Hermite-Hadamard type fractional integral inequalities for generalized beta (r, g) -preinvex functions

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Abstract

In the present paper, a new class of generalized beta (r, g) -preinvex functions is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving generalized beta (r, g) -preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for generalized beta (r, g) -preinvex functions via Riemann-Liouville fractional integrals are established. These results not only extend the results appeared in the literature (see [1], [2]), but also provide new estimates on these types.

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1. Introduction and Preliminaries

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbf{R} = (-\infty, +\infty)$ and I° to denote the interior of I . For any subset $K \subseteq \mathbf{R}^n$, K° is used to denote the interior of K . $\mathbf{R}^n$ is used to denote a n -dimensional vector space. The nonnegative real numbers are denoted by $\mathbf{R}_0 = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For some recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions please (see 9-17) and the references cited therein.

Fractional calculus (see [22]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.2. *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by $J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$, $x > a$*

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [21], [22]) and the references cited therein. Now, let us recall some definitions of various convex functions.

Definition 1.3. (see [4]) A nonnegative function $f : I \subseteq \mathbf{R} \longrightarrow \mathbf{R}_o$ is said to be P -function or P -convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \quad t \in [0, 1].$$

Definition 1.4. (see [5]) A function $f : \mathbf{R}_o \longrightarrow \mathbf{R}$ is said to be s -convex in the second sense, if

$$(1.2) \quad f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \in \mathbf{R}_o$, $\lambda \in [0, 1]$ and $s \in (0, 1]$.

It is clear that a 1-convex function must be convex on $\mathbf{R}_o$ as usual. The s -convex functions in the second sense have been investigated in (see [5]).

Definition 1.5. (see [6]) A set $K \subseteq \mathbf{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbf{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true. For more details please see (see [6], [7]) and the references therein.

Definition 1.6. (see [8]) The function f defined on the invex set $K \subseteq \mathbf{R}^n$ is said to be preinvex with respect η , if for every $x, y \in K$ and $t \in [0, 1]$, we have that $f(x + t\eta(y, x)) \leq (1-t)f(x) + tf(y)$.

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$(1.3) \quad \int_a^b (x-a)^p (b-x)^q f(x) dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^* |f|,$$

for certain $B_{m,k}$, γ_k and rest $R_m^* |f|$ (see [17]).

Recently, Liu (see [18]) obtained several integral inequalities for the left-hand side of (1.3) under the Definition 1.3 of P -function.

Also in (see [19]), Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of generalized beta (r, g) -preinvex function is introduced and some new integral inequalities for the left-hand side of (1.3) involving generalized beta (r, g) -preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type inequalities for generalized beta (r, g) -preinvex functions via fractional integrals are given. These general inequalities give us some new estimates for the left-hand side of Gauss-Jacobi type quadrature formula and Hermite-Hadamard type fractional integral inequalities.

2. New integral inequalities for generalized beta (r, g) -preinvex functions

Definition 2.1. (see [3]) A set $K \subseteq \mathbf{R}^n$ is said to be m -invex with respect to the mapping $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m -invex set degenerates an invex set on K .

Definition 2.3. (see [20]) A positive function f on the invex set K is said to be logarithmically preinvex, if

$$f(u + t\eta(v, u)) \leq f^{1-t}(u) f^t(v)$$

for all $u, v \in K$ and $t \in [0, 1]$.

Definition 2.4. (see [20]) The function f on the invex set K is said to be r -preinvex with respect to η , if

$$f(u + t\eta(v, u)) \leq M_r(f(u), f(v); t)$$

holds for all $u, v \in K$ and $t \in [0, 1]$, where

$$M_r(x, y; t) = \begin{cases} \left[(1-t)x^r + ty^r \right]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ x^{1-t}y^t, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers x and y .

We next give new definition, to be referred as generalized beta (r, g) -preinvex function.

