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A weakened version of Davis-Choi-Jensen's inequality for normalised positive linear maps

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Abstract

In this paper we show that the celebrated Davis-Choi-Jensen's inequality for normalised positive linear maps can be extended in a weakened form for convex functions. A reverse inequality and applications for important instances of convex (concave) functions are also given.

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1. Introduction

The following result that provides an vector operator version for the Jensen inequality is well known, see for instance [6] or [7, p. 5]:

Theorem 1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\text{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$(1.1) \quad f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 1 we have the *Hölder-McCarthy inequality* [5]: Let A be a selfadjoint positive operator on a Hilbert space H , then

- (i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;
- (ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;
- (iii) If A is invertible, then $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r < 0$ and $x \in H$ with $\|x\| = 1$.

In [2] (see also [3, p. 16]) we obtained the following additive reverse of (1.1):

Theorem 2. *Let I be an interval and $f : I \rightarrow \mathbf{R}$ be a convex and differentiable function on $\overset{\circ}{I}$ (the interior of I) whose derivative f' is continuous on I . If A is a selfadjoint operators on the Hilbert space H with $\text{Sp}(A) \subset I$, then*

$$(1.2) \quad \begin{aligned} & (0 \leq) \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \\ & \leq \langle f'(A)Ax, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A)x, x \rangle \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

This is a generalization of the scalar discrete inequality obtained in [4]. For other reverse inequalities of this type see [3, p. 16].

The following particular cases are of interest: If A is a selfadjoint operator on H , then we have the inequality:

$$(1.3) \quad \begin{aligned} & (0 \leq) \langle \exp(A)x, x \rangle - \exp(\langle Ax, x \rangle) \\ & \leq \langle A \exp(A)x, x \rangle - \langle Ax, x \rangle \langle \exp(A)x, x \rangle, \end{aligned}$$

for each $x \in H$ with $\|x\| = 1$.

Let A be a positive definite operator on the Hilbert space H . Then we have the following inequality for the logarithm:

$$(1.4) \quad \begin{aligned} (0 \leq) \ln(\langle Ax, x \rangle) - \langle \ln(A)x, x \rangle \\ \leq \langle Ax, x \rangle \langle A^{-1}x, x \rangle - 1, \end{aligned}$$

for each $x \in H$ with $\|x\| = 1$.

If $p \geq 1$ and A is a positive operator on H , then

$$(1.5) \quad (0 \leq) \langle A^p x, x \rangle - \langle Ax, x \rangle^p \leq p \left[\langle A^p x, x \rangle - \langle Ax, x \rangle \langle A^{p-1} x, x \rangle \right],$$

for each $x \in H$ with $\|x\| = 1$. If A is positive definite, then the inequality (1.5) also holds for $p < 0$. If $0 < p < 1$ and A is a positive definite operator then the reverse inequality also holds

$$(1.6) \quad (0 \leq) \langle Ax, x \rangle^p - \langle A^p x, x \rangle \leq p \left[\langle Ax, x \rangle \cdot \langle A^{p-1} x, x \rangle - \langle A^p x, x \rangle \right],$$

for each $x \in H$ with $\|x\| = 1$.

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H . We denote by $\mathcal{B}_h(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H .

Let H, K be complex Hilbert spaces. Following [1] (see also [7, p. 18]) we can introduce the following definition:

Definition 1. A map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely $\Phi(\lambda A + \mu B) = \lambda \Phi(A) + \mu \Phi(B)$ for any $\lambda, \mu \in \mathbf{C}$ and $A, B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in P[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e. $\Phi(1_H) = 1_K$. We write $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \leq A \leq \beta 1_H$, then $\alpha 1_K \leq \Phi(A) \leq \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1_H) \Psi \Psi^{-1/2}(1_H)$ we get that $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

A real valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda)f(A) + \lambda f(B)$$

for all $\lambda \in [0, 1]$ and for every selfadjoint operators $A, B \in \mathcal{B}(H)$ whose spectra are contained in I .

