0\), for all \( x, y, z, t \in A \), where \( J(x, y, z) = (xy)z + (yz)x + (zx)y \) and \( G_z(x, y, t) = (yz, x, t) + (yt, x, z) + (zt, x, y) \), Theorem 3.2. We prove that a Jordan algebra satisfies the identity \( G_z(x, y, t) = 0 \) for all \( x, y, z, t \in A \). Conversely if \( \text{char} F \neq 3 \), then every commutative algebra \( G_z(x, y, t) = 0 \) for all \( x, y, z, t \in A \) is a Jordan algebra, Proposition 3.1. Moreover, we prove that if \( A \) is a commutative algebra, then \( J(x, z, t)y = J(y, z, t)x \) for all \( x, y, z, t \in A \), if and only if \( A \) is a generalized almost-Jordan algebra for \( \beta = 1 \) and \( \gamma = -3 \), that is, \( A \) satisfies the identity \((x^2y)x + 2((yx)x)x - 3x^3y = 0\), Proposition 3.4. We also prove that if \( A \) is a commutative algebra, then \( G_y(x, z, t) = G_z(x, y, t) \), for all \( x, y, z, t \in A \), if and only if \( A \) is an almost-Jordan algebra, Proposition 3.5. Finally, we give some new identities, Theorem 3.13 and Proposition 3.15 for commutative algebras satisfying the identity \((x^2y)x + 2((yx)x)x - 3x^3y = 0\).
2. Preliminaries

In this section we found relationships among generalized almost-Jordan algebras and alternative algebras, Jordan algebras, baric algebras or b-algebras.

Proposition 2.1. Let \( A \) be a commutative right alternative algebra. Then \( A \) is a generalized almost-Jordan algebra, for \( \beta = \gamma = 1 \).

Proof: Since \( A \) is a right alternative algebra, then \( A \) is an alternative algebra and \((x, y, z) = -(x, z, y), \) so by (1.2) we have, \((x^2, y, x) + (x^2, x, y) = (x^2, y, x) - (x^2, y, x) = 0 = 2(yx, x, x)\).

If \( A \) is a \( F \)-algebra, then we will define a new algebra \( A' = Fe \oplus A \), as vector space, and the multiplication given by:

\[(\alpha e + u)(\beta e + v) = \alpha \beta e + uv,\]

where \( e \) is an idempotent, \( \alpha, \beta \in F \) and \( u, v \in A \).

Proposition 2.2. Let \( A \) be a generalized almost-Jordan algebra. Then \( A' \) is a generalized almost-Jordan algebra and \( \omega: A' \rightarrow F \), given by \( \omega(\alpha e + u) = \alpha \), is a nonzero homomorphism of algebras.

Proof: Let \( \alpha, \beta \in F, u, v \in A, x = \alpha e + u \) and \( y = \beta e + v \). Since \( ez = 0 \) and \( e^2 = e \) for all \( z \in A \), then \((a, b, c) = 0, \) if \( a, b, c \in A \cup \{ e \} \), and at least one of them is equal to \( e \), so

\[x^2 = \alpha^2 e + u^2, \quad yx = \alpha \beta e + vu, \]

\[(x^2, y, x) = (\alpha^2 e + u^2, \beta e + v, \alpha e + u) = \alpha^3 \beta (e, e, e) + (u^2, v, u) = (u^2, v, u) \]

\[(x^2, x, y) = (u^2, u, v) \quad \text{and} \quad (yx, x, x) = (vu, u, u).\]

therefore

\[\beta(x^2, y, x) + \gamma(x^2, x, y) = \beta(u^2, v, u) + \gamma(u^2, u, v) = (\beta + \gamma)(vu, u, u) = (\beta + \gamma)(yx, x, x).\]

Definition 2.3. Let \( A \) be a \( F \)-algebra. If \( \omega: A \rightarrow F \) is a nonzero algebra homomorphism, then the ordered pair \((A, \omega)\) is called a baric algebra or b-algebra. When a b-algebra \((A, \omega)\) is a generalized almost-Jordan algebra, then we call it generalized almost-Jordan b-algebra.