Definition 2.5. Let $K \subseteq R$ be an open m -invex set with respect to $\eta : K \times K \times (0, 1] \longrightarrow R$, $g : [0, 1] \longrightarrow [0, 1]$ be a differentiable function and

$\varphi : I \longrightarrow K$ is a continuous function. The function $f : K \longrightarrow (0, \infty)$ is said to be generalized beta (r, g) -preinvex with respect to η , if

$$f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) \leq M_r(f(\varphi(x)), f(\varphi(y)), m, p, q; g(t)) \quad (2.1)$$

holds for some fixed $m \in (0, 1]$, for any fixed $p, q > -1$ and for all $x, y \in I, t \in [0, 1]$, where

$$M_r(f(\varphi(x)), f(\varphi(y)), m, p, q; g(t)) = \begin{cases} \left[mg^p(t)(1-g(t))^q f^r(\varphi(x)) + g^q(t)(1-g(t))^p f^r(\varphi(y)) \right]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ f(\varphi(x))^{mg^p(t)(1-g(t))^q} f(\varphi(y))^{g^q(t)(1-g(t))^p}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.

Remark 2.6. In Definition 2.5, it is worthwhile to note that the class of generalized beta (r, g) -preinvex function is a generalization of the class of s -convex in the second sense function given in Definition 1.4. Also, for $r = 1, p = 0, q = s, g(t) = t, \forall t \in [0, 1]$ and $\varphi(x) = x, \forall x \in I$, we get the notion of generalized (s, m) -preinvex function (see [3]).

In this section, in order to prove our main results regarding some new integral inequalities involving generalized beta (r, g) -preinvex functions, we need the following new Lemma:

Lemma 2.7. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbf{R}$ is a continuous function on K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$, for $\eta(\varphi(b), \varphi(a), m) > 0$. Then for some fixed $m \in (0, 1]$ and $p, q > 0$, we have

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a) + \eta(\varphi(b), \varphi(a), m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\ &= \eta(\varphi(b), \varphi(a), m)^{p+q+1} \int_0^1 g^p(t)(1-g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)]. \end{aligned}$$

Proof. It is easy to observe that

$$\begin{aligned}
& \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\
&= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p \\
&\quad \times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - g(t)\eta(\varphi(b), \varphi(a), m))^q \\
&\quad \times f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
&= \eta(\varphi(b), \varphi(a), m)^{p+q+1} \int_0^1 g^p(t)(1-g(t))^q f(m\varphi(a)+g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)].
\end{aligned}$$

□

Theorem 2.8. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, \infty)$ is a continuous function on K° with $\eta(\varphi(b), \varphi(a), m) > 0$. Let $k > 1$ and $0 < r \leq 1$. If $f^{\frac{k}{k-1}}$ is a generalized beta (r, g) -preinvex function on an open m -invex set K with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$ where $s, \gamma > -1$, then for any fixed $p, q > 0$, we have

$$\begin{aligned}
& \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x) dx \\
&\leq \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^{\frac{1}{k}}(g(t); k, p, q) \\
&\quad \times \left[m f^{\frac{rk}{k-1}}(\varphi(a)) B^r \left(g(t); \frac{1}{r}, \gamma, s \right) + f^{\frac{rk}{k-1}}(\varphi(b)) B^r \left(g(t); \frac{1}{r}, s, \gamma \right) \right]^{\frac{k-1}{rk}},
\end{aligned}
\tag{2.2}$$

where $B(g(t); k, p, q) = \int_0^1 g^{kp}(t)(1-g(t))^{kq} d[g(t)]$.

Proof. Let $k > 1$ and $0 < r \leq 1$. Since $f^{\frac{k}{k-1}}$ is a generalized beta (r, g) -preinvex function on K , combining with Lemma 2.7, Hölder inequality and Minkowski inequality for all $t \in [0, 1]$ and for some fixed $m \in (0, 1]$, we get

