Pairwise generalized $b$-$R_0$ spaces in bitopological spaces

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Received: January 2017. Accepted: May 2017

Abstract

The main purpose of this paper is to introduce pairwise generalized $b$-$R_0$ spaces in bitopological spaces with the help of generalized $b$-open sets in bitopological spaces and give several characterizations of this spaces. We also introduce generalized $b$-kernel of a set and investigate some properties of it and study the relationship between this space and other bitopological spaces.

Key Words: Bitopological spaces; pairwise $gb$-$R_0$ spaces; pairwise $gb$-$R_1$ spaces; $(i,j)$-$gb$-kernel; $(i,j)$-$gb$-open sets.

AMS Classification: 54A10; 54C08; 54C10; 54D15.
1. Introduction

A triplet \((X, \tau_1, \tau_2)\), where \(X\) is a non-empty set and \(\tau_1, \tau_2\) are topologies on \(X\) is called a bitopological space. Kelly [9] initiated the study of bitopological spaces. Later on several authors turned their attention towards generalizations of various concepts of topology by considering bitopological spaces. Andrijevic [6] introduced a new class of generalized open sets called \(b\)-open sets in the field of topology and studied several fundamental and interesting properties. Later on Khadra and Nasef [1] and Al-Hawary and Al-Omari [4] defined the notions of \(b\)-open sets in bitopological spaces. Al-Hawary ([2], [3]) studied about the pre-open sets. Tripathy and Sarma ([18], [19], [20], [21], [22]) have done some works on bitopological spaces using this notion. Ganster and Steiner [8] introduced the concept of generalized \(b\)-closed sets in topological spaces. There after Tripathy and Sarma [22] have extended this notion to bitopological spaces. It is observed from literature that there has been a considerable work on different relatively weak form of separation axiom like \(R_o\) axiom. For instance, semi-\(R_o\), pre-\(R_o\), \(b\)-\(R_o\) are some of the variant form of \(R_o\) property that have been investigated by different researchers as separate entities. Khaleefa [10] has introduced and studied new types of separation axioms termed by generalized \(b\)-\(R_o\) and generalized \(b\)-\(R_1\) by using generalized \(b\)-open sets due to Ganster and Steiner [8]. Tripathy and Acharjee [20] have done some works on bitopological spaces.

In this paper, we introduce the notion of pairwise \(gb\)-\(R_o\) spaces and pairwise \(gb\)-\(R_1\) spaces in bitopological spaces and investigate some of their properties. In particular, the notion of \((i, j)\)-\(gb\)-kernel of a set is also defined in bitopological spaces.

2. Preliminaries

Throughout this paper, \((X, \tau_1, \tau_2)\) denotes a bitopological space on which no separation axioms are assumed. For a subset \(A\) of \(X\), \(i\)-\(int(A)\) and \(j\)-\(cl(A)\) denotes the \(i\)-interior and \(j\)-closure of \(A\) with respect to the topology \(\tau_i\) and \(\tau_j\) respectively, where \(i, j \in \{1, 2\}, i \neq j\).

**Definition 2.1.** A subset \(A\) of \((X, \tau)\) is called \(b\)-open, if \(A \subset int(cl(A)) \cup cl(int(A))\) and called \(b\)-closed if \(X \setminus A\) is \(b\)-open.
One may refer to Andrijevic [6] for the above definition.

The following definitions are due to Al-Hawary and Al-Omari [4].

**Definition 2.2.** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(i, j)$-open if $A \subset i\text{-}int(j\text{-}cl(A)) \cup j\text{-}cl(i\text{-}int(A))$. The complement of an $(i, j)$-open set is $(i, j)$-b-closed.

By $(i, j)$ we mean the pair of topologies $(\tau_i, \tau_j)$.

**Definition 2.3.** Let $A$ be a subset of a bitopological space $(X, \tau_1, \tau_2)$. Then

(i) the $(i, j)$-b-closure of $A$ denoted by $(i, j)$-b\,cl$(A)$, is defined by the intersection of all $(i, j)$-b-closed sets containing $A$.

