Jensen's and the quadratic functional equations with an endomorphism

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Abstract

We determine the solutions $f : S \rightarrow H$ of the generalized Jensen's functional equation

$$f(x + y) + f(x + \varphi(y)) = 2f(x), \quad x, y \in S,$$

and the solutions $f : S \rightarrow H$ of the generalized quadratic functional equation

$$f(x + y) + f(x + \varphi(y)) = 2f(x) + 2f(y), \quad x, y \in S,$$

where S is a commutative semigroup, H is an abelian group (2-torsion free in the first equation and uniquely 2-divisible in the second) and φ is an endomorphism of S .

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1. Set up, notation and terminology

Throughout the paper we work in the following framework and with the following notation and terminology. We use it without explicit mentioning.

S is a commutative semigroup [a set equipped with an associative composition rule $(x, y) \mapsto x + y$], $\varphi : S \rightarrow S$ is an endomorphism and $(H, +)$ denotes an abelian group with neutral element 0. We say that H is 2-torsion free if $[h \in H \text{ and } 2h = 0] \Rightarrow h = 0$. H is said to be uniquely 2-divisible if for any $h \in H$ the equation $2x = h$ has exactly one solution $x \in H$.

A function $A : S \rightarrow H$ is said to be additive if $A(x + y) = A(x) + A(y)$ for all $x, y \in S$.

We recall that the Cauchy difference Cf of a function $f : S \rightarrow H$ is defined by

$$Cf(x, y) := f(x + y) - f(x) - f(y), \quad x, y \in S.$$

2. Introduction

Let $\sigma \in \text{Hom}(S, S)$ satisfy $\sigma^2 = \text{id}$. In [11], Sinopoulos determined the general solution $f : S \rightarrow H$, where H is 2-torsion free, of Jensen's functional equation

$$(2.1) \quad f(x + y) + f(x + \sigma(y)) = 2f(x), \quad x, y \in S,$$

and the general solution $f : S \rightarrow H$, where H is uniquely 2-divisible, of the quadratic functional equation

$$(2.2) \quad f(x + y) + f(x + \sigma(y)) = 2f(x) + 2f(y), \quad x, y \in S.$$

The purpose of the present paper is to solve the functional equations (2.1) and (2.2) on semigroups without using the condition $\sigma^2 = \text{id}$. More precisely, we solve the functional equations

$$(2.3) \quad f(x + y) + f(x + \varphi(y)) = 2f(x), \quad x, y \in S,$$

$$(2.4) \quad f(x + y) + f(x + \varphi(y)) = 2f(x) + 2f(y), \quad x, y \in S,$$

where φ is an endomorphism of S . By elementary methods we show that our formulas for the solutions of (2.3) (resp. (2.4)) on semigroups are the same as those of (2.1) (resp. (2.2)), so that our results constitute a natural extension of earlier results of, e.g., [11], from Jensen's (resp. the quadratic) functional equation with an involutive automorphism to that

with an endomorphism. Note that the equation (2.3) has been solved in [3] under the assumption that φ is surjective. Here we remove this restriction.

A similar functional equation that has been studied is

$$(2.5) \quad f(x+y) + f(x+\varphi(y)) = 2f(x)f(y), \quad x, y \in S,$$

where $f : S \rightarrow C$ is the function to determine. Eq. (2.5) was solved in a more general framework (see [3]).

3. The generalized Jensen's functional equation

In this section, we solve the functional equation (2.3) by expressing its solutions in terms of additive functions.

Lemma 3.1. *Let $f : S \rightarrow H$ be a solution of the functional equation (2.3). Then*

$$(3.1) \quad f(x + \varphi^2(y)) = f(x + y) \quad \text{for all } x, y \in S.$$

Proof. Replacing y by $\varphi(y)$ in (2.3), we get that

$$f(x + \varphi(y)) + f(x + \varphi^2(y)) = 2f(x).$$

Using this equation and (2.3), we obtain (3.1). $\square$

Lemma 3.2. *Let S be a semigroup (not necessarily abelian), $\phi : S \rightarrow S$ an endomorphism. If $f : S \rightarrow H$ and $\Phi : S \times S \rightarrow H$ satisfy*

$$(3.2) \quad f(xy) + f(\phi(y)x) = \Phi(x, y) \quad \text{for all } x, y \in S,$$

then

$$(3.3) \quad 2f(xyz) = \Phi(x, yz) - \Phi(\phi(z)x, y) + \Phi(xy, z) \quad \text{for all } x, y, z \in S.$$