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x) dx \\ & \leq \eta(\varphi(b),\varphi(a),m)^{p+q+1} \left[\int_0^1 g^{kp}(t)(1-g(t))^{kq} d[g(t)] \right]^{\frac{1}{k}} \\ & \quad \times \left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a)+g(t)\eta(\varphi(b),\varphi(a),m)) d[g(t)] \right]^{\frac{k-1}{k}} \\ & \leq \eta(\varphi(b),\varphi(a),m)^{p+q+1} B^{\frac{1}{k}}(g(t); k, p, q) \\ & \quad \times \left[\int_0^1 \left(mg^\gamma(t)(1-g(t))^s f^r(\varphi(a))^{\frac{k}{k-1}} + g^s(t)(1-g(t))^\gamma f^r(\varphi(b))^{\frac{k}{k-1}} \right)^{\frac{1}{r}} d[g(t)] \right]^{\frac{k-1}{k}} \\ & \leq \eta(\varphi(b),\varphi(a),m)^{p+q+1} B^{\frac{1}{k}}(g(t); k, p, q) \\ & \quad \times \left[\left(\int_0^1 m^{\frac{1}{r}} g^{\frac{\gamma}{r}}(t)(1-g(t))^{\frac{s}{r}} f^{\frac{k}{k-1}}(\varphi(a)) d[g(t)] \right)^r \right. \\ & \quad \left. + \left(\int_0^1 g^{\frac{s}{r}}(t)(1-g(t))^{\frac{\gamma}{r}} f^{\frac{k}{k-1}}(\varphi(b)) d[g(t)] \right)^r \right]^{\frac{k-1}{rk}} \\ & = \eta(\varphi(b),\varphi(a),m)^{p+q+1} B^{\frac{1}{k}}(g(t); k, p, q) \\ & \quad \times \left[m f^{\frac{rk}{k-1}}(\varphi(a)) B^r \left(g(t); \frac{1}{r}, \gamma, s \right) + f^{\frac{rk}{k-1}}(\varphi(b)) B^r \left(g(t); \frac{1}{r}, s, \gamma \right) \right]^{\frac{k-1}{rk}}. \quad \square \end{aligned}$$

Corollary 2.9. Under the same conditions as in Theorem 2.8 for $\gamma = 0$, $r = 1$ and $g(t) = t$, we get (see [1], Theorem 2.2).

Theorem 2.10. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ is a differentiable function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, \infty)$ is a continuous function on K° with $\eta(\varphi(b), \varphi(a), m) > 0$. Let $l \geq 1$ and $0 < r \leq 1$. If f^l is a generalized beta (r, g) -preinvex function on an open m -invex set K with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$ where $s, \gamma > -1$, then for any fixed $p, q > 0$, we have

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x) dx$$

$$\leq \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^{\frac{l-1}{l}}(g(t); 1, p, q)$$

$$\times \left[m f^{rl}(\varphi(a)) B^r \left(g(t); \frac{1}{r}, pr + \gamma, qr + s \right) + f^{rl}(\varphi(b)) B^r \left(g(t); \frac{1}{r}, pr + s, qr + \gamma \right) \right]^{\frac{1}{rl}}.$$

(2.3)

Proof. Let $l \geq 1$ and $0 < r \leq 1$. Since f^l is a generalized beta (r, g) -preinvex function on K , combining with Lemma 2.7, the well-known power mean inequality and Minkowski inequality for all $t \in [0, 1]$ and for some fixed $m \in (0, 1]$, we get

$$\begin{aligned} & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x) dx \\ &= \eta(\varphi(b), \varphi(a), m)^{p+q+1} \\ & \times \int_0^1 \left[g^p(t)(1-g(t))^q \right]^{\frac{l-1}{l}} \left[g^p(t)(1-g(t))^q \right]^{\frac{1}{l}} f(m\varphi(a)+g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\ & \leq \eta(\varphi(b), \varphi(a), m)^{p+q+1} \left[\int_0^1 g^p(t)(1-g(t))^q d[g(t)] \right]^{\frac{l-1}{l}} \\ & \times \left[\int_0^1 g^p(t)(1-g(t))^q f^l(m\varphi(a)+g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \right]^{\frac{1}{l}} \\ & \leq \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^{\frac{l-1}{l}}(g(t); 1, p, q) \\ & \times \left[\int_0^1 g^p(t)(1-g(t))^q \left(m g^\gamma(t)(1-g(t))^s f^r(\varphi(a))^l + g^s(t)(1-g(t))^\gamma f^r(\varphi(b))^l \right)^{\frac{1}{r}} d[g(t)] \right]^{\frac{1}{l}} \\ & \leq \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^{\frac{l-1}{l}}(g(t); 1, p, q) \\ & \times \left[\left(\int_0^1 m^{\frac{1}{r}} g^{p+\frac{\gamma}{r}}(t)(1-g(t))^{q+\frac{s}{r}} f^l(\varphi(a)) d[g(t)] \right)^r \right. \end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 g^{p+\frac{s}{r}}(t)(1-g(t))^{q+\frac{\gamma}{r}} f^l(\varphi(b)) d[g(t)] \right)^r \Bigg]^{\frac{1}{r}} \\
& = \eta(\varphi(b), \varphi(a), m)^{p+q+1} B^{\frac{l-1}{r}}(g(t); 1, p, q) \\
& \times \left[m f^{rl}(\varphi(a)) B^r \left(g(t); \frac{1}{r}, pr + \gamma, qr + s \right) \right. \\
& \left. + f^{rl}(\varphi(b)) B^r \left(g(t); \frac{1}{r}, pr + s, qr + \gamma \right) \right]^{\frac{1}{r}}. \quad \square
\end{aligned}$$

