Abstract

For an edge $xy$ in a connected graph $G$ of order $p \geq 3$, a set $S \subseteq V(G)$ is an $xy$-monophonic set of $G$ if each vertex $v \in V(G)$ lies on an $x-u$ monophonic path or a $y-u$ monophonic path for some element $u$ in $S$. The minimum cardinality of an $xy$-monophonic set of $G$ is defined as the $xy$-monophonic number of $G$, denoted by $m_{xy}(G)$. An $xy$-monophonic set of cardinality $m_{xy}(G)$ is called an $m_{xy}$-set of $G$. We determine bounds for it and find the same for special classes of graphs. It is shown that for any three positive integers $r$, $d$ and $n \geq 2$ with $2 \leq r \leq d$, there exists a connected graph $G$ with monophonic radius $r$, monophonic diameter $d$ and $m_{xy}(G) = n$ for some edge $xy$ in $G$.

Key Words: Monophonic path, vertex monophonic number, edge fixed monophonic number.

Mathematics Subject Classification: 05C12.
1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. For basic graph theoretic terminology we refer to [1, 2]. For vertices $x$ and $y$ in a connected graph $G$, the distance $d(x, y)$ is the length of a shortest $x - y$ path in $G$. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. The neighborhood of a vertex $v$ is the set $N(v)$ consisting of all vertices $u$ which are adjacent with $v$. A vertex $v$ is a simplicial vertex if the subgraph induced by its neighbors is complete. A non-separable graph is connected, non-trivial, and has no cut-vertices. A block of a graph is a maximal non-separable subgraph. A connected block graph is a connected graph in which each of its blocks is complete. A caterpillar is a tree for which the removal of all the end vertices gives a path.

A chord of a path $P$ is an edge joining two non-adjacent vertices of $P$. A path $P$ is called monophonic if it is a chordless path. The closed interval $I_m[x, y]$ consists of all vertices lying on some $x - y$ monophonic of $G$. For any two vertices $u$ and $v$ in a connected graph $G$, the monophonic distance $d_m(u, v)$ from $u$ to $v$ is defined as the length of a longest $u - v$ monophonic path in $G$. The monophonic eccentricity $e_m(v)$ of a vertex $v$ in $G$ is $e_m(v) = \max \{d_m(v, u) : u \in V(G)\}$. The monophonic radius, $rad_m(G)$ of $G$ is $rad_m \{G\} = \min \{e_m(v) : v \in V(G)\}$ and the monophonic diameter, $diam_m \{G\}$ of $G$ is $diam_m \{G\} = \max \{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced in [3] and further studied in [4]. The concept of vertex monophonic number was introduced by Santhakumaran and Titus [5]. A set $S$ of vertices of $G$ is an $x$-monophonic set if each vertex $v$ of $G$ lies on an $x - y$ monophonic path in $G$ for some element $y$ in $S$. The minimum cardinality of an $x$-monophonic set of $G$ is defined as the $x$-monophonic number of $G$ and is denoted by $m_x(G)$ or simply $m_x$. An $x$-monophonic set of cardinality $m_x(G)$ is called a $m_x$-set of $G$.

The following theorems will be used in the sequel.

**Theorem 1.1.** [2] Let $v$ be a vertex of a connected graph $G$. The following statements are equivalent:

i) $v$ is a cut-vertex of $G$.

ii) There exist vertices $u$ and $w$ distinct from $v$ such that $v$ is on every $u - w$ path.

iii) There exists a partition of the set of vertices $V - \{v\}$ into subsets $U$ and $W$ such that for any vertices $u \in U$ and $w \in W$, the vertex $v$ is on every $u - w$ path.

**Theorem 1.2.** [2] Every non-trivial connected graph has at least two vertices which are not cut-vertices.

**Theorem 1.3.** [2] Let $G$ be a connected graph with at least three vertices. The following statements are equivalent:

i) $G$ is a block.

ii) Every two vertices of $G$ lie on a common cycle.

Throughout this paper $G$ denotes a connected graph with at least three vertices.
2. Edge fixed monophonic number

Definition 2.1. Let \( e = xy \) be any edge of a connected graph \( G \) of order at least three. A set \( S \) of vertices of \( G \) is an \( xy \)-monophonic set if every vertex of \( G \) lies on either an \( x - u \) monophonic path or a \( y - u \) monophonic path in \( G \) for some element \( u \) in \( S \). The minimum cardinality of an \( xy \)-monophonic set of \( G \) is defined as the \( xy \)-monophonic number of \( G \) and is denoted by \( m_{xy}(G) \) or \( m_e(G) \). An \( xy \)-monophonic set of cardinality \( m_{xy}(G) \) is called a \( m_{xy} \)-set or \( m_e \)-set of \( G \).

