A generalization of variant of Wilson's type Hilbert space valued functional equations

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Abstract

In the present paper we characterize, in terms of characters, multiplicative functions, the continuous solutions of some functional equations for mappings defined on a monoid and taking their values in a complex Hilbert space with the Hadamard product. In addition, we investigate a superstability result for these equations.

Keywords : D’Alembert’s functional equation, Hilbert space, Hadamard product, superstability.

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1. Introduction

Let $M$ be a monoid i.e., is a semigroup with an identify element that we denote by $e$ and $\sigma, \tau : M \to M$ are two involutive automorphisms. That is $\sigma(xy) = \sigma(x)\sigma(y)$, $\tau(xy) = \tau(x)\tau(y)$ and $\sigma(\sigma(x)) = x$, $\tau(\tau(x)) = x$ for all $x, y \in M$. By a variant of Wilson’s functional equation on $M$ we mean the functional equation

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x)g(y), \quad x, y \in M, \quad (1.1)$$

where $f, g : M \to \mathbb{C}$ are the unknown functions. A special case of Wilson’s functional equation is d’Alembert’s functional equation:

$$f(x\sigma(y)) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in M, \quad (1.2)$$

The solutions of equation (1.2) are known [2]. Further contextual and historical discussion on the functional equation (1.1) and (1.2) can be found, e.g., in [6.2].

The present paper studies an extension to a situation where the unknown functions $f, g$ map a possibly non-abelian group or monoid into a complex Hilbert space $H$ with the Hadamard product. Our considerations refer mainly to results by Rezaei [4], Zeglami [11]. It has been proved [3] that the functional equation (1.2) with $\sigma = id$ is superstable in the class of functions $f : G \to \mathbb{C}$, if every such function satisfies the inequality

$$|f(xy) + f(\tau(y)x) - 2f(x)f(y)| \leq \epsilon \text{ for all } x, y \in G,$$

where $\epsilon$ is a fixed positive real number. Then either $f$ is a bounded function or

$$f(xy) + f(\tau(y)x) = 2f(x)f(y), \quad x, y \in G.$$

Let $H$ be a separable Hilbert space with a orthonormal basis $\{e_n, n \in \mathbb{N}\}$. For two vectors $x, y \in H$, the Hadamard product, also known as the entrywise product on the Hilbert space $H$ is defined by

$$x \ast y = \sum_{n=0}^{\infty} \langle x, e_n \rangle \langle y, e_n \rangle e_n, \quad x, y \in H. \quad (1.3)$$

The Cauchy-Schwarz inequality together with the Parseval identity ensure that the Hadamard multiplication is well defined. In fact,

$$\|x \ast y\| \leq \left( \sum_{n=0}^{\infty} |\langle x, e_n \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{n=0}^{\infty} |\langle y, e_n \rangle|^2 \right)^{\frac{1}{2}} = \|x\| \|y\| \quad (1.4)$$
The purpose of this work is first to give a characterization, in terms of multiplicative functions, the solutions of the Hilbert space valued functional equation by Hadamard product:

\[(1.5) \quad f(x\sigma(y)) + f(\tau(y)x) = 2g(x) * f(y), \quad x, y \in M.\]

When \(f\) we determine the solutions of the functional equation

\[(1.6) \quad f(x\sigma(y)) + f(\tau(y)x) = 2f(x) * g(y), \quad x, y \in M,\]

where \(f, g : M \to H\) are the unknown functions. Second, we determine a characterization of the following d’Alembert-Hilbert-valued functional equation:

\[(1.7) \quad f(x\sigma(y)) + f(\tau(y)x) = 2f(x) * f(y), \quad x, y \in M.\]

Throughout the paper, \(\mathbb{N}, \mathbb{R}\) and \(\mathbb{C}\) stand for the sets of positive integers, real numbers and complex numbers, respectively. We let \(G\) denote a group and \(S\) denote a semigroup i.e., a set with an associative composition rule.

