Odd vertex equitable even labeling of graphs

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Abstract

In this paper, we introduce a new labeling called odd vertex equitable even labeling. Let $G$ be a graph with $p$ vertices and $q$ edges and $A = \{1, 3, \ldots, q\}$ if $q$ is odd or $A = \{1, 3, \ldots, q + 1\}$ if $q$ is even. A graph $G$ is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling $f^* : E(G) \to A$ defined by $f^*(uv) = f(u) + f(v)$ for all edges $uv$ such that for all $a$ and $b$ in $A$, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \ldots, 2q$ where $v_f(a)$ be the number of vertices $v$ with $f(v) = a$ for $a \in A$. A graph that admits odd vertex equitable even labeling is called odd vertex equitable even graph. We investigate the odd vertex equitable even behavior of some standard graphs.

Keywords : Mean labeling; odd mean labeling; $k$-equitable labeling; vertex equitable labeling; odd vertex equitable even labeling; odd vertex equitable even graph.

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1. Introduction

All graphs considered here are simple, finite, connected and undirected. Let \( G(V, E) \) be a graph with \( p \) vertices and \( q \) edges. We follow the basic notations and terminologies of graph theory as in [3]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling and a detailed survey of graph labeling can be found in [2]. The concept of mean labeling was introduced in [8].

A graph \( G(V, E) \) with \( p \) vertices and \( q \) edges is called a mean graph if there is an injective function \( f \) that maps \( V(G) \) to \( \{0, 1, 2, \ldots, q\} \) such that for each edge \( uv \), labeled with \( \frac{f(u) + f(v)}{2} \) if \( f(u) + f(v) \) is even and \( \frac{f(u) + f(v) + 1}{2} \) if \( f(u) + f(v) \) is odd. Then the resulting edge labels are distinct. The concept of k-equitable labeling was introduced by Cahit [1]. Let \( G \) be a graph. A labeling \( f : V(G) \to \{0, 1, \ldots, k - 1\} \) is called \( k \)-equitable labeling if the condition \(|v_f(i) - v_f(j)| \leq 1, |e_f(i) - e_f(j)| \leq 1, i \neq j, i, j = 0, 1, \ldots, k - 1\) is satisfied, where as before the induced edge labeling is given by \( f(u, v) = |f(u) - f(v)| \) and \( v_f(x) \) and \( e_f(x), x \in \{0, 1, \ldots, k - 1\} \) is the number of vertices and edges of \( G \) respectively with label \( x \). The notion of odd mean labeling was due to Manickam and Marudai [6]. Let \( G(V, E) \) be a graph with \( p \) vertices and \( q \) edges. A graph \( G \) is said to be odd mean graph if there exists a function \( f : V(G) \to \{0, 1, 2, 3, \ldots, 2q - 1\} \) satisfying \( f \) is 1-1 and the induced map \( f^* : E(G) \to \{1, 3, 5, \ldots, 2q - 1\} \) defined by \( f^*(uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases} \) is a bijection. The function \( f \) is called an odd mean labeling.

The concept of vertex equitable labeling was due to Lourdusamy and Seenivasan in [5]. Let \( G \) be a graph with \( p \) vertices and \( q \) edges and \( A = \{0, 1, 2, \ldots, \lceil \frac{q}{2} \rceil\} \). A graph \( G \) is said to be vertex equitable if there exists a vertex labeling \( f : V(G) \to A \) induces an edge labeling \( f^* \) defined by \( f^*(uv) = f(u) + f(v) \) for all edges \( uv \) such that for all \( a \) and \( b \) in \( A \), \(|v_f(a) - v_f(b)| \leq 1 \) and the induced edge labels are \( 1, 2, 3, \ldots, q \), where \( v_f(a) \) be the number of vertices \( v \) with \( f(v) = a \) for \( a \in A \). The vertex labeling \( f \) is known as vertex equitable labeling. Motivated by the concepts of \( k \)-equitable labeling [1], odd mean labeling [6] and vertex equitable labeling [5] of graphs, we define a new labeling called odd vertex equitable even labeling.