(ii) the $(i, j)$-b-interior of $A$ denoted by $(i, j)$-b\,int$(A)$, is defined by the union of all $(i, j)$-b-open sets contained in $A$.

The following Definitions and results are due to Tripathy and Sarma [19].

**Definition 2.4.** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(i, j)$-generalized b-closed (in short, $(i, j)$-gb-closed) set if $(j, i)$-bcl$(A) \subset U$ whenever $A \subset U$ and $U$ is $\tau_i$-open in $X$.

We denote the family of all $(i, j)$-gb-closed sets and $(i, j)$-gb-open sets in $(X, \tau_1, \tau_2)$ by $GBC(i, j)$ and $GBO(i, j)$ respectively.

**Definition 2.5.** The $(i, j)$-generalized b-closure of a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is the intersection of all $(i, j)$-gb-closed sets containing $A$ and is denoted by $(i, j)$-gbcl$(A)$.

**Lemma 2.1.** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is $(i, j)$-gb-open if and only if $U \subset (j, i)$-bint$(A)$, whenever $U$ is $\tau_i$-closed and $U \subset A$.

**Lemma 2.2.** For any subset $A$ of a bitopological space $(X, \tau_1, \tau_2), A \subset (i, j)$-gbcl$(A)$. 
Lemma 2.3. Let \((X, \tau_1, \tau_2)\) be a bitopological space. If \(A\) is \((i, j)\)-gb-closed subset of \(X\), then \(A = (i, j)\)-gbcl\((A)\).

Lemma 2.4. A point \(x \in (i, j)\)-gbcl\((A)\) if and only if for every \((i, j)\)-gb-open set \(U\) containing \(x\), \(U \cap A \neq \emptyset\).

3. Pairwise gb-\(R_o\) Spaces

Definition 3.1. A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise generalized \(b-R_o\) (in short, pairwise \(gb-R_o\)) spaces if \((j, i)\)-gbcl\(\{x\}\) \(\subset\) \(U\), for every \((i, j)\)-gb-open set \(U\) containing \(x\) and \(i, j = 1, 2, i \neq j\).

Theorem 3.1. Let \((X, \tau_1, \tau_2)\) be a bitopological space. Then the following statements are equivalent:

(a) \((X, \tau_1, \tau_2)\) is pairwise gb-\(R_o\) space.

(b) For any \((i, j)\)-gb-closed set \(V\) and \(x \notin V\), there exist a \((j, i)\)-gb-open set \(U\) such that \(x \notin U\) and \(V \subset U\), for \(i, j = 1, 2, i \neq j\).

(c) For any \((i, j)\)-gb-closed set \(V\) and \(x \notin V\), \((j, i)\)-gbcl\(\{x\}\) \(\cap\) \(V = \emptyset\) for \(i, j = 1, 2, i \neq j\).

Proof. (a) \(\Rightarrow\) (b) Let \(V\) be an \((i, j)\)-gb-closed set and \(x \notin V\). By (a), we have \((j, i)\)-gbcl\(\{x\}\) \(\subset\) \(X \setminus V\). Put \(U = X \setminus (j, i)\)-gbcl\(\{x\}\). Then \(U\) is \((j, i)\)-gb-open and \(V \subset X \setminus (j, i)\)-gbcl\(\{x\}\) = \(U\). Thus \(V \subset U\) and \(x \notin U\).

(b) \(\Rightarrow\) (c) Let \(V\) be a \((i, j)\)-gb-closed set and \(x \notin V\). By hypothesis, there exists a \((j, i)\)-gb-open set \(U\) such that \(x \notin U\) and \(V \subset U\). Which implies \(U \cap (j, i)\)-gbcl\(\{x\}\) = \(\emptyset\) since \(U\) is \((j, i)\)-gb-open. Hence \(V \cap (j, i)\)-gbcl\(\{x\}\) = \(\emptyset\).