A function \(A : M \to \mathbb{C}\) is called additive, if it satisfies \(A(xy) = A(x) + A(y)\) for all \(x, y \in M\).

A multiplicative function on \(M\) is a map \(\chi : M \to \mathbb{C}\) such that \(\chi(xy) = \chi(x)\chi(y)\) for all \(x, y \in M\).

A monoid \(M\) is generated by its squares if for every \(x \in I_\chi\), \(x = x_1^2x_2^2\cdots x_n^2\) for some \(x_1, x_2, \cdots, x_n \in M\).

A character on a group \(G\) is a homomorphism from \(G\) into the multiplicative of non-zero complex numbers. While a non-zero multiplicative function on a group can never take the value 0, it is possible for a multiplicative function on a monoid \(M\) to take the value 0 on a proper, non-empty subset of \(M\). If \(\chi : M \to \mathbb{C}\) is multiplicative and \(\chi \neq 0\), then

\[I_\chi = \{x \in M/\chi(x) = 0\}\]

is either empty or a proper subset of \(M\). The fact that \(\chi\) is multiplicative establishes that \(I_\chi\) is a two-sided ideal in \(M\) if not empty (for us an ideal is never the empty set). It follows also that \(M \setminus I_\chi\) is a subsemigroup of \(M\).

Let \(C(M)\) denote the algebra of continuous functions from \(M\) into \(\mathbb{C}\).
2. Solutions of (1.5) and (1.6)

In this section, we solve the functional equation (1.5) by expressing its solutions in terms of multiplicative functions.

**Theorem 2.1.** Let \( M \) be a monoid, let \( \sigma, \tau : M \rightarrow M \) be involutive automorphisms. Assume that the functions \( f, g : M \rightarrow H \) satisfy (1.5). Then, there exists a positive integer \( N \) such that

\[
f(x) = \sum_{n=1}^{N} \langle f(x), e_n \rangle e_n \quad \text{and} \quad x \rightarrow \langle g(x), e_{N+k} \rangle \quad \text{is arbitrary}
\]

for all \( x \in M \) and \( k > 0 \). Furthermore, for every \( k \in \{1, 2, ..., N\} \), we have the following possibilities:

\[
\begin{cases}
\langle g(x), e_k \rangle = \frac{\chi_k(x) + \chi_k \circ \sigma \circ \tau(x)}{2} & \text{if } \langle g(x), e_k \rangle \text{ is an arbitrary function}, \\
\langle f(x), e_k \rangle = \alpha_k \chi_k(x) + \chi_k \circ \sigma \circ \tau(x) & \text{if } \langle f(x), e_k \rangle = 0
\end{cases}
\]

for all \( x \in M \), where \( \chi_k \) is a non-zero multiplicative function of \( M \) such that \( \chi_k \circ \sigma \circ \tau = \chi_k \circ \tau \circ \sigma \) and \( \alpha_k \in \mathbb{C} \setminus \{0\} \). If \( M \) is a topological monoid and \( f \in C(M) \), then \( \chi_k, \chi_k \circ \sigma \circ \tau \in C(M) \).

**Proof.** For every integer \( k \geq 0 \), consider the functions \( f_k, g_k : M \rightarrow \mathbb{C} \) defined by

\[
f_k(x) = \langle f(x), e_k \rangle \quad \text{and} \quad g_k(x) = \langle g(x), e_k \rangle \quad \text{for all } x \in M.
\]

Since \((f, g)\) satisfies (1.5), for all \( x, y \in M \), we have

\[
\sum_{k=0}^{\infty} \langle f(x \sigma(y)), e_k \rangle + \langle f(\tau(y)x), e_k \rangle \rangle e_k = \sum_{k=0}^{\infty} \langle f(x \sigma(y)) + f(\tau(y)x) \rangle \rangle e_k
\]

\[
= f(x \sigma(y)) + f(\tau(y)x)
\]

\[
= 2g(x) * f(y)
\]

\[
= 2 \sum_{k=0}^{\infty} \langle g(x), e_k \rangle \langle f \rangle,
\]