Let \( G \) be a graph with \( p \) vertices and \( q \) edges and \( A = \{1, 3, \ldots, q\} \) if \( q \) is odd or \( A = \{1, 3, \ldots, q + 1\} \) if \( q \) is even. A graph \( G \) is said to admit odd
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A graph admits odd vertex equitable even labeling if there exists a vertex labeling \( f : V(G) \to A \) that induces an edge labeling \( f^* \) defined by \( f^*(uv) = f(u) + f(v) \) for all edges \( uv \) such that for all \( a \) and \( b \) in \( A \), \(|v_f(a) - v_f(b)| \leq 1 \) and the induced edge labels are \( 2, 4, \ldots, 2q \) where \( v_f(a) \) be the number of vertices \( v \) with \( f(v) = a \) for \( a \in A \). A graph that admits odd vertex equitable even labeling then \( G \) is called odd vertex equitable even graph.

We observe that \( K_{1,3} \) and \( K_3 \) are vertex equitable graphs but not odd vertex equitable even graphs. We use the following definitions in the subsequent section.

**Definition 1.1.** The disjoint union of two graphs \( G_1 \) and \( G_2 \) is a graph \( G_1 \cup G_2 \) with \( V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \).

**Definition 1.2.** The corona \( G_1 \circ G_2 \) of the graphs \( G_1 \) and \( G_2 \) is defined as a graph obtained by taking one copy of \( G_1 \) (with \( p \) vertices) and \( p \) copies of \( G_2 \) and then joining the \( i^{th} \) vertex of \( G_1 \) to every vertex of the \( i^{th} \) copy of \( G_2 \).

**Definition 1.3.** [7] Let \( G \) be a graph with \( n \) vertices and \( t \) edges. A graph \( H \) is said to be a super subdivision of \( G \) if \( H \) is obtained from \( G \) by replacing every edge \( e_i \) of \( G \) by a complete bipartite graph \( K_{2,m_i} \) for some integer \( m_i, 1 \leq i \leq t \) in such a way that ends of \( e_i \) are merged with two vertices of the 2-vertices part of \( K_{2,m_i} \) after removing the edge \( e_i \) from \( G \). A super subdivision \( H \) of a graph \( G \) is said to be an arbitrary super subdivision of a graph \( G \) if every edge of \( G \) is replaced by an arbitrary \( K_{2,m} \) (\( m \) may vary for each edge arbitrarily).

**Definition 1.4.** [4] Let \( T \) be a tree and \( u_0 \) and \( v_0 \) be the two adjacent vertices in \( T \). Let \( u \) and \( v \) be the two pendant vertices of \( T \) such that the length of the path \( u_0-u \) is equal to the length of the path \( v_0-v \). If the edge \( u_0v_0 \) is deleted from \( T \) and \( u \) and \( v \) are joined by an edge \( uv \), then such a transformation of \( T \) is called an elementary parallel transformation (or an ept) and the edge \( u_0v_0 \) is called transformable edge. If by the sequence of epts, \( T \) can be reduced to a path, then \( T \) is called a \( T_p \)-tree (transformed tree) and such sequence regarded as a composition of mappings (epts) denoted by \( P \), is called a parallel transformation of \( T \). The path, the image of \( T \) under \( P \) is denoted as \( P(T) \). A \( T_p \)-tree and the sequence of two epts reducing it to a path are illustrated in Figure 1.
Definition 1.5. The graph $P_n@P_m$ is obtained by identifying the pendant vertex of a copy of path $P_m$ at each vertex of the path $P_n$.

2. Main Results

Theorem 2.1. Any path is an odd vertex equitable even graph.

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path $P_n$ and it has $n$ vertices and $n - 1$ edges. Let $A = \begin{cases} 1, 3, \ldots, n - 1 & \text{if } n - 1 \text{ is odd} \\ 1, 3, \ldots, n & \text{if } n - 1 \text{ is even} \end{cases}.$

Define a vertex labeling $f : V(P_n) \rightarrow A$ as follows:

For $1 \leq i \leq n$, $f(u_i) = \begin{cases} i & \text{if } i \text{ is odd} \\ i - 1 & \text{if } i \text{ is even} \end{cases}.$

It can be verified that the induced edge labels of $P_n$ are $2, 4, \ldots, 2n - 2$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly $f$ is an odd vertex equitable even labeling of $P_n$. Thus $P_n$ is an odd vertex equitable even graph. \qed

Theorem 2.2. The graph $P_n@P_m$ is an odd vertex equitable even graph for any $n, m \geq 1$. 