Proof. Making the substitutions (x, yz) , $(\phi(z)x, y)$, and (xy, z) in (3.2), we get respectively

$$\begin{aligned} f(xyz) + f(\phi(yz)x) &= \Phi(x, yz), \\ f(\phi(z)xy) + f(\phi(yz)x) &= \Phi(\phi(z)x, y) \\ f(xyz) + f(\phi(z)xy) &= \Phi(xy, z). \end{aligned}$$

Subtracting the middle identity from the sum of the other two we get (3.3).

$\square$

Theorem 3.3. *Suppose that H is 2-torsion free. The general solution $f : S \rightarrow H$ of the functional equation (2.3) is $f = A + c$, where $A : S \rightarrow H$ is an additive map such that $A \circ \varphi = -A$, and where $c \in H$ is a constant.*

Proof. The method used here is closely related to and inspired by the one in [11, Proof of Theorem 2]. Assume that $f : S \rightarrow H$ is a solution of (2.3). Replacing y by $y + \varphi(y)$ in (2.3) and using Lemma 3.1, we get

$$(3.4) \quad f(x + y + \varphi(y)) = f(x).$$

Using lemma 3.2 with $\Phi(x, y) := 2f(x)$, we find after division by 2 that

$$\begin{aligned} f(x + y + z) &= f(x) - f(\varphi(z) + x) + f(x + y) \\ &= f(x) - [f(x + z) + f(\varphi(z) + x)] + f(x + z) + f(x + y) \\ &= f(x + y) + f(x + z) - f(x). \end{aligned}$$

Setting here $z = \varphi(x)$ and using (3.4), we get

$$(3.5) \quad f(y) + f(x) = f(x + \varphi(x)) + f(x + y).$$

Interchanging x and y in (3.5), we get that $f(x + \varphi(x)) = f(y + \varphi(y))$ for all $x, y \in S$. So $f(x + \varphi(x))$ is a constant, say c . By using (3.5), we infer that the function $A(x) := f(x) - c$ is additive. Substituting f into (2.3) we see that $A \circ \varphi = -A$.

The other direction of the proof is trivial to verify. $\square$

As an immediate consequence of Theorem 3.3, we have the following result.

Corollary 3.4 (2, Theorem 3.2). *Suppose that H is 2-torsion free and let $\sigma, \tau \in \text{Hom}(S, S)$ such that $\sigma^2 = \tau^2 = \text{id}$. The general solution $f : S \rightarrow H$ of the functional equation $f(x + \sigma(y)) + f(x + \tau(y)) = 2f(x)$, $x, y \in S$, is $f = A + c$, where $A : S \rightarrow H$ is an additive map such that $A \circ \tau = -A \circ \sigma$, and where $c \in H$ is a constant.*

Proof. The proof follows on putting $\varphi = \tau \circ \sigma$ in Theorem 3.3. $\square$

4. The generalized quadratic functional equation

In this section, we generalize Sinopoulos's result [11, Theorem 3] on semi-groups by solving the functional equation (2.4).

The following lemma lists pertinent basic properties of any solution $f : S \rightarrow H$ of (2.4).

Lemma 4.1. *Suppose that H is 2-torsion free and let $f : S \rightarrow H$ be a solution of the functional equation (2.4).*

(a) $f \circ \varphi = f$.

(b) For all $x, y, z \in S$, we have

$$(4.1) \quad \begin{aligned} f(x + y + z) &= f(x + y) + f(x + z) \\ &\quad + f(y + z) - f(x) - f(y) - f(z). \end{aligned}$$

(c) $Cf : S \times S \rightarrow H$ is a symmetric, bi-additive map satisfying

$$Cf(x, \varphi(y)) = -Cf(x, y) \quad \text{for all } x, y \in S.$$

(d) Let $A : S \rightarrow H$ be $A(x) := f(x + \varphi(x))$, $x \in S$. Then $A \circ \varphi = A$ and A is additive.

(e) $2f(x) = Cf(x, x) + A(x)$ for all $x \in S$.