This yields for all \( k \in \mathbb{N} \),

\[
f_k(x \sigma(y)) + f_k(\tau(y)x) = 2g_k(x)f_k(y) \quad \text{for all } x, y \in M.
\]

(2.1)
If we put \( y = e \) in (2.1), we find that \( f_k(x) = f_k(e)g_k(x) \). So, if we take \( \alpha_k = f_k(e) \), equation (2.1) can be written as follows:

\[
\alpha_k g_k(x\sigma(y)) + \alpha_k g_k(\tau(y)x) = 2\alpha_k g_k(x)g_k(y) \quad \text{for all } x, y \in M.
\]

Then, either \( \alpha_k = 0 \) or \( g_k \) is a solution of equation (1.6). In view of [2, Theorem 3.2], one of the following statements holds:

(a) We have that

\[ f_k = 0 \text{ and } g_k \text{ is an arbitrary function.} \]

(b) There exists a multiplicative function \( \chi_k \) such that \( g_k(x) = \chi_k(x) + \chi_k(\sigma \circ \tau(x)) \) and \( f_k(x) = \frac{\alpha_k(\chi_k(x) + \chi_k(\sigma \circ \tau(x)))}{2} \) for \( x \in M \).

If \( H \) is infinite-dimensional, then

\[
\langle g(x), e_k \rangle = g_k(x) \to 0 \text{ as } k \to +\infty
\]

for every \( x \in M \). Since \( g_k(e) = 1 \), statement (b) is not possible for infinitely many positive integers \( k \). Hence, there exists some positive integer \( N \) such that \( f_k = 0 \) for every \( k > N \). Thus, \( g_k \) is an arbitrary function for any \( k > N \), \( f \) can be represented as

\[
f(x) = \sum_{n=1}^{N} \langle f(x), e_n \rangle e_n,
\]

and the expressions of the component functions \( f_n \) and \( g_n \), \( 1 \leq n \leq N \), of \( f \) and \( g \) come from statements (a) and (b) above. In the case where \( H \) is finite-dimensional, the proof is clear.

As a consequence of Theorem 2.1 we derive formulas for the solutions of d’Alembert’s Hilbert space valued functional equation (1.7).

**Corollary 2.2.** Let \( M \) be a monoid, let \( \sigma, \tau : M \to M \) be involutive automorphisms. Assume that the functions \( g : M \to H \) satisfy (1.7). Then, there exists a positive integer \( N \) such that

\[
f(x) = \sum_{n=1}^{N} \langle f(x), e_n \rangle e_n \quad \text{and} \quad x \to \langle g(x), e_{N+k} \rangle \text{ is arbitrary}
\]

for all \( x \in M \) and \( k > 0 \). Furthermore, for every \( k \in \{1, 2, ..., N\} \), such that

\[
g(x) = \frac{1}{2} \sum_{k=1}^{N} \epsilon_k (\chi_k(x) + \chi_k \circ \sigma \circ \tau(x)) e_k, \quad x \in M,
\]
where \( \epsilon_k = 1 \) or \( 0 \) for every \( k \in \{1, 2, \ldots, N\} \), for all \( x \in M \), where \( \chi_k \) is a non-zero multiplicative function of \( M \) such that \( \chi_k \circ \sigma \circ \tau = \chi_k \circ \tau \circ \sigma \).

If \( M \) is a topological monoid and \( f \in C(M) \), then \( \chi_k, \chi_k \circ \sigma \circ \tau \in C(M) \).