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Figure 1

a) A $T_p$-tree $T$  b) An ept $P_1(T)$  c) Second ept $P_2(T)$
Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path $P_n$. Let $v_{i1}, v_{i2}, \ldots, v_{im}$ be the vertices on the $i^{th}$ copy of the path $P_m$ so that $v_i$ is identified with $v_{im}$ for $1 \leq i \leq n$. Clearly $P_n@P_m$ has $mn$ vertices and $mn - 1$ edges.

Let $A = \begin{cases} 1,3,\ldots,mn-1 & \text{if } mn - 1 \text{ is odd} \\ 1,3,\ldots,mn & \text{if } mn - 1 \text{ is even.} \end{cases}$

Define a vertex labeling $f : V(P_n@P_m) \rightarrow A$ as follows:

For $1 \leq i \leq n, 1 \leq j \leq m$,

If $i$ is odd, $f(v_{ij}) = \begin{cases} m(i-1) + j & \text{if } j \text{ is odd} \\ m(i-1) + j - 1 & \text{if } j \text{ is even.} \end{cases}$

If $i$ is even, $f(v_{ij}) = \begin{cases} mi - j & \text{if } j \text{ is odd} \\ mi - (j-1) & \text{if } j \text{ is even.} \end{cases}$

It can be verified that the induced edge labels of $P_n@P_m$ are $2, 4, \ldots, 2mn-2$ and $|f(v_i) - f(v_j)| \leq 1$ for all $i, j \in A$. Clearly $f$ is an odd vertex equitable even labeling of $P_n@P_m$. Thus, $P_n@P_m$ is an odd vertex equitable even graph. \qed

Corollary 2.3. The graph $P_n \odot K_1$ is an odd vertex equitable even graph for any $n \geq 1$.

Theorem 2.4. The graph $K_{1,n}$ is an odd vertex equitable even graph if only if $n \leq 2$.

Proof. Suppose that $n \leq 2$. When $n = 1, K_{1,n} \cong P_2$ and $n = 2, K_{1,n} \cong P_3$. Hence by Theorem 2.1, $K_{1,n}$ is an odd vertex equitable even graph. Suppose that $n \geq 3$ and $K_{1,n}$ is an odd vertex equitable even graph with odd vertex equitable even labeling $f$. Let $\{V_1, V_2\}$ be the bipartition of $K_{1,n}$ with $V_1 = \{u\}$ and $V_2 = \{u_1, u_2, \ldots, u_n\}$. To get the edge label 2, we have to assign the label 1, to the two adjacent vertices. Thus 1 must be the label of $u$. Since $n \geq 3$, the maximum value of the edge label is either $n + 1$ or $n + 2$ according as $n$ is odd or even. Hence, there is no edge with the induced label 2n. Thus, $K_{1,n}$ is not an odd vertex equitable even graph if $n \geq 3$. \qed

Theorem 2.5. The graph $K_{1,n} \cup K_{1,n-2}$ is an odd vertex equitable even graph for any $n \geq 3$.

Proof. Let $u, v$ be the centre vertices of the two star graphs, $K_{1,n}, K_{1,n-2}$. Assume that $u_1, u_2, \ldots, u_n$ be the vertices incident with $u$ and $v_1, v_2, \ldots, v_{n-2}$ be the vertices incident with $v$. Hence $K_{1,n} \cup K_{1,n-2}$ has $2n + 2$ vertices and $2n - 2$ edges. Let $A = \{1,3,\ldots,2n-1\}$. 
Define a vertex labeling \( f : V(K_{1,n} \cup K_{1,n-2}) \rightarrow A \) as follows:
\[
f(u) = 1, f(v) = 2n - 1, f(u_i) = 2i - 1 \text{ if } 1 \leq i \leq n \text{ and } \]
\[
f(v_i) = 2i + 1 \text{ if } 1 \leq i \leq n - 2.
\]
It can be verified that the induced edge labels of \( K_{1,n} \cup K_{1,n-2} \) are
\( 2, 4, \ldots, 4n - 4 \) and \( |v_f(i) - v_f(j)| \leq 1 \) for all \( i, j \in A \). Clearly \( f \) is an odd vertex equitable even labeling of \( K_{1,n} \cup K_{1,n-2} \). Thus, \( K_{1,n} \cup K_{1,n-2} \) is an odd vertex equitable even graph.