(c) \(\Rightarrow\) (d) Let \(U\) be a \((i, j)\)-gb-open set such that \(x \in U\). Now, \(X \setminus U\) is \((i, j)\)-gb-closed and \(x \notin X \setminus U\). By (c), \((j, i)\)-gbcl\(\{x\}\) \(\cap\) \((X \setminus U) = \emptyset\). Which implies \((j, i)\)-gbcl\(\{x\}\) \(\subset\) \(U\). Hence \((X, \tau_1, \tau_2)\) is pairwise gb-\(R_o\) space.
Theorem 3.2. Let \((X, \tau_1, \tau_2)\) be a bitopological space. Then \(X\) is pairwise \(gb-R_0\) space if and only if for any two distinct points \(x\) and \(y\) of \(X\), either 
\((i, j)\)-\(gbcl\({\{x}\}\) \(\cap\) \((j, i)\)-\(gbcl\({\{y}\}\)) = \emptyset\) or \({x, y}\) \(\subset\) \((i, j)\)-\(gbcl\({\{x}\}\) \(\cap\) \((j, i)\)-\(gbcl\({\{y}\}\))

Proof. Let \((i, j)\)-\(gbcl\({\{x}\}\) \(\cap\) \((j, i)\)-\(gbcl\({\{y}\}\)) \(\neq\) \emptyset\) and \({x, y}\) is not contained in \((i, j)\)-\(gbcl\({\{x}\}\) \(\cap\) \((j, i)\)-\(gbcl\({\{y}\}\))\). Let \(z \in (i, j)\)-\(gbcl\({\{x}\}\) \(\cap\) \((j, i)\)-\(gbcl\({\{y}\}\))\) and \(x \notin (i, j)\)-\(gbcl\({\{x}\}\) \(\cap\) \((j, i)\)-\(gbcl\({\{y}\}\))\). Now, \(x \notin (j, i)\)-\(gbcl\({\{y}\}\)\) implies \(x \in X \setminus (i, j)\)-\(gbcl\({\{y}\}\)\), which is a \((j, i)\)-\(gb\)-open set containing \(x\). Since \(z \in (j, i)\)-\(gbcl\({\{y}\}\)\), so \((i, j)\)-\(gbcl\({\{x}\}\) \(\cap\) \((j, i)\)-\(gbcl\({\{y}\}\)) \(\neq\) \emptyset\).

Conversely, Let \(U\) be a \((i, j)\)-\(gb\)-open set such that \(x \in U\). Suppose \((j, i)\)-\(gbcl\({\{x}\}\) is not contained in \(U\). Then there exists a \(y \in (j, i)\)-\(gbcl\({\{x}\}\) such that \(y \notin U\) and \((i, j)\)-\(gbcl\({\{y}\}\) \(\cap\) \(U\) = \emptyset\), since \(X \setminus U\) is \((i, j)\)-\(gb\)-closed and \(y \in X \setminus U\). Hence \({x, y}\) is not contained in \((i, j)\)-\(gbcl\({\{x}\}\) \(\cap\) \((j, i)\)-\(gbcl\({\{y}\}\))\) and \((i, j)\)-\(gbcl\({\{y}\}\) \(\cap\) \((j, i)\)-\(gbcl\({\{x}\}\)) \(\neq\) \emptyset\).

Now, we introduce the concept of \((i, j)\)-\(gb\)-kernel of a set and utilizing it to characterize the notion of pairwise \(gb-R_0\) space.

Definition 3.2. Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A \subset X\). The intersection of all \((i, j)\)-\(gb\)-open sets containing \(A\) is called the \((i, j)\)-\(gb\)-kernel of \(A\) and is denoted by \((i, j)\)-\(gbker\({A}\)).

The \((i, j)\)-\(gb\)-kernel of a point \(x \in X\) is the set \((i, j)\)-\(gbker\({\{x}\}\) = \cap\{U : U\) is \((i, j)\)-\(gb\)-open and \(x \in U\}\).