The following Jensen's type result is well known:

Theorem 3 (Davis-Choi-Jensen's Inequality). *Let $f : I \rightarrow \mathbf{R}$ be an operator convex function on the interval I and $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, then for any selfadjoint operator A whose spectrum is contained in I we have*

$$(1.7) \quad f(\Phi(A)) \leq \Phi(f(A)).$$

We observe that if $\Psi \in P[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1_H) \Psi \Psi^{-1/2}(1_H)$ in (1.7) we get

$$f(\Psi^{-1/2}(1_H) \Psi(A) \Psi^{-1/2}(1_H)) \leq \Psi^{-1/2}(1_H) \Psi(f(A)) \Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps*

$$(1.8) \quad \begin{aligned} & \Psi^{1/2}(1_H) f(\Psi^{-1/2}(1_H) \Psi(A) \Psi^{-1/2}(1_H)) \Psi^{1/2}(1_H) \\ & \leq \Psi(f(A)). \end{aligned}$$

It is obvious that, by (1.7) we have the vector inequality

$$(1.9) \quad \langle f(\Phi(A))y, y \rangle \leq \langle \Phi(f(A))y, y \rangle$$

for any $y \in K$. By (1.1) we also have

$$(1.10) \quad f(\langle \Phi(A)y, y \rangle) \leq \langle f(\Phi(A))y, y \rangle$$

for any $y \in K$, $\|y\| = 1$. Therefore, for an operator convex function on I we have

$$(1.11) \quad f(\langle \Phi(A)y, y \rangle) \leq \langle f(\Phi(A))y, y \rangle \leq \langle \Phi(f(A))y, y \rangle$$

for any $y \in K$, $\|y\| = 1$.

It is then natural to ask the following question:

Does the inequality between the first and last term in (1.11) remains valid in the more general case of convex functions defined on the interval I ?

A positive answer to this question and some reverse inequalities are provided below. Some applications for important instances of convex (concave) functions are also given.

2. A Jensen's Type Inequality

Suppose that I is an interval of real numbers with interior I and $f : I \rightarrow \mathbf{R}$ is a convex function on I . Then f is continuous on I and has finite left and right derivatives at each point of I . Moreover, if $t, s \in I$ and $t < s$, then $f'_-(t) \leq f'_+(t) \leq f'_-(s) \leq f'_+(s)$ which shows that both f'_- and f'_+ are nondecreasing function on I . It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \rightarrow \mathbf{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \rightarrow [-\infty, \infty]$ such that $\varphi(I) \subset \mathbf{R}$ and

$$(2.1) \quad f(t) \geq f(a) + (t - a)\varphi(a) \text{ for any } t, a \in I.$$

It is also well known that if f is convex on I , then ∂f is nonempty, $f'_-, f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(t) \leq \varphi(t) \leq f'_+(t) \text{ for any } t \in I.$$

In particular, φ is a nondecreasing function. If f is differentiable and convex on I , then $\partial f = \{f'\}$.

We have:

Theorem 1. *Let $f : I \rightarrow \mathbf{R}$ be a convex function on the interval I and $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ a normalised positive linear map. Then for any selfadjoint operator A whose spectrum $\text{Sp}(A)$ is contained in I we have*

$$(2.2) \quad f(\langle \Phi(A)y, y \rangle) \leq \langle \Phi(f(A))y, y \rangle$$

for any $y \in K$, $\|y\| = 1$.

Proof. Let m, M with $m < M$ and such that $\text{Sp}(A) \subseteq [m, M] \subset I$. Then $m1_H \leq A \leq M1_H$ and since $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have that $m1_K \leq \Phi(A) \leq M1_K$ showing that $\langle \Phi(A)y, y \rangle \in [m, M]$ for any $y \in K$, $\|y\| = 1$.

By the gradient inequality (2.1) we have for $a = \langle \Phi(A)y, y \rangle \in [m, M]$ that

$$f(t) \geq f(\langle \Phi(A)y, y \rangle) + (t - \langle \Phi(A)y, y \rangle) \varphi(\langle \Phi(A)y, y \rangle)$$

for any $t \in I$.

Using the continuous functional calculus for the operator A we have for a fixed $y \in K$ with $\|y\| = 1$ that

$$f(A) \geq f(\langle \Phi(A)y, y \rangle) 1_H + \varphi(\langle \Phi(A)y, y \rangle) (A - \langle \Phi(A)y, y \rangle 1_H). \quad (2.3)$$

Since $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by taking the functional Φ in the inequality (2.3) we get

$$\Phi(f(A)) \geq f(\langle \Phi(A)y, y \rangle) 1_K + \varphi(\langle \Phi(A)y, y \rangle) (\Phi(A) - \langle \Phi(A)y, y \rangle 1_K) \quad (2.4)$$

for any $y \in K$ with $\|y\| = 1$.

This inequality is of interest in itself.