**Theorem 2.6.** The graph \( K_{2,n} \) is an odd vertex equitable even graph for all \( n \).

**Proof.** Let \( \{V_1, V_2\} \) be the bipartition of \( K_{2,n} \) with \( V_1 = \{u, v\} \) and \( V_2 = \{u_1, u_2, \ldots, u_n\} \). It has \( n+2 \) vertices and \( 2n \) edges. Let \( A = \{1, 3, \ldots, 2n + 1\} \).

Define a vertex labeling \( f : V(K_{2,n}) \rightarrow A \) as follows:
\[
f(u) = 1, f(v) = 2n + 1 \text{ and } f(u_i) = 2i - 1 \text{ if } 1 \leq i \leq n.
\]
It can be verified that the induced edge labels of \( K_{2,n} \) are \( 2, 4, \ldots, 4n \) and \( |v_f(i) - v_f(j)| \leq 1 \) for all \( i, j \in A \). Clearly \( f \) is an odd vertex equitable even labeling of \( K_{2,n} \). Thus, \( K_{2,n} \) is an odd vertex equitable even graph.

**Theorem 2.7.** Let \( G \) be a graph with \( p \) vertices and \( q \) edges and \( p \leq \left[ \frac{q}{2} \right] + 1 \) then \( G \) is not an odd vertex equitable even graph.

**Proof.** Let \( G \) be a graph with \( p \) vertices and \( q \) edges.

**Case (i):** Let \( q = 2m + 1 \).
Suppose \( G \) is an odd vertex equitable even graph. Let \( A = \{1, 3, \ldots, 2m + 1\} \). To get an edge label 2, there must be two adjacent vertices \( u \) and \( v \) with label 1. Also to get the edge label \( 4m + 2 \), there must be two adjacent vertices \( x \) and \( y \) with label \( 2m + 1 \). Hence, the number of vertices must be greater than or equal to \( m + 3 \). Then \( G \) is not an odd vertex equitable even graph.

**Case (ii):** Let \( q = 2m \).
Suppose \( G \) is an odd vertex equitable even graph. Let \( A = \{1, 3, \ldots, 2m + 1\} \). To get the edge label 2, there must be two adjacent vertices \( u \) and \( v \) each has the label 1. The number of vertices must be greater than or equal to \( m + 2 \). Then \( G \) is not an odd vertex equitable even graph. ∎
Corollary 2.8. The graph $K_{m,n}$ is not an odd vertex equitable even graph if $m, n \geq 3$.

Theorem 2.9. Every $T_p$-tree is an odd vertex equitable even graph.

Proof. Let $T$ be a $T_p$-tree with $n$ vertices. By the definition of a transformed tree there exists a parallel transformation $P$ of $T$ such that for the path $P(T)$ we have (i) $V(P(T)) = V(T)$ (ii) $E(P(T)) = (E(T) - E_d) \cup E_p$

where $E_d$ is the set of edges deleted from $T$ and $E_p$ is the set of edges newly added through the sequence $P = (P_1, P_2, \ldots, P_k)$ of the epts $P$ used to arrive the path $P(T)$. Clearly, $E_d$ and $E_p$ have the same number of edges.

Now denote the vertices of $P(T)$ successively as $v_1, v_2, \ldots, v_n$ starting from one pendant vertex of $P(T)$ right up to the other.

For $1 \leq i \leq n$, define the labeling $f$ as $f(v_i) = \begin{cases} i & \text{if } i \text{ is odd} \\ i - 1 & \text{if } i \text{ is even.} \end{cases}$

Then $f$ is an odd vertex equitable even labeling of the path $P(T)$.

Let $v_i v_j$ be an edge in $T$ for some indices $i$ and $j$ with $1 \leq i < j \leq n$. Let $P_1$ be the ept that delete the edge $v_i v_j$ and add an edge $v_{i+t} v_{j-t}$ where $t$ is the distance of $v_i$ from $v_{i+t}$ and the distance of $v_j$ from $v_{j-t}$. Let $P$ be a parallel transformation of $T$ that contains $P_1$ as one of the constituent epts.

Since $v_{i+t} v_{j-t}$ is an edge of the path $P(T)$, it follows that $i+t+1 = j-t$ which implies $j = i + 2t + 1$. Therefore $i$ and $j$ are of opposite parity.