\(= \{y : x \in (i, j)\)-\(gbcl\({\{y}\}\}\}.

Theorem 3.3. Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A\) be a subset of \(X\). Then \((i, j)\)-\(gbker\({A}\) = \{x \in X : \) \((i, j)\)-\(gbcl\({\{x}\}\) \(\cap\) \(A\) \(\neq\) \emptyset\}.

Proof. Let \(x \in (i, j)\)-\(gbker\({A}\) and \((i, j)\)-\(gbcl\({\{x}\}\) \(\cap\) \(A\) = \emptyset\). Therefore \((i, j)\)-\(gbcl\({\{x}\}\) \(\subset\) \(X \setminus A\) and so \(A \subset X \setminus (i, j)\)-\(gbcl\({\{x}\}\)\). But \(x \notin X \setminus (i, j)\)-\(gbcl\({\{x}\}\)\), which is a \((i, j)\)-\(gb\)-open sets containing \(A\). Thus \(x \notin (i, j)\)-\(gbker\({A}\)\), a contradiction. Consequently, \((i, j)\)-\(gbcl\({\{x}\}\) \(\cap\) \(A\) \(\neq\) \emptyset\).
Theorem 3.5. Let \((X, \tau_1, \tau_2)\) be a bitopological space. Then \(\bigcap\{(i,j)\text{-}gbcl(\{x\}) : x \in X\} = \emptyset\) if and only if \((i,j)\text{-}gb\ker(\{x\}) \neq X\), for every \(x \in X\).

Proof. Assume that \(\bigcap\{(i,j)\text{-}gbcl(\{x\}) : x \in X\} = \emptyset\). Let \((i,j)\text{-}gb\ker(\{x\}) = X\). If there is some \(y \in X\), then \(X\) is the only \((i,j)\text{-}gb\)-open set containing \(y\). Which shows \(y \in (i,j)\text{-}gbcl(\{x\})\), for every \(x \in X\). Therefore \(\bigcap\{(i,j)\text{-}gbcl(\{x\}) : x \in X\} \neq \emptyset\), a contradiction. Hence \((i,j)\text{-}gb\ker(\{x\}) \neq X\), for every \(x \in X\).

Conversely assume that \((i,j)\text{-}gb\ker(\{x\}) \neq X\), for every \(x \in X\). Let \(\bigcap\{(i,j)\text{-}gbcl(\{x\}) : x \in X\} \neq \emptyset\). If there is some \(y \in X\) such that \(y \in \bigcap\{(i,j)\text{-}gbcl(\{x\}) : x \in X\}\), then every \((i,j)\text{-}gb\)-open set containing \(y\) must contain every point of \(X\). This shows that \(X\) is the only \((i,j)\text{-}gb\)-open set containing \(y\). Therefore \((i,j)\text{-}gb\ker(\{x\}) = X\), a contradiction. Hence \(\bigcap\{(i,j)\text{-}gbcl(\{x\}) : x \in X\} = \emptyset\).

Theorem 3.5. Let \((X, \tau_1, \tau_2)\) be a bitopological space. Then the following statements are equivalent:

(a) \((X, \tau_1, \tau_2)\) is pairwise gb-\(R_0\) space.
(b) For any \(x \in X\), \((i,j)\text{-}gbcl(\{x\}) = (j,i)\text{-}gb\ker(\{x\})\), for \(i, j = 1, 2\) and \(i \neq j\).
(c) For any \(x \in X\), \((i,j)\text{-}gbcl(\{x\}) \subset (j,i)\text{-}gb\ker(\{x\})\), for \(i, j = 1, 2\) and \(i \neq j\).
(d) For any \(x, y \in X\), \(y \in (i,j)\text{-}gb\ker(\{x\})\) if and only if \(x \in (j,i)\text{-}gb\ker(\{y\})\), for \(i, j = 1, 2\) and \(i \neq j\).
(e) For any \(x, y \in X\), \(y \in (i,j)\text{-}gbcl(\{x\})\) if and only if \(x \in (j,i)\text{-}gbcl(\{y\})\), for \(i, j = 1, 2\) and \(i \neq j\).
(f) For any \((i,j)\text{-}gb\)-closed set \(V\) and \(x \notin V\), there exist a \((j,i)\text{-}gb\)-open set \(U\) such that \(x \notin U\) and \(V \subseteq U\), for \(i, j = 1, 2\) and \(i \neq j\).
(g) For each \((i,j)\text{-}gb\)-closed set \(V\), \(V = \bigcap\{U : U\) is \((j,i)\text{-}gb\)-open and
$V \subset U$, for $i, j = 1, 2$ and $i \neq j$.