Corollary 2.4. Let \( A \) be a generalized almost-Jordan algebra. Then \((A', \omega)\) is a generalized almost-Jordan b-algebra.
If $A$ is a $F$-algebra, then we will define a new algebra $A^# = F \oplus A$, as vector space, and the multiplication given by:

$$(\alpha + u)(\beta + v) = \alpha \beta + \alpha v + \beta u + uv,$$

where $\alpha, \beta \in F$ and $u, v \in A$, $A^#$ has unit element $1 + 0 = 1$.

**Theorem 2.5.** Let $A$ be a generalized almost-Jordan algebra. Then $A^#$ is a generalized almost-Jordan algebra if and only if $\beta + 3\gamma = 0$ or $A$ is an alternative algebra.

**Proof:** Let $\alpha, \beta \in F, u, v \in A, x = \alpha + u$ and $y = \beta + v$. We note that $(1, a, b) = (a, 1, b) = (a, b, 1) = 0 = (a, b, a)$, for all $a, b \in A^#$, so

$$x^2 = \alpha^2 + 2\alpha u + u^2, \quad yx = \alpha \beta + \alpha v + \beta u + vu,$$

$$(x^2, y, x) = (\alpha^2 + 2\alpha u + u^2, \beta + v, \alpha + u) = 2\alpha(u, v, u) + (u^2, v, u) = (u^2, v, u),$$

$$(x^2, x, y) = (\alpha^2 + 2\alpha u + u^2, \alpha + u, \beta + v) = 2\alpha(u, u, v) + (u^2, u, v),$$

$$(yx, x, x) = (\alpha \beta + \alpha v + \beta u + vu, \alpha + u, \alpha + u) = \alpha(v, u, u) + (vu, u, u),$$

$$(u, u, v) = u^2v - u(uv) = -(vu)u - vu^2) = -(v, u, u).$$

therefore

$$\beta(x^2, y, x) + \gamma(x^2, x, y) - (\beta + \gamma)(yx, x, x) = \beta(u^2, v, u) + 2\alpha \gamma(u, u, v) + \gamma(u^2, u, v) - \alpha(\beta + \gamma)(v, u, u) - (\beta + \gamma)(vu, u, u) = 2\alpha \gamma(u, u, v) - \alpha(\beta + \gamma)(v, u, u) - \alpha(\beta + \gamma)(v, u, u) = -\alpha(3\gamma + \beta)(v, u, u).$$

Since $\alpha$ is arbitrary, then the Theorem follows. \qed

**Corollary 2.6.** Let $A$ be an almost-Jordan algebra. Then $A^#$ is an almost-Jordan algebra.

**Corollary 2.7.** If $A$ is an almost-Jordan algebra and $\omega: A^# \to F$ is given by $\omega(\alpha + u) = \alpha$. Then $(A^#, \omega)$ is an almost-Jordan $b$-algebra.

**Example 2.8.** Let $F$ be a field of characteristic not 2 and $A$ be a commutative $F$-algebra of basis $\{s, t\}$ with the multiplication:

<table>
<thead>
<tr>
<th></th>
<th>$s$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>$s$</td>
<td>$\frac{1}{2}t$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\frac{1}{2}t$</td>
<td>0</td>
</tr>
</tbody>
</table>

This algebra is an almost-Jordan algebra, but is not a Jordan algebra, see [7]. Moreover, it is a $b$-algebra and the only idempotent is zero.

In fact, let $\omega: A \to F$ given by $\omega(as + bt) = a$, where $a, b \in F$, then
\( \omega((as + bt)(a's + b't)) = \omega\left((aa's + \frac{1}{2}(2aa' + ab' + a'b)t\right) = aa' = \\
\omega(as + bt)\omega(a's + b't), \text{ for all } a, a', b, b' \in F, \)

so, this algebra is a \( b \)-algebra.

If \( e = as + bt \in A \), such that \( e^2 = e \), then

\( as + bt = a^2s + \frac{1}{2}(2a^2 + 2ab)t = a^2s + (a^2 + ab)t, \)

so \( a = a^2 \) and \( b = a^2 + ab \), therefore, \( a = b = 0 \), then \( e = 0 \) is the only idempotent of \( A \).