The induced label of the edge $v_i v_j$ is given by $f^*(v_i v_j) = f^*(v_{i+t} v_{j-t+1}) = f(v_i) + f(v_{i+2t+1}) = 2(i+t), 1 \leq i \leq n$. Now $f^*(v_{i+t} v_{j-t}) = f^*(v_{i+t} v_{j+t+1}) = f(v_{i+t} + f(v_{i+t+1}) = 2(i + t), 1 \leq i \leq n$. Therefore, we have $f^*(v_i v_j) = f^*(v_{i+t} v_{j-t+1})$ and hence $f$ is an odd vertex equitable even labeling of the $T_p$-tree $T$. \square

Theorem 2.10. If every edge of a graph $G$ is an edge of a triangle, then $G$ is not an odd vertex equitable even graph.

Proof. Let $G$ be a graph in which every edge is an edge of a triangle. Suppose $G$ is an odd vertex equitable even graph with odd vertex equitable even labeling $f$. To get 2 as an edge label, there must be two adjacent vertices $u$ and $v$ such that $f(u) = 1$ and $f(v) = 1$. Let $uwvw$ be a triangle. To get 4 as an edge label, there must be $f(w) = 3$, then $uw$ and $vw$ get the same edge label. This is contradiction to $f$ is an odd vertex equitable even labeling. Hence $G$ is not an odd vertex equitable even graph. \square
Corollary 2.11. The complete graph $K_n$ where $n \geq 3$, the wheel $W_n$, the triangular snake, double triangular snake, triangular ladder, flower graph $FL_n$, fan graph $P_n + K_1$, $n \geq 2$, double fan graph $P_n + K_2$, $n \geq 2$, friendship graph $C_n^3$, windmill $K_m^n$, $m > 3$, $K_2 + mK_1$, square graph $B_{2n}^2$, total graph $T(P_n)$ and composition graph $P_n[P_2]$ are not odd vertex equitable even graphs.

Theorem 2.12. The cycle $C_n$ is an odd vertex equitable even graph if $n \equiv 0 \text{ or } 1 \pmod{4}$.

Proof. Suppose $n \equiv 0$ or $1 \pmod{4}$. Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle $C_n$. Let $A = \{1, 3, \ldots, n \}$ if $n$ is odd

$$A = \{1, 3, \ldots, n + 1 \}$$ if $n$ is even.

Define a vertex labeling $f : V(C_n) \rightarrow A$ as follows:

$$f(u_i) = i$$ if $i$ is odd and

$$f(u_i) = \begin{cases} i - 1 & \text{if } i \text{ is even and } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ i + 1 & \text{if } i \text{ is even and } \left\lfloor \frac{n}{2} \right\rfloor \leq i \leq n \end{cases}$$

It can be verified that the induced edge labels of cycle are $2, 4, \ldots, 2n$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly $f$ is an odd vertex equitable even labeling of cycle. Thus, the cycle $C_n$ is an odd vertex equitable even graph if $n \equiv 0 \text{ or } 1 \pmod{4}$. \Box

Theorem 2.13. A quadrilateral snake $Q_n$ is an odd vertex equitable even graph.

Proof. A quadrilateral snake is obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i, u_{i+1}$ to the new vertices $v_i, w_i$ respectively and joining $v_i$ and $w_i, 1 \leq i \leq n - 1$. It has $3n - 2$ vertices and $4n - 4$ edges. Let $A = \{1, 3, \ldots, 4n - 3 \}$.

Define a vertex labeling $f : V(Q_n) \rightarrow A$ as follows:

$$f(u_i) = 4i - 3 \text{ if } 1 \leq i \leq n, f(v_i) = 4i - 3 \text{ and } f(w_i) = 4i - 1 \text{ if } 1 \leq i \leq n - 1.$$ 

It can be verified that the induced edge labels of quadrilateral snake are $2, 4, \ldots, 8n - 8$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly $f$ is an odd vertex equitable even labeling of quadrilateral snake. Thus, quadrilateral snake is an odd vertex equitable even graph. \Box

Theorem 2.14. The ladder graph $L_n$ is an odd vertex equitable even graph for all $n$. 


**Proof.** Let \( u_i \) and \( v_i \) be the vertices of \( L_n \). Then \( E(L_n) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{v_{i+1}v_{i+2} : 1 \leq i \leq n-1\} \). Then \( L_n \) has \( 2n \) vertices and \( 3n - 2 \) edges.