Taking the inner product in (2.4) we have for any $y \in K$ with $\|y\| = 1$ that

$$\begin{aligned} & \langle \Phi(f(A))y, y \rangle \\ & \geq f(\langle \Phi(A)y, y \rangle) \|y\|^2 + \varphi(\langle \Phi(A)y, y \rangle) (\langle \Phi(A)y, y \rangle - \langle \Phi(A)y, y \rangle \|y\|^2) \\ & = f(\langle \Phi(A)y, y \rangle) \end{aligned}$$

and the inequality (2.2) is proved. $\square$

Corollary 1. Let $f : I \rightarrow \mathbf{R}$ be a convex function on the interval I and $\Psi \in P[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$. Then for any selfadjoint operator A whose spectrum $\text{Sp}(A)$ is contained in I we have

$$(2.5) \quad f \left(\frac{\langle \Psi(A) v, v \rangle}{\langle \Psi(1_H) v, v \rangle} \right) \leq \frac{\langle \Psi(f(A)) v, v \rangle}{\langle \Psi(1_H) v, v \rangle}$$

for any $v \in K$ with $v \neq 0$.

Proof. If we write the inequality (2.2) for $\Phi = \Psi^{-1/2}(1_H) \Psi \Psi^{-1/2}(1_H)$ we have

$$\begin{aligned} & f \left(\left\langle \Psi^{-1/2}(1_H) \Psi(A) \Psi^{-1/2}(1_H) y, y \right\rangle \right) \\ & \leq \left\langle \Psi^{-1/2}(1_H) \Psi(f(A)) \Psi^{-1/2}(1_H) y, y \right\rangle \end{aligned}$$

for any $y \in K$, $\|y\| = 1$.

Now, let $v \in K$ with $v \neq 0$ and take $y = \frac{1}{\|\Psi^{1/2}(1_H)v\|} \Psi^{1/2}(1_H)v$ in (2) to get

$$\begin{aligned} & f \left(\left\langle \Psi^{-1/2}(1_H) \Psi(A) \Psi^{-1/2}(1_H) \frac{\Psi^{1/2}(1_H)v}{\|\Psi^{1/2}(1_H)v\|}, \frac{\Psi^{1/2}(1_H)v}{\|\Psi^{1/2}(1_H)v\|} \right\rangle \right) \\ & \leq \left\langle \Psi^{-1/2}(1_H) \Psi(f(A)) \Psi^{-1/2}(1_H) \frac{\Psi^{1/2}(1_H)v}{\|\Psi^{1/2}(1_H)v\|}, \frac{\Psi^{1/2}(1_H)v}{\|\Psi^{1/2}(1_H)v\|} \right\rangle \end{aligned}$$

that is equivalent to

$$f \left(\left\langle \frac{\Psi(A)v}{\|\Psi^{1/2}(1_H)v\|}, \frac{v}{\|\Psi^{1/2}(1_H)v\|} \right\rangle \right) \leq \left\langle \frac{\Psi(f(A))v}{\|\Psi^{1/2}(1_H)v\|}, \frac{v}{\|\Psi^{1/2}(1_H)v\|} \right\rangle$$

and since

$$\|\Psi^{1/2}(1_H)v\|^2 = \langle \Psi(1_H)v, v \rangle$$

for $v \in K$ with $v \neq 0$, then we obtain the desired inequality (2.5). $\square$

By taking some example of fundamental convex (concave) functions, we can state the following results:

Let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a normalised positive linear map.

(i) If A is a selfadjoint operator on H and $r \geq 1$, then we have

$$(2.6) \quad |\langle \Phi(A) y, y \rangle|^r \leq \langle \Phi(|A|^r) y, y \rangle$$

and in particular

$$(2.7) \quad |\langle \Phi(A)y, y \rangle| \leq \langle \Phi(|A|)y, y \rangle$$

for all $y \in K$, $\|y\| = 1$. We have the norm inequality

$$(2.8) \quad \|\Phi(A)\|^r \leq \|\Phi(|A|^r)\|.$$

(ii) If A is a positive operator on a Hilbert space H , then for any $p \geq 1$ ($p \in (0, 1)$) we have

$$(2.9) \quad \langle \Phi(A)y, y \rangle^p \leq (\geq) \langle \Phi(A^p)y, y \rangle$$

for all $y \in K$, $\|y\| = 1$. We have the norm inequality

$$(2.10) \quad \|\Phi(A)\|^p \leq (\geq) \|\Phi(A^p)\|.$$

If A is a positive definite operator on a Hilbert space H , then for any $p < 0$ we have

$$(2.11) \quad \langle \Phi(A)y, y \rangle^p \leq \langle \Phi(A^p)y, y \rangle$$

for all $y \in K$, $\|y\| = 1$.