(h) For each $(i, j)$-gb-open set $U$, $U = \bigcup\{V : V$ is $(j, i)$-gb-closed and $V \subset U\}$, for $i, j = 1, 2$ and $i \neq j$.

(i) For every non-empty subset $A$ of $X$ and for any $(i, j)$-gb-open set $U$ such that $A \cap U \neq \emptyset$, there exists a $(j, i)$-gb-closed $V$ such that $A \cap V \neq \emptyset$ and $V \subset U$, for $i, j = 1, 2$ and $i \neq j$.

(j) For any $(j, i)$-gb-closed set $V$ and $x \notin V$, $(j, i)$-gbcl($\{x\}$) $\cap V = \emptyset$, for $i, j = 1, 2$ and $i \neq j$.

**Proof.**

(a) $\Rightarrow$ (b) Let $x, y \in X$. Then by Definition 3.2, $y \in (j, i)$-gb-ker($\{x\}$) $\iff x \in (j, i)$-gbcl($\{y\}$). Since $X$ is pairwise gb-Ro space, therefore by Theorem 3.2, we have $x \in (j, i)$-gbcl($\{y\}$) $\iff y \in (i, j)$-gbcl($\{x\}$). Thus $y \in (j, i)$-gb-ker($\{x\}$) $\iff x \in (j, i)$-gbcl($\{y\}$) $\iff y \in (i, j)$-gbcl($\{x\}$). Hence $(i, j)$-gbcl($\{x\}$) = $(j, i)$-gb-ker($\{x\}$).

(b) $\Rightarrow$ (c) It is obvious.

(c) $\Rightarrow$ (d) Let $x, y \in X$ and $y \in (i, j)$-gb-ker($\{x\}$). Then by Definition 3.2, $x \in (i, j)$-gbcl($\{y\}$). Therefore by (c), $x \in (i, j)$-gbcl($\{y\}$) $\subset (j, i)$-gbker($\{y\}$). Thus $x \in (j, i)$-gb-ker($\{y\}$). Similarly, we can prove the other part also.

(d) $\Rightarrow$ (e) Let $x, y \in X$ and $y \in (i, j)$-gbcl($\{x\}$). Then by Definition 3.2, $x \in (i, j)$-gb-ker($\{y\}$). Therefore by (d), $y \in (j, i)$-gb-ker($\{x\}$) and so $x \in (j, i)$-gbcl($\{y\}$). Similarly, we can prove the other part also.

(e) $\Rightarrow$ (f) Let $V$ be a $(i, j)$-gb-closed set and $x \notin V$. Then for any $y \in V$, we have $(i, j)$-gbcl($\{y\}$) $\subset V$ and $x \notin (i, j)$-gbcl($\{y\}$). Therefore by (e), $y \notin (j, i)$-gbcl($\{x\}$). That is there exists a $(j, i)$-gb-open set $U_y$ such that $y \in U_y$ and $x \notin U_y$. Let $U = \bigcup_{y \in V}$ $\{U_y : (j, i)$-gb-open, $y \in U_y$ and $x \notin U_y\}$. Hence $U$ is $(j, i)$-gb-open set such that $x \notin U$ and $V \subset U$.

