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On fractional powers of double band matrices

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Abstract

In the present article, we determine the explicit formula for finding the fractional powers of a double band matrix and in particular, we establish the formula for finding the n th root of the matrix. Some examples are also given for supporting the new formulas.

Keywords: *Forward difference operator; Backward difference operator; Double band matrix.*

1. Introduction

Let $\mathbf{R}$ be the set of all real numbers and $\mathbf{N}$ denote the set of all positive integers. Let w be the space of all real valued sequences. Let X and Y be two subspaces of w , then we define a matrix mapping $A : X \rightarrow Y$, as

$$(1.1) \quad (Ax)_n := \sum_k a_{nk} x_k, (n \in \mathbf{N}).$$

In fact, for $x = (x_k) \in X$, Ax is called as the A -transform of x provided the series in the right hand side of (1.1) converges for each $n \in \mathbf{N}$. Moreover, the matrix $A = (a_{ij})(i, j \in \mathbf{N})$ is also regarded as a linear operator. Let $x = (x_k)$ be any sequence in w and $r(\neq 0), s$ be two real numbers, then we define the generalized first forward and backward difference operators as

$$(B_+(r, s)x)_k = rx_k + sx_{k+1}, (k \in \mathbf{N})$$

and

$$(B_-(r, s)x)_k = rx_k + sx_{k-1}, (k \in \mathbf{N}).$$

Clearly, the operators $B_+(r, s)$ and $B_-(r, s)$ represent an upper and a lower double band matrices, respectively as follows:

$$B_+(r, s) = \begin{pmatrix} r & s & 0 & 0 & \dots \\ 0 & r & s & 0 & \dots \\ 0 & 0 & r & s & \dots \\ 0 & 0 & 0 & r & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, B_-(r, s) = \begin{pmatrix} r & 0 & 0 & 0 & \dots \\ s & r & 0 & 0 & \dots \\ 0 & s & r & 0 & \dots \\ 0 & 0 & s & r & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The forward difference operator Δ was initially studied by Kızmaz [9] by defining the difference sequence $(\Delta x_k) = (x_k - x_{k+1}), k \in \mathbf{N}$. In matrix notation, it can also be expressed as double band matrix where main and super diagonal elements are 1 and -1, respectively. Ahmad and Mursaleen [1] have studied the backward difference operator $\Delta^{(1)}$, defined by $(\Delta^{(1)}x_k) = (x_k - x_{k-1}), k \in \mathbf{N}$ and the corresponding double band matrix contains 1 and -1, respectively in main and sub diagonal positions. It is noted that any integral power of the difference matrices Δ and $\Delta^{(1)}$ can be directly found out by taking the difference operators Δ^m and $\Delta^{(m)}$, $m \in \mathbf{N}$ introduced by Et and Colak [7] and Ahmad and Mursaleen [1], respectively. Subsequently, the fractional powers of these matrices are directly evaluated by using the operator Δ^α and $\Delta^{(\alpha)}$ introduced by Baliarsingh [4] (see also [5, 6]). The double band matrix $B(r, s)$ was studied by Altay and Başar

[3](see also [8]). The primary goal of this work is to evaluate the integral and fractional powers of the matrix $B(r, s)$.

Since both of the operators $B_+(r, s)$ and $B_-(r, s)$ are linear and now taking α times successive transformations, we write

$$(1.2) \quad (B_+^\alpha(r, s)x)_k = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} r^{\alpha-i} s^i x_{k+i}, (k \in \mathbf{N})$$

$$(1.3) \quad (B_-^\alpha(r, s)x)_k = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} r^{\alpha-i} s^i x_{k-i}, (k \in \mathbf{N}),$$

where $\Gamma(\alpha)$ denotes the well known Gamma function of a real number α and $\alpha \notin \{0, -1, -2, -3, \dots\}$. For any integral value of α , Eqns.(1.2) and (1.3) reduce to finite sums. It is noticed that operators $B_+^\alpha(r, s)$ and $B_-^\alpha(r, s)$ includes several difference operators such as

- (i) Forward and backward difference operators Δ and $\Delta^{(1)}$ for $\alpha = 1, r = 1$ and $s = -1$ (see [9, 2]).
- (ii) Forward and backward difference operators Δ^m and $\Delta^{(m)}$ for $\alpha = m \in \mathbf{N}, r = 1$ and $s = -1$ (see [7, 1]).
- (iii) Forward and backward difference operators Δ^α and $\Delta^{(\alpha)}$ for $r = 1$ and $s = -1$ (see [4, 5, 6]).
- (iv) Difference operator $B(r, s)$ for $\alpha = 1$ (see [3]).