(iii) If A is a selfadjoint operator on H then we have

$$(2.12) \quad \exp(\langle \Phi(A)y, y \rangle) \leq \langle \Phi(\exp(A))y, y \rangle$$

for all $y \in K$, $\|y\| = 1$. We have the norm inequality

$$(2.13) \quad \exp(\|\Phi(A)\|) \leq \|\Phi(\exp(A))\|.$$

Let $P_j \in \mathcal{B}(H)$, $j = 1, \dots, k$ be contractions with

$$(2.14) \quad \sum_{j=1}^k P_j^* P_j = 1_H.$$

The map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ defined by [7]

$$\Phi(A) := \sum_{j=1}^k P_j^* A P_j$$

is a normalized positive linear map on $\mathcal{B}(H)$. Therefore, if $f : I \rightarrow \mathbf{R}$ be a convex function on the interval I and A is selfadjoint operator whose spectrum $\text{Sp}(A)$ is contained in I , we have by (2.2) that

$$(2.15) \quad f \left(\sum_{j=1}^k \langle P_j^* A P_j y, y \rangle \right) \leq \left\langle \sum_{j=1}^k P_j^* f(A) P_j y, y \right\rangle$$

for all $y \in K$, $\|y\| = 1$.

If we take $k = 1$ and $P_1 = 1_H$ in (2.15), then we recapture Jensen's inequality (1.1).

We then have for any selfadjoint operator A and $r \geq 1$ that

$$(2.16) \quad \left| \sum_{j=1}^k \langle P_j^* A P_j y, y \rangle \right|^r \leq \left\langle \sum_{j=1}^k P_j^* |A|^r P_j y, y \right\rangle$$

and

$$(2.17) \quad \exp \left(\sum_{j=1}^k \langle P_j^* A P_j y, y \rangle \right) \leq \left\langle \sum_{j=1}^k P_j^* (\exp A) P_j y, y \right\rangle$$

for all $y \in K$, $\|y\| = 1$. In the case $r = 1$ we have

$$(2.18) \quad \left| \sum_{j=1}^k \langle P_j^* A P_j y, y \rangle \right| \leq \left\langle \sum_{j=1}^k P_j^* |A| P_j y, y \right\rangle.$$

By taking the supremum over $y \in K$, $\|y\| = 1$ we also have the norm inequalities

$$(2.19) \quad \left\| \sum_{j=1}^k P_j^* A P_j \right\|^r \leq \left\| \sum_{j=1}^k P_j^* |A|^r P_j \right\|, \quad r \geq 1$$

and

$$(2.20) \quad \exp \left(\left\| \sum_{j=1}^k P_j^* A P_j \right\| \right) \leq \left\| \sum_{j=1}^k P_j^* (\exp A) P_j \right\|.$$

In the case $r = 1$ we have

$$(2.21) \quad \left\| \sum_{j=1}^k P_j^* A P_j \right\| \leq \left\| \sum_{j=1}^k P_j^* |A| P_j \right\|.$$

If A is a positive operator on a Hilbert space H , then for any $p \in (-\infty, 0) \cup [1, \infty)$ ($p \in (0, 1)$) we have by (2.15) for power function that

$$(2.22) \quad \left\langle \sum_{j=1}^k P_j^* A P_j y, y \right\rangle^p \leq (\geq) \left\langle \sum_{j=1}^k P_j^* A^p P_j y, y \right\rangle$$

for all $y \in K$, $\|y\| = 1$.

If we take $k = 1$ and $P_1 = 1_H$ in (2.22), then we recapture Hölder-McCarthy's inequality.

By taking the supremum over $y \in K$, $\|y\| = 1$ we also have the norm inequality

$$(2.23) \quad \left\| \sum_{j=1}^k P_j^* A P_j \right\|^p \leq (\geq) \left\| \sum_{j=1}^k P_j^* A^p P_j \right\|,$$

where $p \geq 1$ ($p \in (0, 1)$).