**Example 2.9.** Let \( F \) be a field and \( A \) be a commutative \( F \)-algebra of basis \( \{x_1, x_2, x_3, x_4\} \) with the multiplication:

<table>
<thead>
<tr>
<th></th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>( x_2 )</td>
<td>( x_3 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>( x_3 )</td>
<td>( x_3 )</td>
<td>0</td>
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<tr>
<td>( x_3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>0</td>
<td>( x_3 )</td>
<td>0</td>
<td>( x_2 + x_3 )</td>
</tr>
</tbody>
</table>

In [1] the authors prove that this algebra is a generalized almost-Jordan algebra for all \( \beta, \gamma \in F \), because \( (x^2y)x = ((yx)x)x = x^3y = 0 \) for all \( x, y \in A \). Since \( (x_1, x_1, x_2) = x_1^2x_2 - x_1(x_1x_2) = x_3 \), then \( A \) is not alternative algebra.

We will to prove that this algebra is not a \( b \)-algebra.

Let \( \omega: A \to F \) be an algebra homomorphism, since \( x_2^2 = 0, x_2^2 = x_3, x_1^2 = x_2 \) and \( x_3^2 = x_2 + x_3 \), then \( \omega(x_3) = \omega(x_2) = \omega(x_1) = \omega(x_4) = 0 \), so \( A \) is not a \( b \)-algebra.

**3. Main Results**

Let \( A \) be a generalized almost-Jordan algebra.

Linearising (1.3) we have,

\[
\beta \left( (x^2, y, z) + 2(xz, y, x) \right) + \gamma \left( (x^2, z, y) + 2(xz, x, y) \right) = \\
= (\beta + \gamma) \left( (yx, x, z) + (yz, x, z) \right)
\]

(3.1)
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\[ 2\beta \left( (tx, y, z) + (xz, y, t) + (tz, y, x) \right) + 2\gamma \left( (tx, z, y) + (xz, t, y) + (tz, x, y) \right) = \]
\[ = (\beta + \gamma) \left( (yx, t, z) + (yx, z, t) + (yt, x, z) + (yt, z, x) + (yz, x, t) + (yz, t, x) \right) \]

(3.2)

Let \( G_x : A \times A \times A \to A \) given by

\[ G_x(y, z, t) = (yz, x, t) + (yt, x, z) + (zt, x, y) \]

It is easy to see that, \( G_x \) is 3-lineal function and symmetric in every two variables. Moreover, the complete linearization of the \((x^2, y, x)\) is \(2G_x(y, z, t)\). If \( A \) is a Jordan algebra, then \( G_x(y, z, t) = 0 \), for all \( x, y, z, t \in A \). Conversely we have.

**Proposition 3.1.** Let \( A \) be a commutative algebra over a field of characteristic not 3, such that

\[ G_x(y, z, t) = 0, \]

for all \( x, y, z, t \in A \). Then \( A \) is a Jordan algebra.

**Proof:** Setting \( z = t = y \) in \( G_x(y, z, t) = (yz, x, t) + (yt, x, z) + (zt, x, y) = 0 \), we get \((y^2, x, y) + (y^2, x, y) + (y^2, x, y) = 0\), so \( 3(y^2, x, y) = 0 \), then \( A \) is a Jordan algebra. \( \square \)

**Theorem 3.2.** \( A \) is a generalized almost-Jordan algebra, if and only if \( A \) is a commutative algebra satisfying

\[ (3\beta + \gamma) \left( G_y(x, z, t) - G_x(y, z, t) \right) + (\beta + 3\gamma) \left( J(x, z, t)y - J(y, z, t)x \right) = 0, \]

for all \( x, y, z, t \in A \), where \( J(a, b, c) = (ab)c + (bc)a + (ca)b \).