Example 2.2. For the graph \( G \) given in Figure 2.1, the minimum edge fixed monophonic sets and the edge fixed monophonic numbers are given in Table 2.1.

\[
\begin{array}{|c|c|}
\hline
\text{Edge} & \text{Monophonic Number} \\
\hline
xy & m_{xy}(G) \\
\hline
\end{array}
\]

Figure 2.1: \( G \)

Theorem 2.3. For any edge \( xy \) in a connected graph \( G \) of order at least three, the vertices \( x \) and \( y \) do not belong to any minimum \( xy \)-monophonic set of \( G \).

Proof. Suppose that \( x \) belongs to a minimum \( xy \)-monophonic set, say \( S \), of \( G \). Since \( G \) is a connected graph with at least three vertices and \( xy \) in an edge, it follows from the definition of an \( xy \)-monophonic set that \( S \) contains a vertex \( v \) different from \( x \) and \( y \). Since the vertex \( x \) lies on every \( x - v \) monophonic path in \( G \), it follows that \( T = S - \{x\} \) is an \( xy \)-monophonic set of \( G \), which is a contradiction to \( S \) a minimum \( xy \)-monophonic set of \( G \). Similarly, \( y \) does not belong to any minimum \( xy \)-monophonic set of \( G \). \( \square \)
Table 2.1: The Edge Fixed Monophonic Number of a Graph

<table>
<thead>
<tr>
<th>Edge $e$</th>
<th>minimum $e$-monophonic sets</th>
<th>$e$-monophonic number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1v_2$</td>
<td>${v_4, v_6}, {v_5, v_6}$</td>
<td>2</td>
</tr>
<tr>
<td>$v_2v_3$</td>
<td>${v_4, v_6}, {v_5, v_6}$</td>
<td>2</td>
</tr>
<tr>
<td>$v_3v_4$</td>
<td>${v_2, v_6}$</td>
<td>2</td>
</tr>
<tr>
<td>$v_4v_5$</td>
<td>${v_2, v_6}$</td>
<td>2</td>
</tr>
<tr>
<td>$v_5v_1$</td>
<td>${v_2, v_6}$</td>
<td>2</td>
</tr>
<tr>
<td>$v_1v_6$</td>
<td>${v_2, v_4}$</td>
<td>2</td>
</tr>
<tr>
<td>$v_1v_3$</td>
<td>${v_2, v_6, v_4}, {v_2, v_6, v_5}$</td>
<td>3</td>
</tr>
</tbody>
</table>

**Theorem 2.4.** Let $xy$ be any edge of a connected graph $G$ of order at least three. Then

i) every simplicial vertex of $G$ other than the vertices $x$ and $y$ (whether $x$ or $y$ is simplicial or not) belongs to every $m_{xy}$-set.

ii) no cut-vertex of $G$ belongs to any $m_{xy}$-set.

**Proof.** (i) By Theorem 2.3, the vertices $x$ and $y$ do not belong to any $m_{xy}$-set. So, let $u \neq x, y$ be a simplicial vertex of $G$. Let $S$ be a $m_{xy}$-set of $G$ such that $u \notin S$. Then $u$ is an internal vertex of either an $x - v$ monophonic path or a $y - v$ monophonic path for some element $v$ in $S$. Without loss of generality, let $P$ be an $x - v$ monophonic path with $u$ is an internal vertex. Then both the neighbors of $u$ on $P$ are not adjacent and hence $u$ is not a simplicial vertex, which is a contradiction.

(ii) Let $v$ be a cut-vertex of $G$. Then by Theorem 1.1, there exists a partition of the set of vertices $V - \{v\}$ into subsets $U$ and $W$ such that for any vertex $u \in U$ and $w \in W$, the vertex $v$ lies on every $u - w$ path. Let $S$ be a $m_{xy}$-set of $G$. We consider three cases.