3. A Reverse Inequality

We have:

Theorem 1. *Let I be an interval and $f : I \rightarrow \mathbf{R}$ be a convex and differentiable function on I whose derivative f' is continuous on I . If $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is a normalised positive linear map and A is a selfadjoint operators on the Hilbert space H with $\text{Sp}(A) \subset I$, then*

$$(3.1) \quad \begin{aligned} 0 &\leq \langle \Phi(f(A)) y, y \rangle - f(\langle \Phi(A) y, y \rangle) \\ &\leq \langle \Phi(A f'(A)) y, y \rangle - \langle \Phi(A) y, y \rangle \langle \Phi(f'(A)) y, y \rangle \end{aligned}$$

for any $y \in K$, $\|y\| = 1$.

Proof. From the gradient inequality (2.1) we have

$$(3.2) \quad f(t) \geq f(s) + (t - s) f'(s)$$

for any $t, s \in I$.

Let $y \in K$, $\|y\| = 1$. If we take in (3.2) $t = \langle \Phi(A) y, y \rangle \in I$, then we get

$$f(\langle \Phi(A) y, y \rangle) \geq f(s) + (\langle \Phi(A) y, y \rangle - s) f'(s)$$

for any $s \in I$ that can be written as

$$(s - \langle \Phi(A) y, y \rangle) f'(s) \geq f(s) - f(\langle \Phi(A) y, y \rangle)$$

for any $s \in I$.

Let $y \in K$, $\|y\| = 1$. Using the continuous functional calculus for the operator A we have

$$(3.3) \quad Af'(A) - \langle \Phi(A)y, y \rangle f'(A) \geq f(A) - f(\langle \Phi(A)y, y \rangle) 1_H.$$

Since $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by taking the functional Φ in the inequality (3.3) we have

$$(3.4) \quad \begin{aligned} & \Phi(Af'(A)) - \langle \Phi(A)y, y \rangle \Phi(f'(A)) \\ & \geq \Phi(f(A)) - f(\langle \Phi(A)y, y \rangle) 1_K, \end{aligned}$$

for any $y \in K$, $\|y\| = 1$.

This is an inequality of interest in itself.

Taking the inner product in (3.4) we obtain the desired result (3.1). $\square$

Corollary 2. *Let I be an interval and $f : I \rightarrow \mathbf{R}$ be a convex and differentiable function on I whose derivative f' is continuous on I . If $\Psi \in P[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$ and A is a selfadjoint operators on the Hilbert space H with $\text{Sp}(A) \subset I$, then*

$$(3.5) \quad \begin{aligned} 0 & \leq \frac{\langle \Psi(f(A))v, v \rangle}{\langle \Psi(1_H)v, v \rangle} - f\left(\frac{\langle \Psi(A)v, v \rangle}{\langle \Psi(1_H)v, v \rangle}\right) \\ & \leq \frac{\langle \Psi(Af'(A))v, v \rangle}{\langle \Psi(1_H)v, v \rangle} - \frac{\langle \Psi(A)v, v \rangle}{\langle \Psi(1_H)v, v \rangle} \frac{\langle \Psi(f'(A))v, v \rangle}{\langle \Psi(1_H)v, v \rangle} \end{aligned}$$

for any $v \in K$ with $v \neq 0$.

The proof follows from the inequality (3.1) by a similar argument to the one from the proof of Corollary 1 and the details are omitted.

Let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a normalised positive linear map.

(i) If A is a selfadjoint operator on H , then we have

$$(3.6) \quad \begin{aligned} 0 & \leq \langle \Phi(\exp(A))y, y \rangle - \exp(\langle \Phi(A)y, y \rangle) \\ & \leq \langle \Phi(A \exp(A))y, y \rangle - \langle \Phi(A)y, y \rangle \langle \Phi(\exp(A))y, y \rangle \end{aligned}$$

for all $y \in K$, $\|y\| = 1$.

(ii) If A is a positive (positive definite) operator on a Hilbert space H , then for any $p \geq 1$ ($p \in (-\infty, 0)$) we have

$$(3.7) \quad \begin{aligned} 0 & \leq \langle \Phi(A^p)y, y \rangle - \langle \Phi(A)y, y \rangle^p \\ & \leq p [\langle \Phi(A^p)y, y \rangle - \langle \Phi(A)y, y \rangle \langle \Phi(A^{p-1})y, y \rangle] \end{aligned}$$

for all $y \in K$, $\|y\| = 1$.

If A is a positive operator on a Hilbert space H , then for any $p \in (0, 1)$ we have

$$(3.8) \quad \begin{aligned} 0 &\leq \langle \Phi(A)y, y \rangle^p - \langle \Phi(A^p)y, y \rangle \\ &\leq p [\langle \Phi(A)y, y \rangle \langle \Phi(A^{p-1})y, y \rangle - \langle \Phi(A^p)y, y \rangle] \end{aligned}$$

for all $y \in K$, $\|y\| = 1$.