Keeping in view Eqns.(1.2) and (1.3), the inverse operators of $B_+^\alpha(r, s)$ and $B_-^\alpha(r, s)$ are $B_+^{-\alpha}(r, s)$ and $B_-^{-\alpha}(r, s)$, respectively, where

$$(1.4) \quad (B_+^{-\alpha}(r, s)x)_k = \sum_{i=0}^{\infty} \frac{\Gamma(-\alpha+1)}{i!\Gamma(-\alpha-i+1)} r^{-\alpha-i} s^i x_{k+i}, (k \in \mathbf{N})$$

$$(1.5) \quad (B_-^{-\alpha}(r, s)x)_k = \sum_{i=0}^{\infty} \frac{\Gamma(-\alpha+1)}{i!\Gamma(-\alpha-i+1)} r^{-\alpha-i} s^i x_{k-i}, (k \in \mathbf{N}),$$

It is remarked that for $\alpha = 1$, Eqns. (1.4) and (1.5) are being simplified by using formula $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and finally we obtain

$$(1.6) \quad (B_+^{-1}(r, s)x)_k = \sum_{i=0}^{\infty} (-1)^i \frac{s^i}{r^{i+1}} x_{k+i}, (k \in \mathbf{N})$$

$$(1.7) \quad (B_-^{-1}(r, s)x)_k = \sum_{i=0}^{\infty} (-1)^i \frac{s^i}{r^{i+1}} x_{k-i}, (k \in \mathbf{N}),$$

2. Main results

In the present section, we establish some new results on the operators $B_+^\alpha(r, s)$ and $B_-^\alpha(r, s)$ and their application to the matrix theory involving the integral and non integral powers of double band matrices.

Theorem 1. For any reals α, β and $k \in \mathbf{N}$,

- (i) $\left(B_+^\alpha(r, s) \left(B_+^\beta(r, s)x\right)\right)_k = \left(B_+^\beta(r, s) \left(B_+^\alpha(r, s)x\right)\right)_k = \left(B_+^{\alpha+\beta}(r, s)x\right)_k,$
- (ii) $\left(B_-^\alpha(r, s) \left(B_-^\beta(r, s)x\right)\right)_k = \left(B_-^\beta(r, s) \left(B_-^\alpha(r, s)x\right)\right)_k = \left(B_-^{\alpha+\beta}(r, s)x\right)_k,$
- (iii) $\|B_+^\alpha(r, s)\| = \|B_-^\alpha(r, s)\| = (|r| + |s|)^\alpha.$

Proof. The proof is a routine verification, hence omitted. $\square$

Corollary 1. For any reals α, β and $k \in \mathbf{N}$,

- (i) $\left(B_+^\alpha(r, s) \left(B_+^{-\alpha}(r, s)x\right)\right)_k = \left(B_+^{-\alpha}(r, s) \left(B_+^\alpha(r, s)x\right)\right)_k = x_k,$
- (ii) $\left(B_-^\alpha(r, s) \left(B_-^{-\alpha}(r, s)x\right)\right)_k = \left(B_-^{-\alpha}(r, s) \left(B_-^\alpha(r, s)x\right)\right)_k = x_k,$
- (iii) $\left(B_\pm^{-2}(r, s)x\right)_k = \sum_{i=0}^{\infty} (-1)^i (i+1) \frac{s^i}{r^{i+1}} x_{k \pm i},$
- (iv) $\left(B_\pm^{-3}(r, s)x\right)_k = \sum_{i=0}^{\infty} (-1)^i (i+1)(i+2) \frac{s^i}{2r^{i+1}} x_{k \pm i}.$