**Case (i):** Both $x$ and $y$ belong to $U$. Suppose that $S \cap W = \emptyset$. Let $w_1 \in W$. Since $S$ is an $xy$-monophonic set, there exists an element $z$ in $S$ such that $w_1$ lies on either an $x - z$ monophonic path or a $y - z$ monophonic path in $G$. Suppose that $w_1$ lies on an $x - z$ monophonic path $P : x = z_0, z_1, \ldots, w_1, \ldots, z_n = z$ in $G$. Then the $x - w_1$ subpath of $P$ and $w_1 - z$ subpath of $P$ both contain $v$ so that $P$ is not a path in $G$, which is a contradiction. Hence $S \cap W \neq \emptyset$. Let $w_2 \in S \cap W$. Then $v$ is an internal vertex of any $x - w_2$ monophonic path and $v$ is also an internal vertex of any $y - w_2$ monophonic path. If $v \in S$, then let $S' = S - \{v\}$. It is clear that every vertex that lies on an $x - v$ monophonic path also lies on an $x - w_2$ monophonic path. Hence it follows that $S'$ is an $xy$-monophonic set of $G$, which is a contradiction to $S$ a minimum $xy$-monophonic set of $G$. Thus $v$ does not belong to any minimum $xy$-monophonic set of $G$.

**Case (ii):** Both $x$ and $y$ belong to $W$. It is simillar to Case (i).

**Case (iii):** Either $x = v$ or $y = v$. By Theorem 2.3, $v$ does not belong to any $m_{xy}$-set. □

**Corollary 2.5.** Let $T$ be a tree with $k$ end vertices. Then $m_{xy}(T) = k - 1$ or $k$ according as $xy$ is an end edge or cut-edge.
Proof. This follows from Theorem 2.4. \(\square\)

Corollary 2.6. Let \(K_{1,n}(n \geq 2)\) be a star. Then \(m_{xy}(K_{1,n}) = n - 1\) for any edge \(xy\) in \(K_{1,n}\).

Corollary 2.7. Let \(G\) be a complete graph \(K_p(p \geq 3)\). Then \(m_{xy}(G) = p - 2\) for any edge \(xy\) in \(G\).

Theorem 2.8. For any edge \(xy\) in the cube \(Q_n(n \geq 3)\), \(m_{xy}(Q_n) = 1\).

Proof. Let \(e = xy\) be an edge in \(Q_n\) and let \(x = (a_1, a_2, \ldots, a_n)\), where \(a_i \in \{0, 1\}\). Let \(x' = (a'_1, a'_2, \ldots, a'_n)\) be another vertex of \(Q_n\) such that \(a'_i\) is the complement of \(a_i\). Let \(u\) be any vertex in \(Q_n\). For convenience, let \(u = (a_1, a_2, a_3, \ldots, a_n)\). Then \(u\) lies on an \(x - x'\) monophonic path \(P: x = (a_1, a_2, \ldots, a_n), (a_1, a'_2, a_3, \ldots, a_n), (a'_1, a'_2, a'_3, \ldots, a'_n)\). Hence \(\{x'\}\) is an \(xy\)-monophonic set of \(Q_n\) and so \(m_{xy}(Q_n) = 1\). \(\square\)

Theorem 2.9. \(i)\) For any edge \(xy\) in the wheel \(W_n = K_1 + C_{n-1}(n \geq 5)\), \(m_{xy}(W_n) = 1\).

\(ii)\) For any edge \(xy\) in the complete bipartite graph \(K_{m,n}(1 \leq m \leq n)\),

\[
m_{xy}(K_{m,n}) = \begin{cases} n - 1 & \text{if } m = 1 \\ 1 & \text{if } m = 2 \\ 2 & \text{if } m \geq 3. \end{cases}
\]

Proof. \(i)\) Let \(xy\) be an edge in \(W_n\). Then either \(x\) or \(y\) is a vertex of \(C_{n-1}\). Let \(x \in V(C_{n-1})\) and let \(z\) be a non-adjacent vertex of \(x\) in \(C_{n-1}\). It is clear that every vertex of \(W_n\) lies on an \(x - z\) monophonic path. Hence \(\{z\}\) is a \(m_{xy}\)-set of \(W_n\) and so \(m_{xy}(W_n) = 1\).