Corollary 2.11. Under the same conditions as in Theorem 2.10 for $\gamma = 0$, $r = 1$ and $g(t) = t$, we get (see [1], Theorem 2.3).

3. Hermite-Hadamard type fractional integral inequalities for generalized beta (r, g) -preinvex functions

In this section, we prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for generalized beta (r, g) -preinvex functions via fractional integrals.

Theorem 3.1. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ is a differentiable function. Suppose $K \subseteq \mathbf{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$ with $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, \infty)$ be a generalized beta (r, g) -preinvex function on an open m -invex set K° . Then for $\alpha > 0$, $p, q > -1$ and $0 < r \leq 1$, we have

$$\begin{aligned}
& \frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t) dt \\
& \leq \left[m f^r(\varphi(a)) B^r \left(g(t); \frac{1}{r}, p + r(\alpha - 1), q \right) + f^r(\varphi(b)) B^r \left(g(t); \frac{1}{r}, q + r(\alpha - 1), p \right) \right]^{\frac{1}{r}}.
\end{aligned}
\tag{3.1}$$

Proof. Let $0 < r \leq 1$. Since f is a generalized beta (r, g) -preinvex function on an open m -invex set K° , combining with Minkowski inequality for all $t \in [0, 1]$ and for some fixed $m \in (0, 1]$, we get

$$\begin{aligned} & \frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t) dt \\ &= \int_0^1 g^{\alpha-1}(t) f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\ &\leq \int_0^1 g^{\alpha-1}(t) \left[mg^p(t)(1-g(t))^q f^r(\varphi(a)) + g^q(t)(1-g(t))^p f^r(\varphi(b)) \right]^{\frac{1}{r}} d[g(t)] \\ &\leq \left\{ \left[\int_0^1 m^{\frac{1}{r}} g^{\alpha-1+\frac{p}{r}}(t)(1-g(t))^{\frac{q}{r}} f(\varphi(a)) d[g(t)] \right]^r \right. \\ &\quad \left. + \left[\int_0^1 g^{\alpha-1+\frac{q}{r}}(t)(1-g(t))^{\frac{p}{r}} f(\varphi(b)) d[g(t)] \right]^r \right\}^{\frac{1}{r}} \\ &= \left[m f^r(\varphi(a)) B^r \left(g(t); \frac{1}{r}, p + r(\alpha - 1), q \right) + f^r(\varphi(b)) B^r \left(g(t); \frac{1}{r}, q + r(\alpha - 1), p \right) \right]^{\frac{1}{r}}. \end{aligned}$$

□

Corollary 3.2. Under the same conditions as in Theorem 3.1 for $p = 0, m = q = 1, \varphi(x) = x, \eta(\varphi(b), \varphi(a), m) = \eta(b, a)$ and $g(t) = t$, we get (see [2], Theorem 3.1).

Theorem 3.3. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ is a differentiable function. Suppose $K \subseteq \mathbf{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$ with $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, \infty)$ are respectively generalized beta (r, g) -preinvex function and generalized beta (l, g) -preinvex function on an open m -invex set K° . Then for $\alpha > 0, p, q > -1, r > 1$ and $r^{-1} + l^{-1} = 1$, we have

$$\begin{aligned} & \frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t) h(t) dt \\ (3.2) \quad & \leq \frac{1}{2} \left\{ \left[m f^r(\varphi(a)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2(\alpha - 1 + p), 2q \right) \right. \right. \end{aligned}$$

$$\begin{aligned}
& + f^r(\varphi(b)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2(\alpha - 1 + q), 2p \right) \Bigg]^{\frac{2}{r}} \\
& + \left[m h^l(\varphi(a)) B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2(\alpha - 1 + p), 2q \right) \right. \\
& \left. + h^l(\varphi(b)) B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2(\alpha - 1 + q), 2p \right) \right]^{\frac{2}{l}} \Bigg\}.
\end{aligned}$$