(iii) If A is a positive definite operator on a Hilbert space H , then

$$(3.9) \quad \begin{aligned} 0 &\leq \ln(\langle \Phi(A)y, y \rangle) - \langle \Phi(\ln A)y, y \rangle \\ &\leq \langle \Phi(A)y, y \rangle \langle \Phi(A^{-1})y, y \rangle - 1 \end{aligned}$$

for all $y \in K$, $\|y\| = 1$.

Let $P_j \in \mathcal{B}(H)$, $j = 1, \dots, k$ be contractions with the property (2.14). If $f : I \rightarrow \mathbf{R}$ is a convex function on the interval I and A is selfadjoint operator whose spectrum $\text{Sp}(A)$ is contained in I , then we have by (3.1) that

$$(3.10) \quad \begin{aligned} 0 &\leq \left\langle \sum_{j=1}^k P_j^* f(A) P_j y, y \right\rangle - f \left(\sum_{j=1}^k \left\langle P_j^* A P_j y, y \right\rangle \right) \\ &\leq \left\langle \sum_{j=1}^k P_j^* A f'(A) P_j y, y \right\rangle - \left\langle \sum_{j=1}^k P_j^* A P_j y, y \right\rangle \left\langle \sum_{j=1}^k P_j^* f'(A) P_j y, y \right\rangle \end{aligned}$$

for all $y \in K$, $\|y\| = 1$. This is a generalization of (1.2).

In particular, if A is a selfadjoint operator on H , then we have

$$(3.11) \quad \begin{aligned} 0 &\leq \left\langle \sum_{j=1}^k P_j^* \exp(A) P_j y, y \right\rangle - \exp \left(\sum_{j=1}^k \left\langle P_j^* A P_j y, y \right\rangle \right) \\ &\leq \left\langle \sum_{j=1}^k P_j^* A \exp(A) P_j y, y \right\rangle - \left\langle \sum_{j=1}^k P_j^* A P_j y, y \right\rangle \left\langle \sum_{j=1}^k P_j^* \exp(A) P_j y, y \right\rangle \end{aligned}$$

for all $y \in K$, $\|y\| = 1$.

If A is a positive (positive definite) operator on a Hilbert space H , then for any $p \geq 1$ ($p \in (-\infty, 0)$) we have

$$\begin{aligned} 0 &\leq \left\langle \sum_{j=1}^k P_j^* A^p P_j y, y \right\rangle - \left(\sum_{j=1}^k \left\langle P_j^* A P_j y, y \right\rangle \right)^p \\ &\leq p \left[\left\langle \sum_{j=1}^k P_j^* A^p P_j y, y \right\rangle - \left\langle \sum_{j=1}^k P_j^* A P_j y, y \right\rangle \left\langle \sum_{j=1}^k P_j^* A^{p-1} P_j y, y \right\rangle \right], \end{aligned}$$

(3.12)

for all $y \in K$, $\|y\| = 1$. However, when $p \in (0, 1)$ and A is a positive, then

$$\begin{aligned} 0 &\leq \left(\sum_{j=1}^k \langle P_j^* A P_j y, y \rangle \right)^p - \left\langle \sum_{j=1}^k P_j^* A^p P_j y, y \right\rangle \\ &\leq p \left[\left\langle \sum_{j=1}^k P_j^* A P_j y, y \right\rangle \left\langle \sum_{j=1}^k P_j^* A^{p-1} P_j y, y \right\rangle - \left\langle \sum_{j=1}^k P_j^* A^p P_j y, y \right\rangle \right], \end{aligned} \quad (3.13)$$

for all $y \in K$, $\|y\| = 1$.

If A is a positive definite operator on H , then

$$\begin{aligned} 0 &\leq \ln \left(\sum_{j=1}^k \langle P_j^* A P_j y, y \rangle \right) - \left\langle \sum_{j=1}^k P_j^* (\ln A) P_j y, y \right\rangle \\ &\leq \left\langle \sum_{j=1}^k P_j^* A P_j y, y \right\rangle \left\langle \sum_{j=1}^k P_j^* A^{-1} P_j y, y \right\rangle - 1 \end{aligned} \quad (3.14)$$

for all $y \in K$, $\|y\| = 1$.

These inequalities generalize the corresponding results from (1.4)-(1.6).

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