(f) $\Rightarrow$ (g) Let $V$ be an $(i, j)$-gb-closed set in $X$ and $W = \bigcap\{U : U$ is $(j, i)$-gb-open and $V \subset U\}$. Clearly, $V \subset W$. Suppose that, $x \notin V$. Therefore by (f), there is a $(j, i)$-gb-open set $U$ such that $x \notin U$ and $V \subset U$. So $x \notin W$ and thus $W \subset V$. Hence $V = W = \bigcap\{U : U$ is $(j, i)$-gb-open and $V \subset U\}$.

(g) $\Rightarrow$ (h) It is obvious.
(h) \Rightarrow (i) Let \( A \) be a non-empty subset of \( X \) and \( U \) be a \((i, j)\)-gb-open set in \( X \) such that \( A \cap U \neq \emptyset \). Let \( x \in A \cap U \). By (h), \( U = \bigcup \{ V : V \text{ is } (j, i)\)-gb-closed and \( V \subset U \} \). Then there is a \((j, i)\)-gb-closed \( V \) such that \( x \in V \subset U \). Therefore \( x \in A \cap V \) and so \( A \cap V \neq \emptyset \).

(i) \Rightarrow (j) Let \( V \) be a \((i, j)\)-gb-closed set such that \( x \notin V \). Then \( X \setminus V \) is \((i, j)\)-gb-open set containing \( x \) and \( \{ x \} \cap (X \setminus V) \neq \emptyset \). Therefore by (i), there is a \((j, i)\)-gb-closed set \( W \) such that \( W \subset X \setminus V \) and \( \{ x \} \cap W \neq \emptyset \).

Hence \((j, i)\)-gbcl(\{x\}) \subset X \setminus V \) and so \((j, i)\)-gbcl(\{x\}) \cap V = \emptyset \).

(j) \Rightarrow (a) Follows from Theorem 3.1.

**Theorem 3.6.** In a pairwise gb-R_0 space \((X, \tau_1, \tau_2)\), for any \( x \in X \), \((i, j)\)-gbcl(\{x\}) \cap (j, i)-gb-ker(\{x\}) = \{x\} holds for \( i, j = 1, 2 \) and \( i \neq j \), then \((i, j)\)-gbcl(\{x\}) = \{x\}.

**Proof.** Since \((X, \tau_1, \tau_2)\) is pairwise gb-R_0 space, therefore by Theorem 3.5 (b), we have \((i, j)\)-gbcl(\{x\}) = (j, i)-gb-ker(\{x\}). Hence the result follows.

**Theorem 3.7.** If \((X, \tau_1, \tau_2)\) is a pairwise gb-R_0 space, then for any \( x, y \in X \), either \((i, j)\)-gbcl(\{x\}) \cap (j, i)-gbcl(\{x\}) = \{x\} or \((i, j)\)-gbcl(\{x\}) \cap (j, i)-gbcl(\{y\}) = \emptyset \).

**Proof.** Let \((X, \tau_1, \tau_2)\) is a pairwise gb-R_0 space. Suppose that \{(i, j)\}-gbcl(\{x\}) \cap (j, i)-gbcl(\{x\}) \cap \{(i, j)\}-gbcl(\{y\}) \cap (j, i)-gbcl(\{y\}) \neq \emptyset \). Let \( z \in \{(i, j)\}-gbcl(\{x\}) \cap (j, i)-gbcl(\{y\}) \cap (i, j)-gbcl(\{y\}) \). Then \((i, j)-gbcl(\{z\}) \subset \{(i, j)\}-gbcl(\{x\}) \cap (j, i)-gbcl(\{x\}) \subset \{(i, j)\}-gbcl(\{y\}) \cap (j, i)-gbcl(\{y\}) \). Similarly, \( z \in \{(j, i)\}-gbcl(\{x\}) \) implies \((i, j)\)-gbcl(\{x\}) \subset \{(i, j)\}-gbcl(\{y\}) \). Hence the result follows.
Theorem 3.8. If \((X, \tau_1, \tau_2)\) is a pairwise \(gb-R_0\) space, then for any \(x, y \in X\), either \((i, j)-gb\)-\(ker\(\{x\}\)) \(\cap\) \((j, i)-gb\)-\(ker\(\{x\}\)) = \((i, j)-gb\)-\(ker\(\{y\}\)) \(\cap\) \((j, i)-gb\)-\(ker\(\{y\}\)) or \((i, j)-gb\)-\(ker\(\{x\}\)) \(\cap\) \((j, i)-gb\)-\(ker\(\{x\}\)) \(\cap\) \((j, i)-gb\)-\(ker\(\{y\}\)) \(\cap\) \((j, i)-gb\)-\(ker\(\{y\}\)) = \emptyset.