Proof. This is a direct application of Theorem 1. $\square$

Theorem 2. Let the lower double band matrix $A = (a_{nk})$ be defined by

$$a_{nk} = \begin{cases} r, & (k = n) \\ s, & (k = n-1) \\ 0, & (\text{otherwise}) \end{cases},$$

then for $\alpha \in \mathbf{R}$, $A^\alpha = (a_{nk}^\alpha)$ is given by

$$a_{nk}^\alpha = \begin{cases} r^\alpha, & (k = n) \\ \frac{\Gamma(\alpha+1)}{(n-k)! \Gamma(\alpha-n+k+1)} r^{\alpha-n+k} s^{n-k}, & (0 \leq k < n) \\ 0, & (k > n) \end{cases},$$

Proof. We prove this theorem by induction. Since $(Ax)_k = (B_-(r, s)x)_k = rx_k + sx_{k-1}$, hence we obtain that

$$\begin{aligned}(A^2x)_k &= A(rx_k + sx_{k-1}) = r^2x_k + 2rsx_{k-1} + s^2x_{k-2}, \\ (A^3x)_k &= A(r^2x_k + 2rsx_{k-1} + s^2x_{k-2}) = r^3x_k + 3r^2sx_{k-1} + 3rs^2x_{k-2} + s^3x_{k-3},\end{aligned}$$

Let the theorem be assumed for any arbitrary β and hence

$$(A^\beta x)_k = \sum_{i=0}^{\infty} \frac{\Gamma(\beta+1)}{i!\Gamma(\beta-i+1)} r^{\beta-i} s^i x_{k-i}.$$

Now,

$$\begin{aligned}(A^{\beta+1}x)_k &= A\left(\sum_{i=0}^{\infty} \frac{\Gamma(\beta+1)}{i!\Gamma(\beta-i+1)} r^{\beta-i} s^i x_{k-i}\right), \\ &= A\left(r^\beta x_k + \beta r^{\beta-1} s x_{k-1} + \frac{\beta(\beta-1)}{2!} r^{\beta-2} s^2 x_{k-2} + \dots\right) \\ &= r^\beta (rx_k + sx_{k-1}) + \beta r^{\beta-1} s (rx_{k-1} + sx_{k-2}) \\ &\quad + \frac{\beta(\beta-1)}{2!} r^{\beta-2} s^2 (rx_{k-2} + sx_{k-3}) + \dots \\ &= r^{\beta+1} x_k + (\beta+1) r^\beta s x_{k-1} + \frac{(\beta+1)\beta}{2!} r^{\beta-1} s^2 x_{k-2} + \dots \\ &= \sum_{i=0}^{\infty} \frac{\Gamma(\beta+2)}{i!\Gamma(\beta-i+2)} r^{\beta+1-i} s^i x_{k-i}\end{aligned}$$

This completes the proof. $\square$

Theorem 3. Let the upper double band matrix $A = (a_{nk})$ be defined by

$$a_{nk} = \begin{cases} r, & (k = n) \\ s, & (k = n+1) \\ 0, & (\text{otherwise}) \end{cases},$$

then for $\alpha \in \mathbf{R}$, $A^\alpha = (a_{nk}^\alpha)$ is given by

$$a_{nk}^\alpha = \begin{cases} r^\alpha, & (k = n) \\ \frac{\Gamma(\alpha+1)}{(k-n)!\Gamma(\alpha-k+n+1)} r^{\alpha-k+n} s^{k-n}, & (k > n) \\ 0, & (0 \leq k < n) \end{cases},$$

Proof. The proof is analogous to that of Theorem 2. $\square$

Corollary 2. The explicit formula for m th root of the double band matrix A , defined in Theorem 2 is given by

$$a_{nk}^{\frac{1}{m}} = \begin{cases} r^{1/m}, & (k = n) \\ \frac{(1-m)(1-2m)\dots(1-m(n-k-1))}{(n-k)!m^{n-k}} r^{\frac{1}{m}-n+k} s^{n-k}, & (0 \leq k < n) \\ 0, & (k > n) \end{cases},$$

Proof. Suppose $m \in \mathbf{N}$ and m th root of A is denoted by $A^{1/m} = (a_{nk}^{\frac{1}{m}})$. From Theorem 2, one may write

$$a_{nk}^{1/m} = \begin{cases} r^{1/m}, & (k = n) \\ \frac{\Gamma(1/m+1)}{(n-k)!\Gamma(\frac{1}{m}-n+k+1)} r^{\frac{1}{m}-n+k} s^{n-k}, & (0 \leq k < n) \\ 0, & (k > n) \end{cases},$$