**Proof.** The proof follows by putting \( f = g \) in Theorem 2.1. \( \square \)

**Corollary 2.3.** Let \( M \) be a monoid, let \( \tau : M \to M \) be involutive automorphisms. Assume that the functions \( f, g : M \to H \) satisfy

\[
f(xy) + f(\tau(y)x) = 2g(x) * f(y).
\]

Then, there exists a positive integer \( N \) such that

\[
f(x) = \sum_{n=1}^{N} \langle f(x), e_n \rangle e_n \quad \text{and} \quad x \to \langle g(x), e_{N+k} \rangle \quad \text{is arbitrary}
\]

for all \( x \in M \) and \( k > 0 \). Furthermore, for every \( k \in \{1, 2, \ldots, N\} \), we have the following possibilities:

\[
\begin{align*}
\langle g(x), e_k \rangle &= \frac{\chi_k(x) + \chi_k \circ \tau(x)}{2} & \langle g(x), e_k \rangle \text{ is an arbitrary function}, \\
\langle f(x), e_k \rangle &= \frac{\alpha_k(\chi_k(x) + \chi_k \circ \tau(x))}{2} & \langle f(x), e_k \rangle = 0
\end{align*}
\]

for all \( x \in M \), where \( \chi_k \) is a non-zero multiplicative function of \( M \) and \( \alpha_k \in \mathbb{C} \setminus \{0\} \).

If \( M \) is a topological monoid and \( f \in C(M) \), then \( \chi_k, \chi_k \circ \tau \in C(M) \).

**Proof.** The proof follows by putting \( \sigma = \text{id} \) in Theorem 2.1. \( \square \)

We complete this section with a result concerning Wilson Hilbert space valued functional equation (1.6).

**Theorem 2.4.** Let \( M \) be a monoid which is generated by its squares, let \( \sigma, \tau : M \to M \) be involutive automorphisms. Assume that the pair \( f, g : M \to \mathbb{C} \), satisfy Wilson’s Hilbert valued functional equation (1.6).

Then, there exists a positive integer \( N \) such that

\[
f(x) = \sum_{n=1}^{N} \langle f(x), e_n \rangle e_n \quad \text{and} \quad \langle g(x), e_{N+k} \rangle \quad \text{is arbitrary}
\]

for all \( x \in M \) and \( k > 0 \). Furthermore, for every \( k \in \{1, 2, \ldots, N\} \), we have the following possibilities:
(i) 
\[
\begin{align*}
\langle g(x), e_k \rangle & \text{ is an arbitrary function,} \\
\langle f(x), e_k \rangle & = 0
\end{align*}
\]
where \( \chi_k : M \rightarrow \mathbb{C} \) is a non-zero multiplicative function with \( \chi_k \circ \sigma \circ \tau = \chi_k \circ \tau \circ \sigma \), and for some \( \alpha_k \in \mathbb{C}\setminus\{0\} \).

(ii) There exists a non-zero multiplicative function \( \chi_k : M \rightarrow \mathbb{C} \) with \( \chi_k \circ \sigma \circ \tau = \chi_k \circ \tau \circ \sigma \) such that
\[
g_k = \chi_k + \chi_k \circ \sigma \circ \tau.
\]
Furthermore, we have

1. If \( \chi_k \neq \chi_k \circ \sigma \circ \tau \), then
\[
f_k = \alpha_k \chi_k \circ \sigma + \beta_k \chi_k \circ \tau
\]
for some \( \alpha_k, \beta_k \in \mathbb{C} \setminus \{0\} \).

2. If \( \chi_k = \chi_k \circ \sigma \circ \tau \), then there exists a non-zero additive function \( A_k : M \setminus I_{\chi_k \circ \sigma} \rightarrow \mathbb{C} \) with \( A_k \circ \tau = -A_k \circ \sigma \) such that
\[
f_k(x) = \begin{cases} 
(\alpha_k + A_k(x))\chi_k(\sigma(x)) & \text{for } x \in M \setminus I_{\chi_k \circ \sigma} \\
0 & \text{for } x \in I_{\chi_k \circ \sigma}
\end{cases}
\]
for some \( \alpha_k, \in \mathbb{C} \).

Conversely, if \( f \) and \( g \) have the forms described above, then the pair \((f, g)\) is a solution of equation (1.6). Moreover, if \( M \) is a topological monoid generated by its squares, and \( f, g \in C(M) \), then \( \chi_k, \chi_k \circ \sigma, \chi_k \circ \tau, \chi_k \circ \sigma \circ \tau \in C(M) \), while \( A_k \in C(M \setminus I_{\chi_k \circ \sigma}) \).