Proof. Let $r > 1$ and $r^{-1} + l^{-1} = 1$. Since f and h are respectively generalized beta (r, g) -preinvex function and generalized beta (l, g) -preinvex function on an open m -invex set K° , combining with Cauchy and Minkowski inequalities for all $t \in [0, 1]$ and for some fixed $m \in (0, 1]$, we get

$$\begin{aligned}
& \frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t) h(t) dt \\
& = \int_0^1 g^{(\alpha-1)(\frac{1}{r}+\frac{1}{l})}(t) f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) \\
& \quad \times h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
& \leq \int_0^1 g^{(\alpha-1)(\frac{1}{r}+\frac{1}{l})}(t) \left[m g^p(t) (1-g(t))^q f^r(\varphi(a)) + g^q(t) (1-g(t))^p f^r(\varphi(b)) \right]^{\frac{1}{r}} \\
& \quad \times \left[m g^p(t) (1-g(t))^q h^l(\varphi(a)) + g^q(t) (1-g(t))^p h^l(\varphi(b)) \right]^{\frac{1}{l}} d[g(t)] \\
& \leq \frac{1}{2} \left\{ \int_0^1 \left[m g^{\alpha-1+p}(t) (1-g(t))^q f^r(\varphi(a)) + g^{\alpha-1+q}(t) (1-g(t))^p f^r(\varphi(b)) \right]^{\frac{2}{r}} d[g(t)] \right. \\
& \quad \left. + \int_0^1 \left[m g^{\alpha-1+p}(t) (1-g(t))^q h^l(\varphi(a)) + g^{\alpha-1+q}(t) (1-g(t))^p h^l(\varphi(b)) \right]^{\frac{2}{l}} d[g(t)] \right\} \\
& \leq \frac{1}{2} \left[\left\{ \left(\int_0^1 g^{\frac{2(\alpha-1+q)}{r}}(t) (1-g(t))^{\frac{2p}{r}} f^2(\varphi(b)) d[g(t)] \right)^{\frac{r}{2}} \right. \right. \\
& \quad \left. \left. + \left(\int_0^1 m^{\frac{2}{r}} g^{\frac{2(\alpha-1+p)}{r}}(t) (1-g(t))^{\frac{2q}{r}} f^2(\varphi(a)) d[g(t)] \right)^{\frac{r}{2}} \right\}^{\frac{2}{r}} \right. \\
& \quad \left. + \left\{ \left(\int_0^1 g^{\frac{2(\alpha-1+q)}{l}}(t) (1-g(t))^{\frac{2p}{l}} h^2(\varphi(b)) d[g(t)] \right)^{\frac{l}{2}} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 m^{\frac{2}{t}} g^{\frac{2(\alpha-1+p)}{t}}(t) (1-g(t))^{\frac{2q}{t}} h^2(\varphi(a)) d[g(t)] \right)^{\frac{1}{2}} \left. \right\}^{\frac{2}{t}} \Bigg] \\
& = \frac{1}{2} \left\{ \left[m f^r(\varphi(a)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2(\alpha-1+p), 2q \right) \right. \right. \\
& \quad \left. \left. + f^r(\varphi(b)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2(\alpha-1+q), 2p \right) \right]^{\frac{2}{r}} \right. \\
& \quad \left. + \left[m h^l(\varphi(a)) B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2(\alpha-1+p), 2q \right) \right. \right. \\
& \quad \left. \left. + h^l(\varphi(b)) B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2(\alpha-1+q), 2p \right) \right]^{\frac{2}{l}} \right\}. \quad \square
\end{aligned}$$

Corollary 3.4. Under the same conditions as in Theorem 3.3 for $p = 0$, $m = q = 1$, $\varphi(x) = x$, $\eta(\varphi(b), \varphi(a), m) = \eta(b, a)$ and $g(t) = t$, we get (see [2], Theorem 3.3).