However, for $(0 \leq k < n)$ one may easily calculate

$$\begin{aligned} a_{nk}^{\frac{1}{m}} &= \frac{\Gamma(1/m+1)}{(n-k)!\Gamma(\frac{1}{m}-n+k+1)} r^{\frac{1}{m}-n+k} s^{n-k} \\ &= \frac{\frac{1}{m}(\frac{1}{m}-1)(\frac{1}{m}-2)\dots(\frac{1}{m}-n+k+1)}{(n-k)!} r^{\frac{1}{m}-n+k} s^{n-k} \\ &= \frac{(1-m)(1-2m)\dots(1-m(n-k-1))}{(n-k)!m^{n-k}} r^{\frac{1}{m}-n+k} s^{n-k}. \end{aligned}$$

Since A is a triangle, for $k = n$ and $k > n$, $a_{nk}^{\frac{1}{m}}$ takes the values as $r^{1/m}$ and 0, respectively. $\square$

Corollary 3. The explicit formula for m th root of the double band matrix A , defined in Theorem 3 is given by

$$a_{nk}^{\frac{1}{m}} = \begin{cases} r^{1/m}, & (k = n) \\ \frac{(1-m)(1-2m)\dots(1-m(k-n-1))}{(k-n)!m^{k-n}} r^{\frac{1}{m}-k+n} s^{k-n}, & (k > n) \\ 0, & (0 \leq k < n) \end{cases},$$

Proof. This follows from Theorem 3. $\square$

Now, as applications of Corollaries 2 and 3, we illustrate following counter examples:

Example 1 : Consider a matrix A of order 5, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix}$$

Using Corollary 2, we calculate the square root of A as $a_{nk}^{\frac{1}{2}} = 1$ for all $k = n$ and $a_{21}^{\frac{1}{2}} = a_{32}^{\frac{1}{2}} = a_{43}^{\frac{1}{2}} = a_{54}^{\frac{1}{2}} = 3/2$,
 $a_{31}^{\frac{1}{2}} = a_{42}^{\frac{1}{2}} = a_{53}^{\frac{1}{2}} = -9/8$,
 $a_{41}^{\frac{1}{2}} = a_{52}^{\frac{1}{2}} = 27/16$,
 $a_{51}^{\frac{1}{2}} = -405/128$, Therefore, one can write

$$A^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3/2 & 1 & 0 & 0 & 0 \\ -9/8 & 3/2 & 1 & 0 & 0 \\ 27/16 & -9/8 & 3/2 & 1 & 0 \\ -405/128 & 27/16 & -9/8 & 3/2 & 1 \end{pmatrix}$$

Example 2 : Consider a matrix B of order 5, where

$$B = \begin{pmatrix} 8 & 5 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 & 0 \\ 0 & 0 & 8 & 5 & 0 \\ 0 & 0 & 0 & 8 & 5 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}$$

Using Corollary 3, we calculate the cube root of B as $b_{nk}^{\frac{1}{3}} = 2$ for all $k = n$ and

$$b_{12}^{\frac{1}{3}} = b_{23}^{\frac{1}{3}} = b_{34}^{\frac{1}{3}} = b_{45}^{\frac{1}{3}} = 5/12,$$

$$b_{13}^{\frac{1}{3}} = b_{24}^{\frac{1}{3}} = b_{35}^{\frac{1}{3}} = -28/288,$$

$$b_{14}^{\frac{1}{3}} = b_{25}^{\frac{1}{3}} = 625/20736,$$

$$b_{15}^{\frac{1}{3}} = -6250/497664,$$

Therefore, we write

$$B^{\frac{1}{3}} = \begin{pmatrix} 2 & 5/12 & -28/288 & 625/20736 & -6250/497664 \\ 0 & 2 & 5/12 & -28/288 & 625/20736 \\ 0 & 0 & 2 & 5/12 & -28/288 \\ 0 & 0 & 0 & 2 & 5/12 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Conclusion:

In this study, we derive an explicit formula for finding any arbitrary power of the double band matrix. Being an application of these results, we have also determine the n th root and inverse of the double band matrix of finite order.

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