**Proof:** By (3.2) we have,

\[ 2\beta G_y(x, z, t) + 2\gamma \left( (tx, z, y) + (xz, t, y) + (tz, x, y) \right) = (\beta + \gamma) \left( G_t(x, y, z) + G_x(y, z, t) + G_z(x, y, t) \right) - (\beta + \gamma) \left( (xz, t, y) + (tx, x, y) + (tx, z, y) \right), \]

so

\[ 2\beta G_y(x, z, t) + (3\beta + \gamma) \left( (xz, t, y) + (tx, x, y) + (tx, z, y) \right) = \]
\[ (\beta + \gamma) \left( G_t(x, y, z) + G_x(y, z, t) + G_z(x, y, t) \right). \]

(3.3)
In (3.3), replacing $x$ by $y$ and $y$ by $x$, we have

\[ 2\beta G_x(y, z, t) + (\beta + 3\gamma) \left( (yz, t, x) + (tz, y, x) + (ty, z, x) \right) = (\beta + \gamma) \left( G_t(x, y, z) + G_y(x, z, t) + G_z(x, y, t) \right). \tag{3.4} \]

By (3.3) and (3.4),

\[ 2\beta (G_y(x, z, t) - G_x(y, z, t)) + (\beta + 3\gamma) \left( (xz, t, y) + (tz, x, y) + (tx, z, y) - (yz, t, x) - (tz, y, x) - (ty, z, x) \right) = (\beta + \gamma) \left( G_x(y, z, t) - Hy(x, z, t) \right), \]

so

\[ (3\beta + \gamma) \left( G_y(x, z, t) - G_x(y, z, t) \right) + (\beta + 3\gamma) \left( (xz, t, y) + (tz, x, y) + (tx, z, y) - (yz, t, x) - (tz, y, x) - (ty, z, x) \right) = 0, \]

but

\[ (xz, t, y) + (tz, x, y) + (tx, z, y) - (yz, t, x) - (tz, y, x) - (ty, z, x) = \]

\[ ((xz)t)y - (xz)(ty) + ((tx)x)y - (tx)(xy) + ((tx)z)y - (tx)(zy) - ((yz)t)x + (yz)(tx) - (ty)z + (ty)(xz) = \]

\[ \left( (xz)t + (tz)x + (tx)z \right)y - \left( (yz)t + (tz)y + (ty)z \right)x = J(x, z, t)y - J(y, z, t)x, \]

where $J(a, b, c) = (ab)c + (bc)a + (ca)b$. Therefore,

\[ (3\beta + \gamma) \left( G_y(x, z, t) - G_x(y, z, t) \right) + (\beta + 3\gamma) \left( J(x, z, t)y - J(y, z, t)x \right) = 0. \tag{3.5} \]

Conversely, setting $z = t = x$ in (3.5) we have

\[ (3\beta + \gamma) \left( G_y(x, x, x) - G_x(y, x, x) \right) + (\beta + 3\gamma) \left( J(x, x, x)y - J(y, x, x)x \right) = 0. \tag{*} \]

Then, using the definition of $G_x, G_y$ and the commutativity of the algebra we obtain:
Moreover, $J(x, z, t)y - J(y, z, t)x = 3x^3y - 2((yx)x)x - (x^2y)x$.

Replacing these values in (*) we get

$$(3 \beta + \gamma)(3x^2y) - 2((yx)x)x - x^3y = 3(\beta + 3\gamma)(3x^2y) - 2((yx)x)x - (x^2y)x = 0$$

Reordering these terms we obtain

$$8 \gamma x^3y - (8 \beta + 8 \gamma)((yx)x)x + 8 \beta (x^2y)x = 0.$$ 
Since characteristic of the field is different of 2 we get

$$\gamma x^3y - (\beta + \gamma)((yx)x)x + \beta (x^2y)x = 0,$$
and by identity (2), $A$ is a generalized almost-Jordan algebra. \hfill \Box

In [7], Osborn introduced two mappings,

$$H(y; x, z, t) = (y(xz))t + (y(yz))x + (y(tx))z$$ and

$$K(y, x, z, t) = (xy)(zt) + (yz)(xt) + (yt)(xz),$$

so

$$G_y(x, z, t) = (xz, y, t) + (xt, y, z) + (zt, y, x) = \left( (xz)y \right) t + \left( (xt)y \right) z + \left( (zt)y \right) x - (xz)(yt) - (xt)(yz) - (zt)(yx) = H(y; x, z, t) - K(y, x, z, t),$$

but $K(x, y, z, t) = (yx)(zt) + (xz)(yt) + (xt)(yz) = K(y, x, z, t)$, then

$$(3.6) \quad G_y(x, z, t) - G_x(y, z, t) = H(x; y, z, t) - H(y; x, z, t),$$
for all $x, y, z, t \in A$.