Proof. The proof is similar to that of Theorem 3.7 which follows from Definition of \((i, j)-gb\)-\(ker\(\{x\}\)) and Theorem 3.5.

4. Pairwise \(gb-R_1\) Spaces

Definition 4.1. A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise generalized \(b-R_1\) (in short, pairwise \(gb-R_1\)) if for every pair of distinct points \(x, y\) of \(X\) such that \((i, j)-gb\)-\(cl\(\{x\}\)) \(\neq\) \((j, i)-gb\)-\(cl\(\{y\}\)) there exists a \((j, i)-gb\)-open set \(U\) and an \((i, j)-gb\)-open set \(V\) such that \(U \cap V = \emptyset\) and \((i, j)-gb\)-\(cl\(\{x\}\)) \(\subset\) \(U\), \((j, i)-gb\)-\(cl\(\{y\}\)) \(\subset\) \(V\), for \(i, j = 1, 2\) and \(i \neq j\).

Theorem 4.1. If \((X, \tau_1, \tau_2)\) is pairwise \(gb-R_1\) space, then it is pairwise \(gb-R_0\) space.

Proof. Suppose that \((X, \tau_1, \tau_2)\) is pairwise \(gb-R_1\) space. Let \(U\) be an \((i, j)-gb\)-open set containing \(x\). Then for each \(y \in X \setminus U\), \((j, i)-gb\)-\(cl\(\{x\}\)) \(\neq\) \((i, j)-gb\)-\(cl\(\{y\}\)). Since \((X, \tau_1, \tau_2)\) is pairwise \(gb-R_1\), there exists an \((i, j)-gb\)-open set \(U_y\) and a \((j, i)-gb\)-open set \(V_y\) such that \(U_y \cap V_y = \emptyset\) and \((i, j)-gb\)-\(cl\(\{y\}\)) \(\subset\) \(V_y\), \((j, i)-gb\)-\(cl\(\{x\}\)) \(\subset\) \(U_y\). Let \(A = \bigcup \{V_y : y \in X \setminus U\}\). Then \(X \setminus U \subset A\), \(x \notin A\) and \(A\) is \((j, i)-gb\)-open set. Therefore \((j, i)-gb\)-\(cl\(\{x\}\)) \(\subset\) \(X \setminus A\) \(\subset U\) and hence \((X, \tau_1, \tau_2)\) is pairwise \(gb-R_0\) space.

Theorem 4.2. A bitopological space \((X, \tau_1, \tau_2)\) is pairwise \(gb-R_1\) if and only if for every \(x, y \in X\) such that \((i, j)-gb\)-\(cl\(\{x\}\)) \(\neq\) \((j, i)-gb\)-\(cl\(\{y\}\)) there exists an \((i, j)-gb\)-open set \(U\) and a \((j, i)-gb\)-open set \(V\) such that \(x \in V\), \(y \in U\) and \(U \cap V = \emptyset\) for \(i, j = 1, 2\) and \(i \neq j\).