\(ii)\) Let \(U = \{u_1, u_2, \ldots, u_m\}\) and \(W = \{w_1, w_2, \ldots, w_n\}\) be the vertex subsets of the bipartition of the vertices of \(K_{m,n}\). If \(m = 1\), then by Corollary 2.6, \(m_{xy}(K_{1,n}) = n - 1\) for any edge \(xy\) in \(K_{1,n}\). If \(m = 2\), let \(e\) be an edge in \(K_{m,n}\), say \(e = u_1w_1\). It is clear that every vertex of \(K_{m,n}\) lies on an \(u_1 - u_2\) monophonic path. Hence \(\{u_2\}\) is an \(e\)-monophonic set of \(K_{m,n}\) and so \(m_{e}(K_{m,n}) = 1\). If \(m \geq 3\), then it is clear that no singleton subset of \(V\) is an \(e\)-monophonic set of \(K_{m,n}\) and so \(m_{e}(K_{m,n}) \geq 2\). Without loss of generality, take \(e = u_1w_1\). Let \(S = \{u_2, w_2\}\). Then every vertex of \(U\) lies on a \(u_1 - w_2\) monophonic path and every vertex of \(W\) lies on a \(u_1 - u_2\) monophonic path. Hence \(S\) is an \(e\)-monophonic set of \(K_{m,n}\) and so \(m_{e}(K_{m,n}) = 2\). \(\square\)

Theorem 2.10. For any edge \(xy\) in a connected graph \(G\) of order \(p \geq 3\),

\[
1 \leq m_{xy}(G) \leq p - 2.
\]

Proof. It is clear from the definition of \(m_{xy}\)-set that \(m_{xy}(G) \geq 1\). Also, since the vertices \(x\) and \(y\) do not belong to any \(m_{xy}\)-set, it follows that \(m_{xy}(G) \leq p - 2\). \(\square\)

Remark 2.11. The bounds for \(m_{xy}(G)\) in Theorem 2.10 are sharp. If \(C\) is any cycle, then \(m_{xy}(C) = 1\) for any edge \(xy\) in \(C\). For any edge \(xy\) in a complete graph \(K_p\) \((p \geq 3)\), \(m_{xy}(K_p) = p - 2\).
Now we proceed to characterize graphs for which the upper bound in Theorem 2.10 is
attained.

**Theorem 2.12.** Let $G$ be a connected graph of order at least 3. Then $G$ is either $K_p$ or $K_{1,p-1}$ if and only if $m_{xy}(G) = p - 2$ for every edge $xy$ in $G$.

**Proof.** If $G = K_p$, then by Corollary 2.7, $m_{xy}(G) = p - 2$ for every edge $xy$ in $G$. If $G = K_{1,p-1}$, then by Corollary 2.6, $m_{xy}(G) = p - 2$ for any edge $xy$ in $G$. Conversely, suppose that $m_{xy}(G) = p - 2$ for every edge $xy$ in $G$. By Theorem 1.2, $G$ has at least two vertices which are not cut-vertices. Let $xy$ be an edge of $G$ with $x$ is not a cut-vertex. If $G$ has two or more cut-vertices, then by Theorem 2.4(ii), $m_{xy}(G) \leq p - 3$, which is a contradiction. Thus the number of cut-vertices $k$ of $G$ is at most one.

**Case (i) $k = 0$.** Then the graph $G$ is a block. Now we claim that $G$ is complete. If $G$ is not complete, then there exist two vertices $x$ and $y$ in $G$ such that $d(x, y) \geq 2$. By Theorem 1.3, $x$ and $y$ lie on a common cycle and hence $x$ and $y$ lie on a smallest cycle $C : x, x_1, x_2, \ldots, y_1, \ldots, x_n, x$ of length at least 4. Then $(V(G) - V(C)) \cup \{y\}$ is an $xx_1$-monophonic set of $G$ and so $m_{xx_1}(G) \leq p - 3$, which is a contradiction. Hence $G$ is the complete graph.

**Case (ii) $k = 1$.** Let $x$ be the cut-vertex of $G$. If $p = 3$, then $G = P_3$, a star with three vertices. If $p \geq 4$, we claim that $G = K_{1,p-1}$. It is enough to prove that degree of every vertex other than $x$ is one. Suppose that there exists a vertex, say $y$, with $\deg y \geq 2$. Let $z \neq x$ be an adjacent vertex of $y$ in $G$. Let $c = yz$. Since the vertices $y$ and $z$ do not lie on any minimum $yz$-monophonic set of $G$ and by Theorem 2.4(ii), we have $m_{yz}(G) \leq p - 3$, which is a contradiction. Thus every vertex of $G$ other than $x$ is of degree one. Hence $G$ is a star. \(\Box\)

**Theorem 2.13.** For any edge $xy$ in a connected graph $G$, every $x$-monophonic set of $G$ is an $xy$-monophonic set of $G$.