**Corollary 3.3.** If $A$ satisfies the identity $(x^2)^2 = x^4$ for all $x \in A$ and $\beta + 3\gamma \neq 0$, then $J(x, z, t)y = J(y, z, t)x$.

**Proof:** By [7], we have $H(y; x, z, t) = H(x; y, z, t)$ for all $x, y, z, t \in A$ and by Theorem 3.2, $J(x, z, t)y = J(y, z, t)x.$ \hfill \Box

**Proposition 3.4.** Let $A$ be a commutative algebra. Then the following identities are equivalent:
1. \((x^2y)x + 2((yx)x)x - 3x^3y = 0\),

2. \(J(x, z, t)y = J(y, z, t)x\).

**Proof:** Since \(A\) satisfies the identity \((x^2y)x + 2((yx)x)x - 3x^3y = 0\), then \(A\) is a generalized almost-Jordan algebra for \(\beta = 1, \gamma = -3\) and \(\beta + 3\gamma \neq 0\), so by (3.5), \(J(x, z, t)y = J(y, z, t)x\).

Conversely, setting \(z = t = x\) in \(J(x, z, t)y = J(y, z, t)x\), we get \(J(x, x, x)x = J(y, y, y)x\), that is \(3x^3y = ((yx)x)x + (x^2y)x + ((yx)x)x\), so \(A\) satisfies the identity \((x^2y)x + 2((yx)x)x - 3x^3y = 0\). \(\square\)

**Proposition 3.5.** Let \(A\) be a commutative algebra. Then \(A\) is an almost-Jordan algebra if and only if 

\[G_y(x, z, t) = G_x(y, z, t),\]

for all \(x, y, z, t \in A\).

**Proof:** Since \(A\) is an almost-Jordan algebra, \(\beta + 3\gamma = 0\) so \(3\beta + \gamma \neq 0\), and by (3.5), \(G_y(x, z, t) - G_x(y, z, t) = 0\), so \(G_y(x, z, t) = G_x(y, z, t)\).

Conversely, if \(A\) satisfies the identity, \(G_y(x, z, t) = G_x(y, z, t)\), then developing the associators we have

\[((yz)x - (xz)y)t + [(yt)x - (xt)y]z + ((zt)x)y - ((zt)y)x = 0.\]

Since \((y, z, x) = (yz)x - y(zx)\) and \((y, t, x) = (yt)x - y(tx)\), we get

\((y, z, x)t + (y, t, x)z + ((zt)x)y - ((zt)y)x = 0.\)

Replacing \((zt, x, y) = ((zt)x)y - (zt)(xy)\) and \((zt, y, x) = ((zt)y)x - (zt)(xy)\)
in the above expression we obtain

\[(y, z, x)t + (y, t, x)z + (zt, x, y) - (zt, y, x) = 0.\] (3.7)

Since \(A\) is a commutative algebra, then \((a, b, c) = -(c, b, a)\) and (3.7) becomes

\[(x, y, t) + (x, t, y)z + (y, x, z)t - (x, y, z)t = 0.\] (3.8)

Setting \(z = t = x\) in (3.8), we obtain

\[2(x, x, y)x + (y, x, x^2) - (x, y, x^2) = 0.\]
Developing the associators and using the commutativity we get
\[ 3(x^2y)x - 2(x(xy))x - yx^3 = 0. \]

By identity (1.1), \( A \) is an almost-Jordan algebra. \( \Box \)

By identity (3.6), we have

**Corollary 3.6.** Let \( A \) be a commutative algebra. Then \( A \) is an almost-Jordan algebra if and only if
\[ H(y; x, z, t) = H(x; y, z, t), \]
for all \( x, y, z, t \in A \).

By [7], we have

**Proposition 3.7.** If \( A \) satisfies the identity \((x^2)^2 = x^4\) for all \( x \in A \), then \( A \) is an almost-Jordan algebra.

