Existence of solutions for a nonlinear fractional system with nonlocal boundary conditions

A. Guezane-Lakoud

University Badji Mokhtar Annaba, Algeria

G. Rebiai

University of Guelma, Algeria and

R. Khaldi

University Badji Mokhtar Annaba, Algeria

Received : March 2017. Accepted : April 2017

Abstract

In this paper, we use fixed point theorems to prove the existence and uniqueness of solution for a nonlinear fractional system with boundary conditions. At the end we present two examples illustrating the obtained results.

Keywords: *Fractional Rieman-Liouville derivative, Fractional differential equation, Fixed Point Theorem.*

Mathematics Subject Classification: *34B10, 26A33, 34B15.*

1. Introduction

In recent years, the theory of differential fractional equations has become an interesting field to explore. It is to be noted that such theory has many applications in several events existing in the real world, and also in many sciences such as: engineering, physics, chemistry, biology, etc ..., [13]. Moreover, the study of the systems of fractional differential equations has become more and more popular tool for controlling and modeling different systems [2,7,15-17]. Thus the fixed point theory is a powerful mathematical tool in the study of the existence, uniqueness, positivity and stability of solutions, see [1,3-6,9-14].

In this work, we consider the following system of fractional differential equations with boundary conditions:

$$(FS) \begin{cases} -D_{0+}^{\alpha} u(t) = g(t) f(u(t)), 0 < t < 1, \\ u(0) = u'(0) = 0, au(1) + bu'(1) = 0, \end{cases}$$

where D_{0+}^{α} denotes the Reimann-Liouville fractional derivative, $2 < \alpha < 3$, $u = (u_1, u_2, \dots, u_n)^T$ is an unknown function with

$u_i : [0, 1] \rightarrow \mathbf{R}, g : [0, 1] \rightarrow \mathbf{R}$ is a given function, $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$, $f(u) = (f_1(u_1, u_2, \dots, u_n), \dots, f_n(u_1, u_2, \dots, u_n))^T$, $f_i : \mathbf{R}^n \rightarrow \mathbf{R}$.

This paper is organized as follows: in Section 2, some preliminary materials to be used later are stated. In Section 3, we present and prove our main results consisting of the existence and uniqueness of the solution of (FS). Finally our study is ended by an example illustrating the obtained results.

2. Preliminaries

In this section, we recall the basic definitions and lemmas from the fractional calculus theory, see [13].

Definition 1. The Riemann-Liouville fractional integrals of order α of a function h is defined as

$$I_{a+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{h(s)}{(t-s)^{1-\alpha}} ds.$$

Definition 2. The Riemann-Liouville derivative of fractional order $\alpha > 0$ for a function h is defined as

$$D_{0+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} h(s) ds = \left(\frac{d}{dt} \right)^n I^{n-\alpha} h(t),$$

where $n = [\alpha] + 1$ ($[\alpha]$ denotes the integer part of the real number α).

Lemma 3. For $\alpha > 0$, the general solution of the homogeneous equation

$$D_{0+}^{\alpha} u(t) = 0,$$

is given by

$$u(t) = c_0 t^{\alpha-n} + c_1 t^{\alpha-n-1} + \dots + c_{n-2} t^{\alpha-2} + c_{n-1} t^{\alpha-1},$$

where $c_i, i = 1, 2, \dots, n-1$, are arbitrary real constants.

Lemma 4. Let $p, q \geq 0, f \in L_1[a, b]$. Then

$$I_{0+}^p I_{0+}^q f(t) = I_{0+}^{p+q} f(t) = I_{0+}^q I_{0+}^p f(t).$$

3. Main results

Lemma 5. Let $y \in C([0, 1], \mathbf{R})$. Assume that $a, b \in \mathbf{R}$ such that $a - b(\alpha - 1) \neq 0$, then for $i \in \{1, \dots, n\}$, the linear nonhomogeneous problem

$$(S_i) = \begin{cases} -D_{0+}^{\alpha} u_i(t) = y(t), & 0 < t < 1, \\ u_i(0) = u'_i(0) = 0, & au_i(1) - bu'_i(1) = 0, \quad i \in \{1, \dots, n\}, \end{cases}$$