**Proof.** We proceed as in the proof of Theorem 2.1. For every integer \( k \geq 0 \), we consider the functions \( f_k, g_k : M \rightarrow \mathbb{C} \), defined by
\[
f_k(x) = \langle f(x), e_k \rangle \text{ and } g_k(x) = \langle g(x), e_k \rangle \text{ for } x \in M.
\]
Since the pair \((f, g)\) satisfies (1.6), for all \( k \in \mathbb{N} \) we have
\[
(2.2) \quad f_k(x \sigma(y)) + f_k(\tau(y)x) = 2f_k(x)g_k(y) \quad \text{for all } x, y \in M.
\]
By [6,Theorem 3.4] we infer that there are only the following cases
(a) \( f_k = 0 \) and \( g_k \) is an arbitrary function.
(b) There exists a non-zero multiplicative function $\chi_k : M \to \mathbb{C}$ such that

$$f_k = \alpha_k \chi_k \circ \sigma$$

and

$$g_k = \frac{\chi_k + \chi_k \circ \sigma \circ \tau}{2}$$

for some $\alpha_k \in \mathbb{C}\setminus\{0\}$.

(c) There exists a non-zero multiplicative function $\chi_k : M \to \mathbb{C}$ with

$$\chi_k \circ \sigma \circ \tau = \chi_k \circ \tau \circ \sigma$$

such that

$$g_k = \frac{\chi_k + \chi_k \circ \sigma \circ \tau}{2}.$$

Furthermore, we have.

(i) If $\chi_k \neq \chi_k \circ \sigma \circ \tau$, then

$$f_k = \alpha_k \chi_k \circ \sigma + \beta_k \chi_k \circ \tau$$

for some $\alpha_k, \beta_k \in \mathbb{C}\setminus\{0\}$.

(ii) If $\chi_k = \chi_k \circ \sigma \circ \tau$, then there exists a non-zero additive function $A_k : M \setminus I_{\chi_k \circ \sigma} \to \mathbb{C}$ with $A_k \circ \tau = -A_k \circ \sigma$ such that

$$f_k(x) = \begin{cases} (\alpha_k + A_k(x))\chi_k(\sigma(x)) & \text{for } x \in M \setminus I_{\chi_k \circ \sigma} \\ 0 & \text{for } x \in I_{\chi_k \circ \sigma} \end{cases}$$

for some $\alpha_k \in \mathbb{C}$. Conversely, the functions given with properties satisfy the functional equation (2.2). The continuation of the proof depends on the dimension of $H$. In fact, if $H$ is infinite-dimensional, then

$$\langle g(x), e_k \rangle = g_k(x) \to 0 \text{ as } k \to +\infty$$

for every $x \in M$. Statements (b) and (c) are not possible for infinitely positive integers $n$. Hence, there exists some positive integer $N$ such that $f_k = 0$ for every $k > N$. Thus, $f$ can be represented as

$$f(x) = \sum_{n=1}^{N} \langle f(x), e_n \rangle e_n,$$

$g_k$ is an arbitrary function for any $k > N$, and expressions of the component functions $f_n$ and $g_n$, $1 \leq n \leq N$ of $f$ and $g$ follow from the previous discussion. In the case where $H$ is a finite-dimensional space, the proof is clear. \qed
Corollary 2.5. Let $M$ be a monoid which is generated by its squares, let $	au : M \rightarrow M$ be an involutive automorphism, and let the pair $f, g : M \rightarrow H$ satisfy the functional equation

$$f(xy) + f(\tau(y)x) = 2f(x) * g(y), \quad x, y \in M.$$ 

Then, there exists a positive integer $N$ such that

$$f(x) = \sum_{n=1}^{N} \langle f(x), e_n \rangle e_n \quad \text{and} \quad x \rightarrow \langle g(x), e_{N+k} \rangle \text{ is arbitrary}$$ 

for all $x \in M$ and $k > 0$. Furthermore, for every $k \in \{1, 2, ..., N\}$, we have the following possibilities:

(i) \( \begin{align*} 
\langle g(x), e_k \rangle \text{ is an arbitrary function}, \\
\langle f(x), e_k \rangle = 0 
\end{align*} \)

(ii) There exists a non-zero multiplicative function $\chi_k : M \rightarrow \mathbb{C}$ such that

$$g_k = \frac{\chi_k + \chi_k \circ \tau}{2}.$$ 

Furthermore, we have.

(1) If $\chi_k \neq \chi_k \circ \tau$, then

$$f_k = \alpha_k \chi_k + \beta_k \chi_k \circ \tau,$$

for some $\alpha_k, \beta_k \in \mathbb{C} \setminus \{0\}$. 

(2) If $\chi_k = \chi_k \circ \tau$, then there exists an additive function $A_k : M \setminus I_{\chi_k} \rightarrow \mathbb{C}$ with $A_k \circ \tau = -A_k$ such that

$$f_k(x) = \begin{cases} 
(\alpha_k + A_k(x))\chi_k(x) & \text{for } x \in M \setminus I_{\chi_k} \\
0 & \text{for } x \in I_{\chi_k}
\end{cases}$$

for some $\alpha_k \in \mathbb{C}$.

Conversely, if $f$ and $g$ have the forms described above, then the pair $(f, g)$ is a solution. Moreover, if $M$ is a topological monoid generated by its squares, and $f, g \in C(M)$, then $\chi_k, \chi_k \circ \tau \in C(M)$, while $A_k \in C(M \setminus I_{\chi_k})$.

Proof. The proof follows by putting $\sigma = id$ in Theorem 2.4. \(\square\)
3. Superstability of Hilbert valued cosine type functional equations

The main result of this section is Theorem 3.3 that contains a superstability result for the functional equation (1.6). For the proof of our result we will begin by pointing out a superstability result for the equation

\[(3.1) \quad f(xy) + f(\sigma(y)x) = 2f(x)g(y)\]

where \(f, g : G \to \mathbb{C}\) are the unknown functions.

**Proposition 3.1.** Let \(\delta > 0\) be given, let \(M\) be a monoid and let \(\sigma\) is an involutive morphism of \(M\). Assume that the functions \(f, g : M \to \mathbb{C}\) satisfies the inequality

\[|f(xy) + f(\sigma(y)x) - 2f(x)g(y)| \leq \delta\text{ for all } x, y \in M,\]

and that \(g\) is unbounded. Then, the ordered pair \((f, g)\) satisfies equation (3.1).

**Proof.** The proof is part of the proof of [3, Theorem 2.1 and Theorem 3.7] if we put \(\chi = 1\) that deals with \(M\) being a group. \(\square\)

**Corollary 3.2.** Let \(\delta > 0\) be given and let \(G\) be a monoid. Assume that the function \(f : G \to \mathbb{C}\) satisfies the inequality

\[|f(xy) + f(\sigma(y)x) - 2f(x)f(y)| \leq \delta\text{ for all } x, y \in G.\]

Then, either

\[|f(x)| \leq \frac{1 + \sqrt{1 + 2\delta}}{2}\text{ for all } x \in G,\]

or \(f\) has the form

\[f = \frac{\mu + \mu \circ \sigma}{2},\]

where \(\mu\) is a multiplicative function.