**Remark 3.8.** The converse of the Proposition 3.7 is note true. Let \( A \) be the algebra of Example 2.8, so \( s^4 = s + \frac{7}{4}t \) and \((s^2)^2 = s + 2t \neq s^4 \).

An algebra \( A \) is called power-associative algebra if for all \( x \in A \), the subalgebra \( A(x) \) of \( A \) generated by \( x \) is associative algebra.

**Corollary 3.9.** If \( A \) is a commutative power-associative algebra, then \( A \) is an almost-Jordan algebra.

By Corollaries 3.3 and 3.9, we have

**Proposition 3.10.** If \( A \) is a commutative power-associative algebra and \( \beta + 3\gamma \neq 0 \), then \( A \) is an almost-Jordan algebra and \( J(x, z, t)y = J(y, z, t)x \).

**Corollary 3.11.** If \( A \) is a commutative power-associative algebra and \( \beta + 3\gamma \neq 0 \), then the following identities hold,
1. \( 3(x^2y)x - 2\left((yx)x\right)x - x^3y = 0, \)
2. \( (x^2y)x + 2\left((yx)x\right)x - 3x^3y = 0. \)
for all \( x, y \in A \).
Remark 3.12. The converse of the Corollary 3.11 is not true. Let $A$ be the algebra of Example 2.9, so $A$ satisfies both identities, but $A$ is not power-associative algebra, because $x_1^4 = 0$ and $(x_1^2)^2 = x_3$.

Let $A$ be a commutative algebra which satisfies the identities of Corollary 3.11, then $(x^2y)x = ((yx)x)x = x^3y$ for all $x, y \in A$. In this case the converse is true.

Next, let $A$ be a commutative non necessarily power-associative algebra, so identities (1) and (2) of the above Corollary are not equivalent. Since identity (1), a Lie triple or almost-Jordan algebra has been largely studied we will study an algebra $A$ satisfying

\begin{equation}
(x^2y)x + 2((yx)x)x - 3x^3y = 0, 
\end{equation}

for all $x, y \in A$, which is identity (2) of Corollary 3.11.

It is known (see [1]) that every finite dimensional solvable algebra satisfying (3.9) is nilpotent. If $R$ is radical of $A$ and $A/R$ is solvable, then $A$ has Wedderburn decomposition, (see [2]). Moreover, if $A$ has an idempotent element, then $A = A_0 \oplus A_1 \oplus A_{-2}$, where $A_i = \{x \in A \mid ex = ix\}, i = 0,1,-\frac{3}{2}$, is the Peirce decomposition of $A$. The subspaces $A_i$ satisfies the relations, (see [2]):

\begin{align*}
A_0^2 &\subseteq A_0, & A_1^2 &\subseteq A_1, & A_0A_1 &= \{0\} = A_{-\frac{3}{2}}A_0 = A_{-\frac{3}{2}}, & A_{-\frac{3}{2}}A_1 &\subseteq A_{-\frac{3}{2}}.
\end{align*}

In this work we give same new identities.

Substituting $y = x^k$ in (3.9), we get $(x^2x^k)x + 2x^{k+3} - 3x^3x^k = 0$, so

\begin{equation}
2x^{k+3} = 3x^3x^k - (x^2x^k)x, \quad k \geq 2
\end{equation}

The identity (3.9) is equivalent to

\begin{equation}
(x^2, y, x) + 2(yx, x, x) - 3(x^2, x, y) = 0,
\end{equation}

for all $x, y \in A$.

Linearising (3.11), we have

\begin{equation}
2(xz, y, x) + (x^2, y, z) + 2(yz, x, x) + 2(yx, z, x) + 2(yx, x, z) - 6(xz, x, y) - 3(x^2, z, y) = 0
\end{equation}

Interchanging $y$ and $z$, we have

\begin{align*}
2(xy, z, x) + (x^2, z, y) + 2(yz, x, x) + 2(zx, y, x) + 2(zx, x, y) - 6(xy, x, z) - 3(x^2, y, z) &= 0.
\end{align*}
Subtracting both identities and canceling out by the factor 4, we obtain

\[(3.13) \quad (x^2, y, z) - (x^2, z, y) + 2(yx, x, z) - 2(zx, x, y) = 0, \]

so \((x^2, y, z) + 2(yx, x, z) - 2(zx, x, y) = (x^2, z, y) + 2(zx, x, y)\), and substituting in identity (3.11), we get

\[(3.14) \quad (xz, y, x) + (yz, x, x) + (yx, z, x) - 2(xz, x, z) - 2(xz, x, x) - 2(xz, x, x) = 0. \]