**Proof.** Let $S$ be an $x$-monophonic set of $G$. Then every vertex of $G$ lies on an $x - z$ monophonic path for some $z$ in $S$. It follows that $S$ is an $xy$-monophonic set of $G$. \(\Box\)

**Corollary 2.14.** For any edge $xy$ in a connected graph $G$, $m_{xy}(G) \leq \min\{m_x(G), m_y(G)\}$.

**Theorem 2.15.** For every pair $a, b$ of integers with $1 \leq a \leq b$, there is a connected graph $G$ with $m_{xy}(G) = a$ and $m_x(G) = b$ for some edge $xy$ in $G$.

**Proof.** Let $C_4 : x, y, z, u, x$ be a cycle of order 4. Add $b - 1$ new vertices $v_1, v_2, \ldots, v_{a-1}, w_1, w_2, \ldots, w_{b-a}$ and joining each $v_i(1 \leq i \leq a - 1)$ to $x$ and joining each $w_j(1 \leq j \leq b - a)$ to the vertices $y$ and $u$, thereby producing the graph $G$ given in Figure 2.2. Let $S = \{v_1, v_2, \ldots, v_{a-1}\}$ be the set of all simplicial vertices of $G$. Since $S$ is not an $xy$-monophonic set, it follows from Theorem 2.4(i) that $m_{xy}(G) \geq a$. On the other hand, $S_1 = S \cup \{u\}$
is an $xy$-monophonic set of $G$ and so $m_{xy}(G) = |S_1| = a$. Clearly, $S_2 = \{v_1, v_2, \ldots, v_{a-1}, z, w_1, w_2, \ldots, w_{b-a}\}$ is the unique $x$-monophonic set of $G$ and so $m_x(G) = |S_2| = b$.

![Figure 2.2: $G$](image1)

We have seen that if $G$ is a connected graph of order $p \geq 3$, then $1 \leq m_{xy}(G) \leq p - 2$ for any edge $xy$ in $G$. In the following theorem we give an improved upper bound for the edge fixed monophonic number of a tree in terms of its order and monophonic diameter.

**Theorem 2.16.** If $T$ is a tree of order $p$ and monophonic diameter $d_m$, then $m_{xy}(T) \leq p - d_m + 1$ for any edge $xy$ in $T$.

**Proof.** Let $P : v_0, v_1, v_2, \ldots, v_{d_m}$ be a monophonic path of length $d_m$. Now, let $S = V(G) - \{v_1, v_2, \ldots, v_{d_m-1}\}$. If $e$ is an internal edge of $P$, then clearly $S$ is an $e$-monophonic set of $T$ so that $m_e(T) \leq |S| = p - d_m + 1$. If $e$ is an end edge of $P$, say $e = v_0v_1$, then $S_1 = S - \{v_0\}$ is an $e$-monophonic set of $T$ so that $m_e(T) \leq |S_1| = p - d_m$. If $e = xy$ is an edge lies outside $P$, then $S_2 = S - \{x, y\}$ is an $e$-monophonic set of $T$ so that $m_e(T) \leq |S_2| = p - d_m$. Hence for any edge $xy$ in $T$, $m_{xy}(T) \leq p - d_m + 1$. \qed

**Remark 2.17.** The bound in Theorem 2.16 is not true for any graph. For example, consider the graph $G$ given in Figure 2.3. Here $p = 7$, $d_m(G) = 4$, $m_e(G) = 5$ and $p - d_m + 1 = 4$. Hence $m_e(G) > p - d_m + 1$.

![Figure 2.3: $G$](image2)

**Theorem 2.18.** For any edge $xy$ in a non-trivial tree $T$ of order $p$ and monophonic diameter $d_m$, $m_{xy}(T) = p - d_m$ or $p - d_m + 1$ if and only if $T$ is a caterpillar.
Proof. Let $T$ be any non-trivial tree. Let $P : v_0, v_1, \ldots, v_d$ be a monophonic path of length $d_m$. Let $k$ be the number of end vertices of $T$ and let $l$ be the number of internal vertices of $T$ other than $v_1, v_2, \ldots, v_{d-1}$. Then $d_m - 1 + l + k = p$. By Corollary 2.5, $m_{xy}(T) = k$ or $k - 1$ for any edge $xy$ in $T$ and so $m_{xy}(T) = p - d_m - l + 1$ or $p - d_m - l$ for any edge $xy$ in $T$. Hence $m_{xy}(T) = p - d_m + 1$ or $p - d_m$ for any edge $xy$ in $T$ if and only if $l = 0$, if and only if all the internal vertices of $T$ lie on the monophonic path $P$, if and only if $T$ is a caterpillar.