Proof. (a) Let us first observe that $f \circ \varphi$ is a solution of (2.4). We next replace x by $\varphi(x)$ in (2.4) we find that

$$f(\varphi(x) + y) + f(\varphi(x) + \varphi(y)) = 2f(\varphi(x)) + 2f(y).$$

Adding this equation to (2.4), we get

$$\begin{aligned} [f(x + y) + f(\varphi(x) + y)] + [f(x + \varphi(y)) + f(\varphi(x) + \varphi(y))] \\ = 2f(x) + 2f(\varphi(x)) + 4f(y). \end{aligned}$$

Using (2.4) and the fact that H is 2-torsion free, we obtain

$$[f(x) + f(y)] + [f(x) + f(\varphi(y))] = f(x) + f(\varphi(x)) + 2f(y),$$

i.e.

$$f(x) - f \circ \varphi(x) = f(y) - f \circ \varphi(y) \quad \text{for all } x, y \in S.$$

From this last equation we infer that $f - f \circ \varphi$ is a constant in H , say c . Using the fact that $f - f \circ \varphi$ is a solution of (2.4) and that H is 2-torsion free, we see that $c = 0$.

(b) Putting $\Phi(x, y) := 2f(x) + 2f(y)$ in lemma 3.2, we get after division by 2 that

$$\begin{aligned}
 f(x + y + z) &= f(x) + f(y + z) - f(\varphi(z) + x) - f(y) + f(x + y) + f(z) \\
 &= f(x) + f(y + z) - [f(x + z) + f(\varphi(z) + x)] + f(x + z) \\
 &\quad - f(y) + f(x + y) + f(z) \\
 &= f(x + y) + f(x + z) + f(y + z) - f(x) - f(y) - f(z),
 \end{aligned}$$

we get (4.1).

(c) That Cf is symmetric and bi-additive follows immediately from the very definition of Cf and (4.1). Let $x, y \in S$ be arbitrary. By help of (2.4) and (a), we get that

$$\begin{aligned}
 Cf(x, \varphi(y)) &= f(x + \varphi(y)) - f(x) - f \circ \varphi(y) \\
 &= 2f(x) + 2f(y) - f(x + y) - f(x) - f \circ \varphi(y) \\
 &= f(x) + f(y) - f(x + y) \\
 &= -Cf(x, y).
 \end{aligned}$$

(d) A is φ -even, because

$$\begin{aligned}
 A(\varphi(x)) &= f(\varphi(x) + \varphi^2(x)) = 2f \circ \varphi(x + \varphi(x)) \\
 &= f(x + \varphi(x)) = A(x) \quad \text{for all } x \in S.
 \end{aligned}$$

Next, let $x, y \in S$ be arbitrary. By help of (4.1) and (a), we find

$$\begin{aligned}
 A(x + y) &= f((x + \varphi(x)) + y + \varphi(y)) \\
 &= [f(x + \varphi(x) + y) + f(x + \varphi(x) + \varphi(y))] + f(y + \varphi(y)) \\
 &\quad - f(x + \varphi(x)) - f(y) - f \circ \varphi(y) \\
 &= 2f(x + \varphi(x)) + 2f(y) + f(y + \varphi(y)) - f(x + \varphi(x)) \\
 &\quad - f(y) - f \circ \varphi(y) \\
 &= f(x + \varphi(x)) + f(y + \varphi(y)) \\
 &= A(x) + A(y).
 \end{aligned}$$

(e) Using (2.4), we obtain

$$\begin{aligned}
 Cf(x, x) + A(x) &= f(x + x) + f(x + \varphi(x)) - 2f(x) \\
 &= 4f(x) - 2f(x) = 2f(x) \quad \text{for all } x \in S.
 \end{aligned}$$

The second main theorem of the present paper reads as follows.

Theorem 4.2. Suppose that H is uniquely 2-divisible. The general solution $f : S \rightarrow H$ of the functional equation (2.4) is

$$f(x) = Q(x, x) + A(x), \quad x \in S,$$

where $Q : S \times S \rightarrow H$ is an arbitrary symmetric, bi-additive map such that $Q(x, \varphi(y)) = -Q(x, y)$ for all $x, y \in S$, and where $A : S \rightarrow H$ is an arbitrary additive function such that $A \circ \varphi = A$.

Proof. That all solutions of (2.4) have this form is a consequence of Lemma 4.1 and the fact that H is uniquely 2-divisible. Conversely, simple computations based on the properties of Q and A , show that the indicated functions are solutions. $\square$

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