Let \( A = \begin{cases} 1,3,...,3n-2 & \text{if } n \text{ is odd} \\ 1,3,...,3n-1 & \text{if } n \text{ is even} \end{cases} \).

Define a vertex labeling \( f : V(L_n) \rightarrow A \) as follows:
\[
f(u_{2i-1}) = f(v_{2i-1}) = 6i - 5 \text{ if } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \quad f(u_{2i}) = 6i - 1 \text{ and } \quad f(v_i) = 6i - 3 \text{ if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor.
\]

It can be verified that the induced edge labels of \( L_n \) are \( 2,4,\ldots,6n-4 \) and \(|v_f(i) - v_f(j)| \leq 1\) for all \( i,j \in A \). Clearly \( f \) is an odd vertex equitable even labeling of \( L_n \). Thus, \( L_n \) is an odd vertex equitable even graph. \( \square \)

**Theorem 2.15.** The graph \( L_n \odot K_1 \) is an odd vertex equitable even graph for all \( n \).

**Proof.** Let \( L_n \) be the ladder. Let \( L_n \odot K_1 \) be the graph obtained by joining a pendant edge to each vertex of the ladder. Let \( u_i \) and \( v_i \) be the vertices of \( L_n \). For \( 1 \leq i \leq n, u'_i \) and \( v'_i \) be the new vertices adjacent with \( u_i \) and \( v_i \) respectively. Clearly \( L_n \odot K_1 \) has \( 4n \) vertices and \( 5n - 2 \) edges.

Let \( A = \begin{cases} 1,3,...,5n-2 & \text{if } n \text{ is odd} \\ 1,3,...,5n-1 & \text{if } n \text{ is even} \end{cases} \).

Define a vertex labeling \( f : V(L_n \odot K_1) \rightarrow A \) as follows:
\[
\begin{align*}
&f(u_{2i-1}) = f(u'_{2i-1}) = 10i - 9, f(v_{2i-1}) = f(v'_{2i-1}) = 10i - 7. \\
&f(u_{2i}) = 10i - 1, f(v_{2i}) = 10i - 5, f(u'_{2i}) = f(v'_{2i}) = 10i - 3.
\end{align*}
\]

It can be verified that the induced edge labels of \( L_n \odot K_1 \) are \( 2,4,\ldots,10n-4 \) and \(|v_f(i) - v_f(j)| \leq 1\) for all \( i,j \in A \). Clearly \( f \) is a odd vertex equitable even labeling of \( L_n \odot K_1 \). Thus, \( L_n \odot K_1 \) is an odd vertex equitable even graph. \( \square \)

**Theorem 2.16.** The arbitrary super subdivision of any path \( P_n \) is an odd vertex equitable even graph.

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be the vertices and \( e_i = v_iv_{i+1} \) be the edges of the path \( P_n \) for \( 1 \leq i \leq n-1 \). Let \( G \) be an arbitrary super subdivision of the path \( P_n \). That is, for \( 1 \leq i \leq n-1 \) each edge \( e_i \) of the path \( P_n \) is replaced by a complete bipartite graph \( K_{2,m_i} \) where \( m_i \) is any positive integer. Let \( V(G) = \{v_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq j \leq m_i, 1 \leq i \leq n-1\} \).
Clearly $G$ has $m_1 + m_2 + \ldots + m_{n-1} + n$ vertices and $2(m_1 + m_2 + \ldots + m_{n-1})$ edges. Let $A = \{1, 3, \ldots, 2(m_1 + m_2 + \ldots + m_{n-1}) + 1\}$.

Define a vertex labeling $f : V(G) \rightarrow A$ as follows:

- $f(v_1) = 1$, $f(v_i) = 2(m_1 + m_2 + \ldots + m_i) + 1$ if $2 \leq i \leq n$,
- $f(u_{1j}) = 2j - 1$ if $1 \leq j \leq m_1$ and
- $f(u_{ij}) = f(v_i) + 2j - 2$ if $2 \leq i \leq n - 1, 1 \leq j \leq m_i$.

Therefore the induced edge labels of $G$ are $2, 4, \ldots, 4(m_1 + m_2 + \ldots + m_{n-1})$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly $f$ is an odd vertex equitable even labeling of $G$. Thus, arbitrary super subdivision of any path is an odd vertex equitable even graph. \(\square\)

References


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