**Theorem 3.13.** Let \(A\) be an algebra which satisfies identity (3.9). Then \(A\) satisfies the following identities for \(i, j \geq 2, i \neq j\):

1. \((x^2, x^i, x^j) = 2\left\langle x^{i+2}x^i - x^{i+2}x^j \right\rangle, \)
2. \(2x^{i+2}x^i = (x^{i+1}x^i)x + (x^{i+1}x^i)x + ((x^{i+1}x^i)x)x - (x^2x^i)x^i. \)

**Proof:** Setting \(y = x^i, z = x^j\) and then \(y = x^j, z = x^i\) in identity (3.14), we have,

\[
\begin{align*}
(x+j+1, x^i, x) + (x^jx^i, x^i, x) + (x+j+1, x^j, x) - 2(x^{j+1}, x, x^i) - (x^2, x^j, x^i) &= 0, \\
(x^i+1, x^j, x) + (x^jx^i, x, x) + (x^i+1, x^j, x) - 2(x^{i+1}, x, x^j) - (x^2, x^i, x^j) &= 0.
\end{align*}
\]

Subtracting both identities we obtain

\[-2(x^{j+1}, x, x^i) - (x^2, x^j, x^i) + 2(x^{i+1}, x, x^j) + (x^2, x^i, x^j) = 0,\]

Developing the associators, we obtain

\[-2x^{j+2}x^i + 2x^{i+2}x^j + (x^2x^i)x^j - x^2(x^jx^i) = 0,\]

This is identity (1).

To get identity (2) we use the commutativity and we will develop the associator in the identity: \((x^{j+1}, x^j, x) + (x^jx^i, x^i, x) + (x^{i+1}, x^j, x) - 2(x^{j+1}, x, x^i) - (x^2, x^j, x^i) = 0. \)

**Remark 3.14.** Setting \(y = z = x^i\) in identity (3.14), we have

\[2x^{i+2}x^i = 2(x^{i+1}x^i)x + ((x^i)^2)x - (x^i)^2x^2 - (x^2x^i)x^i + x^2(x^i)^2. \]

**Proposition 3.15.** Let \(A\) be an algebra which satisfies identity (3.9). Then \(A\) satisfies the following identities for \(k \geq 1\):

1. \(2x^4x^k - 2x^{k+2}x^2 + (x^2)^2x^k - x^2(x^2x^k) = 0,\)

2. \(2x^{i+2}x^i = (x^{i+1}x^i)x + (x^{i+1}x^i)x + ((x^{i+1}x^i)x)x - (x^2x^i)x^i. \)
2. $4x^{k+4} = 4(x^3x^k)x + 3x^3x^{k+1} - 2x^{k+2}x^2 - x^2(x^2x^k),$

3. $4x^{k+4} = 4(x^3x^k)x + 3x^3x^{k+1} - 2x^4x^k - (x^2x^k)x.$

**Proof:** Setting $i = 2, j = k$ in (1) of Theorem 3.13, we obtain identity (1).

Setting $i = 2, j = k$ in (2) of Theorem 3.13, we obtain

$$2x^{k+2}x^2 = (x^{k+1}x^2)x + (x^3x^k)x + ((x^2x^k)x)x - (x^2x^k)x^2,$$

Using the identity (3.10), we get

$$2x^{k+2}x^2 = (3x^{k+1}x^3 - 2x^{k+4}) + (x^3x^k)x + (3x^kx^3 - 2x^{k+3})x - (x^2x^k)x^2 = 3x^{k+1}x^3 - 4x^{k+4} + 4(x^3x^k)x - (x^2x^k)x^2,$$

which is identity (2).

Finally identity (3) follows from identities (1) and (2).

**Acknowledgement.** The authors thank the referee for his/her valuable suggestions and comments.

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