**Proof.** The proof follows immediately from Proposition 3.1 and Theorem [1, Theorem 4]. \(\square\)
Theorem 3.3. Let $\delta > 0$ be given and let $M$ be a monoid. Assume that the functions $f, g : M \to H$ satisfy the inequality

$$
||f(xy) + f(\sigma(y)x) - 2f(x) * g(y)|| \leq \delta \text{ for all } x, y \in M.
$$

(3.2)

Then, either

(i) there exists $k \geq 1$ such that the function $x \mapsto \langle g(x), e_k \rangle$ is bounded, or

(ii) the pair $(f, g)$ is a solution of the functional equation:

$$
f(xy) + f(\sigma(y)x) = 2f(x) * g(y).
$$

(3.3)

Proof. Suppose that the pair $(f, g)$ satisfies (3.2). By applying the Parseval identity and the definition of Hadamard product with the inequality (3.2), we find that the scalar valued functions $f_k, g_k$ defined by

$$
f_k(x) = \langle f(x), e_k \rangle \text{ and } g_k(x) = \langle g(x), e_k \rangle \text{ for } x \in M,
$$

satisfy the inequality

$$
|f_k(xy) + f_k(\sigma(y)x) - 2f_k(x)g_k(y)| \leq \delta \text{ for all } x, y \in M.
$$

According to Proposition 3.1, for all $k \in \mathbb{N}$, we have that either the function $x \mapsto \langle g(x), e_k \rangle$ is bounded or the pair $(f_k, g_k)$ is a solution of (3.1). Then, we conclude that the pair $(f, g)$ satisfies equation (3.3) if assertion (i) fails. □

In [4] it was proved that if $g : H \to H$ is surjective, then every component function $x \mapsto \langle g(x), e_n \rangle$ is unbounded. By applying Theorem (3.3), this leads to the following result.

Corollary 3.4. Let $\delta > 0$ be given. Assume that functions $f, g : H \to H$, where $g$ is surjective, satisfy the inequality

$$
||f(xy) + f(\sigma(y)x) - 2f(x) * g(y)|| \leq \delta \text{ for all } x, y \in H.
$$

Then, the pair $(f, g)$ satisfies the equation

$$
f(xy) + f(\sigma(y)x) = 2f(x) * g(y) \text{ for all } x, y \in H.
$$
Proof. Since \( g \) is surjective, then every component function \( x \mapsto \langle g(x), e_n \rangle \) is unbounded. Thus, the proof follows immediately from Theorem 3.3. \( \square \)

**Corollary 3.5.** Let \( \delta > 0 \) be given and let \( G \) be a topological group. Assume that the function \( g : G \to H \) satisfies the inequality
\[
||g(xy) + g(\sigma(y)x) - 2g(x) * g(y)|| \leq \delta \text{ for all } x, y \in G.
\]

Then, either there exists \( k \geq 1 \) such that
\[
||\langle g(x), e_k \rangle|| \leq \frac{1 + \sqrt{1 + 2\delta}}{2} \text{ for all } x \in G
\]
or there exist a multiplicative function \( \chi_k : M \to C \{0\} \) and a positive integer \( N \) such that
\[
g(x) = \frac{1}{2} \sum_{n=1}^{N} \epsilon_n (\chi_k(x) + \chi_k \circ \sigma(x)) e_n, \text{ for all } x \in G,
\]
where \( \epsilon_n = 1 \) or 0 for every \( n \in \{1, 2, \ldots, N\} \).

**Proof.** If we put \( f = g \) in Theorem 3.3, we immediately have that either there exists \( k \geq 1 \) such that the function \( x \mapsto \langle g(x), e_k \rangle \) is bounded or \( g \) is a solution of the equation
\[
g(xy) + g(\sigma(y)x) = 2g(x) * g(y), \quad x, y \in G.
\]

The remainder of the proof follows if we put \( \chi = 1 \) from Corollary [3, Corollary 3.8] and Corollary 2.3. \( \square \)

**Corollary 3.6.** Let \( \delta > 0 \) be given and let \( G \) be a group with identity element. Let \( g : G \to H \) such that
\[
||g(xy) + g(yx) - 2g(x) * g(y)|| \leq \delta \text{ for all } x, y \in G.
\]

Then either \( g \) is bounded or \( g \) is multiplicative.

**Proof.** From Corollary 2.2 and Corollary 2.5 and then using [3, Corollary 3.9]. \( \square \)
References


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