Proof. Suppose that \((X, \tau_1, \tau_2)\) is pairwise \(gb-R_1\) space. Let \(x, y \in X\) such that \((i, j)-gb\)-\(cl\(\{x\}\)) \(\neq\) \((j, i)-gb\)-\(cl\(\{y\}\)). Then there exists an \((i, j)-gb\)-open set \(U\) and a \((j, i)-gb\)-open set \(V\) such that \(x \in (i, j)-gb\)-\(cl\(\{x\}\)) \(\subset\) \(V\) and \(y \in (j, i)-gb\)-\(cl\(\{y\}\)) \(\subset\) \(U\).
Conversely, suppose that there exists an \((i, j)\)-gb-open set \(U\) and a \((j, i)\)-gb-open set \(V\) such that \(x \in V\), \(y \in U\) and \(U \cap V = \emptyset\). Therefore \((i, j)\)-\(\text{gbcl}(\{x\}) \cap (j, i)\)-\(\text{gbcl}(\{y\}) = \emptyset\). So by Theorem 3.2, \((X, \tau_1, \tau_2)\) is pairwise gb-\(R_0\) space. Then \((i, j)\)-\(\text{gbcl}(\{x\}) \subset V\) and \((j, i)\)-\(\text{gbcl}(\{y\}) \subset U\). Hence \((X, \tau_1, \tau_2)\) is pairwise gb-\(R_1\) space.

**Theorem 4.3.** Let \((X, \tau_1, \tau_2)\) be a bitopological space. Then the following are equivalent:

(a) \((X, \tau_1, \tau_2)\) is pairwise gb-\(R_1\) space.

(b) For any \(x, y \in X\), \(x \neq y\) and \((i, j)\)-\(\text{gbcl}(\{x\}) \neq (j, i)\)-\(\text{gbcl}(\{y\})\) implies that there exists an \((i, j)\)-gb-open set \(G_1\) and a \((j, i)\)-gb-open set \(G_2\) such that \(x \in G_1\), \(y \notin G_1\), \(y \in G_2\), \(x \notin G_2\) and \(X = G_1 \cup G_2\), for \(i, j = 1, 2\) and \(i \neq j\).

**Proof.** (a) \(\Rightarrow\) (b) Suppose that \((X, \tau_1, \tau_2)\) is pairwise gb-\(R_1\) space. Let \(x, y \in X\) such that \((i, j)\)-\(\text{gbcl}(\{x\}) \neq (j, i)\)-\(\text{gbcl}(\{y\})\). Therefore by Theorem 4.2, there exists an \((i, j)\)-gb-open set \(V\) and a \((j, i)\)-gb-open set \(U\) such that \(x \in U\), \(y \in V\) and \(U \cap V = \emptyset\). Then \(G_1 = X \setminus V\) is \((i, j)\)-gb-closed and \(G_2 = X \setminus U\) is \((j, i)\)-gb-closed set such that \(x \in G_1\), \(y \notin G_1\), \(y \in G_2\), \(x \notin G_2\) and \(X = G_1 \cup G_2\).

(b) \(\Rightarrow\) (a) Let \(x, y \in X\) such that \((i, j)\)-\(\text{gbcl}(\{x\}) \neq (j, i)\)-\(\text{gbcl}(\{y\})\). Therefore for any \(x, y \in X\), \(x \neq y\), we have \((i, j)\)-\(\text{gbcl}(\{x\}) \cap (j, i)\)-\(\text{gbcl}(\{y\}) = \emptyset\). Then by Theorem 3.2, \((X, \tau_1, \tau_2)\) is pairwise gb-\(R_0\) space. By (b), there is an \((i, j)\)-gb-closed set \(G_1\) and a \((j, i)\)-gb-closed set \(G_2\) such that \(x \in G_1\), \(y \notin G_1\), \(y \in G_2\), \(x \notin G_2\) and \(X = G_1 \cup G_2\). Therefore \(x \in X \setminus G_2 = U\), which is \((j, i)\)-gb-open and \(y \in X \setminus G_1 = V\), which is \((i, j)\)-gb-open. Which implies that \((i, j)\)-\(\text{gbcl}(\{x\}) \subset U\), \((j, i)\)-\(\text{gbcl}(\{y\}) \subset V\) and \(U \cap V = \emptyset\). Hence the result.

**References**

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