Theorem 3.5. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ is a differentiable function. Suppose $K \subseteq \mathbf{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$ with $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, \infty)$ are respectively generalized beta (r, g) -preinvex function and generalized beta (l, g) -preinvex function on an open m -invex set K° . Then for $\alpha > 0$, $p, q > -1$, $r > 1$ and $r^{-1} + l^{-1} = 1$, we have

$$\begin{aligned}
& \frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t) h(t) dt \\
& \leq \left\{ m f^r(\varphi(a)) B(g(t); 1, \alpha-1+p, q) + f^r(\varphi(b)) B(g(t); 1, \alpha-1+q, p) \right\}^{\frac{1}{r}} \\
& (3.3) \\
& + \left\{ m h^l(\varphi(a)) B(g(t); 1, \alpha-1+p, q) + h^l(\varphi(b)) B(g(t); 1, \alpha-1+q, p) \right\}^{\frac{1}{l}}.
\end{aligned}$$

Proof. Let $r > 1$ and $r^{-1} + l^{-1} = 1$. Since f and h are respectively generalized beta (r, g) -preinvex function and generalized beta (l, g) -preinvex function on an open m -invex set K° , combining with Hölder inequality for all $t \in [0, 1]$ and for some fixed $m \in (0, 1]$, we get

$$\begin{aligned} & \frac{1}{\eta^\alpha(\varphi(b), \varphi(a), m)} \int_{m\varphi(a)+g(0)\eta(\varphi(b), \varphi(a), m)}^{m\varphi(a)+g(1)\eta(\varphi(b), \varphi(a), m)} (t - m\varphi(a))^{\alpha-1} f(t) h(t) dt \\ &= \int_0^1 g^{(\alpha-1)(\frac{1}{r}+\frac{1}{l})}(t) f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) \\ & \quad \times h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\ &\leq \left\{ \int_0^1 \left[mg^{\alpha-1+p}(t)(1-g(t))^q f^r(\varphi(a)) + g^{\alpha-1+q}(t)(1-g(t))^p f^r(\varphi(b)) \right]^{\frac{1}{r}} \right. \\ & \quad \times \left. \left[mg^{\alpha-1+p}(t)(1-g(t))^q h^l(\varphi(a)) + g^{\alpha-1+q}(t)(1-g(t))^p h^l(\varphi(b)) \right]^{\frac{1}{l}} d[g(t)] \right\}^{\frac{1}{r}} \\ &\leq \left\{ \int_0^1 \left[mg^{\alpha-1+p}(t)(1-g(t))^q f^r(\varphi(a)) + g^{\alpha-1+q}(t)(1-g(t))^p f^r(\varphi(b)) \right] d[g(t)] \right\}^{\frac{1}{r}} \\ & \quad + \left\{ \int_0^1 \left[mg^{\alpha-1+p}(t)(1-g(t))^q h^l(\varphi(a)) + g^{\alpha-1+q}(t)(1-g(t))^p h^l(\varphi(b)) \right] d[g(t)] \right\}^{\frac{1}{l}} \\ &= \left\{ m f^r(\varphi(a)) B(g(t); 1, \alpha - 1 + p, q) + f^r(\varphi(b)) B(g(t); 1, \alpha - 1 + q, p) \right\}^{\frac{1}{r}} \\ & \quad + \left\{ m h^l(\varphi(a)) B(g(t); 1, \alpha - 1 + p, q) + h^l(\varphi(b)) B(g(t); 1, \alpha - 1 + q, p) \right\}^{\frac{1}{l}}. \end{aligned}$$

□

Corollary 3.6. Under the same conditions as in Theorem 3.5 for $p = 0$, $m = q = 1$, $\varphi(x) = x$, $\eta(\varphi(b), \varphi(a), m) = \eta(b, a)$ and $g(t) = t$, we get (see [2], Theorem 3.9).

Remark 3.7. For different choices of positive values $r, l = \frac{1}{2}, \frac{1}{3}, 2$, etc., for some fixed $m \in (0, 1]$, for any fixed $p, q > -1$, for a particular choices of a differentiable function $g(t) = e^{-t}, \ln(t+1), \sin(\frac{\pi t}{2}), \cos(\frac{\pi t}{2})$, etc., and a

particular choices of a continuous function $\varphi(x) = e^x$ for all $x \in \mathbf{R}$, x^n for all $x > 0$ and for all $n \in \mathbf{N}$, etc., by Theorem 3.1, Theorem 3.3 and Theorem 3.5 we can get some special kinds of Hermite-Hadamard type fractional integral inequalities.

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