For any connected graph $G$, $rad_m(G) \leq diam_m(G)$. It is shown in [3] that every two positive integers $a$ and $b$ with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This result can be extended so that the edge fixed monophonic number can be prescribed. □

Theorem 2.19. For positive integers $r, d$ and $n \geq 2$ with $2 \leq r \leq d$, there exists a connected graph $G$ with $rad_m(G) = r$, $diam_m(G) = d$ and $m_{xy}(G) = n$ for some edge $xy$ in $G$.

Proof. Case (i) $2 \leq r = d$. Let $C_{r+2} : v_1, v_2, \ldots, v_{r+2}, v_1$ be the cycle of order $r+2$. Let $G$ be the graph obtained from $C_{r+2}$ by adding $n$ vertices $u_1, u_2, \ldots, u_n$ and joining each vertex $u_i$ ($1 \leq i \leq n$) to both $v_2$ and $v_{r+2}$, and also adding the edge $v_1u_1$. The graph $G$ is shown in Figure 2.4. It is easily verified that the monophonic eccentricity of each vertex of $G$ is $r$ and so $rad_m(G) = diam_m(G) = r$. Also, for the edge $v_1u_1$, it is clear that $S = \{v_{r+1}, u_2, \ldots, u_n\}$ is a minimum $xy$-monophonic set of $G$ and so $m_{xy}(G) = n$.

Case (ii) $2 \leq r < d \leq 2r$. Let $C_{r+2} = v_1, v_2, \ldots, v_{r+2}, v_1$ be the cycle of order $r + 2$ and let $P_{d-r+1} : u_0, u_1, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let $H$ be the graph obtained from $C_{r+2}$ and $P_{d-r+1}$ by identifying $v_1$ in $C_{r+2}$ and $u_0$ in $P_{d-r+1}$. Let $G$ be the graph obtained from $H$ by adding $n - 1$ new vertices $w_1, w_2, \ldots, w_{n-1}$ and joining each $w_i$ ($1 \leq i \leq n - 1$) with $u_{d-r-1}$. The graph $G$ is shown in Figure 2.5. It is easily verified that $r \leq e_m(x) \leq d$ for any vertex $x$ in $G$, $e_m(v_1) = r$ and $e_m(v_3) = d$. Thus $rad_m(G) = r$ and $diam_m(G) = d$. For the edge $e = u_{d-r-1}u_{d-r}$, $S = \{w_1, w_2, \ldots, w_{n-1}, v_3\}$ is a minimum $e$-monophonic set of $G$ and so $m_e(G) = n$. 

![Figure 2.4: G](image-url)
Case (iii) $d > 2r$. Let $P_{2r-1} : v_1, v_2, \ldots, v_{2r-1}$ be a path of order $2r - 1$. Let $G$ be the graph obtained from the wheel $W = K_1 + C_{d+2}$ and the complete graph $K_n$ by identifying the vertex $v_1$ of $P_{2r-1}$ with the central vertex of $W$, and identifying the vertex $v_{2r-1}$ of $P_{2r-1}$ with a vertex of $K_n$. The graph $G$ is shown in Figure 2.6. Since $d > 2r$, we have $e_m(x) = d$ for any vertex $x \in V(C_{d+2})$. Also, $e_m(x) = 2r$ for any vertex $x \in V(K_n) - v_{2r-1}$; $r \leq e_m(x) \leq 2r - 1$ for any vertex $x \in V(P_{2r-1})$; and $e_m(x) = r$ for the central vertex $x$ of $P_{2r-1}$. Thus $rad_m(G) = r$ and $diam_m(G) = d$.

Let $S = V(K_n) - \{v_{2r-1}\}$ be the set of all simplicial vertices of $G$. Then by Theorem 2.4(i), every $m_e$-set contains $S$ for the edge $e = u_1u_2$. It is clear that $S$ is not an $e$-monophonic set of $G$ and so $m_e(G) > |S| = n - 1$. Since $S' = S \cup \{u_{d+1}\}$ is an $e$-monophonic set of $G$, we have $m_e(G) = n$. □

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