(3.1)

has the following solution

$$(3.2) \quad u_i(t) = \int_0^1 G_i(t, s) y(s) ds, \quad i \in \{1, \dots, n\}$$

where

$$G_i(t, s) = \begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}}{a-b(\alpha-1)} \left(\frac{a}{\Gamma(\alpha)} (1-s)^{\alpha-1} - \frac{b}{\Gamma(\alpha-1)} (1-s)^{\alpha-2} \right), & s \leq t, \\ \frac{t^{\alpha-1}}{a-b(\alpha-1)} \left(\frac{a}{\Gamma(\alpha)} (1-s)^{\alpha-1} - \frac{b}{\Gamma(\alpha-1)} (1-s)^{\alpha-2} \right), & s \geq t. \end{cases}$$

(3.3)

Proof. Let u_i be a solution of the fractional boundary value problem (FS). Using Lemma 3, we obtain

$$(3.4) \quad u_i(t) = -I_{0+}^{\alpha} y(t) + At^{\alpha-1} + Bt^{\alpha-2} + Ct^{\alpha-3},$$

then, by multiplying (3.4) by $t^{3-\alpha}$, it yields

$$t^{3-\alpha} u_i(t) = -I_{0+}^{\alpha} y(t) t^{\alpha-3} + At^2 + Bt + C.$$

According to the condition $u(0) = 0$, we obtain $C = 0$. Therefore, differentiating (3.4), we have

$$(3.5) \quad u_i'(t) = -I_{0+}^{\alpha-1} y(t) + (\alpha-1) At^{\alpha-2} + (\alpha-2) B.$$

Multiplying (3.5) by $t^{3-\alpha}$, we obtain

$$t^{3-\alpha} u_i'(t) = -I_{0+}^{\alpha-1} y(t) t^{3-\alpha} + (\alpha-1) At + (\alpha-2) B.$$

(3.6)

From condition $u_i'(0) = 0$, it follows $B = 0$, thus,

$$(3.7) \quad u_i(t) = -I_{0+}^{\alpha} y(t) + At^{\alpha-1}.$$

Since $au_i(1) - bu_i'(1) = 0$, then

$$(3.8) \quad A = \frac{a}{a-b(\alpha-1)} I_{0+}^{\alpha} y(1) - \frac{b}{a-b(\alpha-1)} I_{0+}^{\alpha-1} y(1).$$

By substituting A in (3.7), we get

$$u_i(t) = \int_0^1 G_i(t, s) y(s) ds. \quad \square$$

Lemma 6. If $a > 0$ and $b < 0$, then the functions G_i are nonnegative, continuous and

$$(3.9) \quad G_i(t, s) \leq \frac{1}{\Gamma(\alpha-1)}, \forall s, t \in [0, 1], i \in \{1, \dots, n\}.$$

Proof. The proof is direct, we omit it. $\square$

Let X be the Banach space of all functions $u \in C^n[0, 1] = C[0, 1] \times \dots \times C[0, 1]$ with the norm $\| \cdot \|$ defined by $\|u\| = \sum_{i=1}^n \max_{t \in [0, 1]} |u_i(t)|$. Define the integral operator $T : X \rightarrow X$ by $T(u) = (T_1 u, T_2 u, \dots, T_n u)$, where

$$(3.10) \quad (T_i u)(t) = \int_0^1 G_i(t, s) g(s) f_i(u(s)) ds, i = 1, \dots, n.$$

Lemma 7. The function $u \in X$ is a solution of the system (FS) if and only if $T_i u(t) = u(t)$, for all $t \in [0, 1]$, $\forall i \in \{1, \dots, n\}$.

The first main statement in this work is the uniqueness of solution of the boundary problem (FS) .

Theorem 8. Assume that

- i) $f_i \in C(\mathbf{R}^n, \mathbf{R})$, $g \in L^1([0, 1], \mathbf{R})$
- ii) There exists a constant $L > 0$ such that

$$(3.11) \quad |f_i(x_1, \dots, x_n) - f_i(y_1, \dots, y_n)| \leq L \sum_{i=1}^n |x_i - y_i|$$

and

$$(3.12) \quad \frac{nL \|g\|_{L^1[0,1]}}{\Gamma(\alpha - 1)} < 1,$$

for all $t \in [0, 1]$ and for all $x_i, y_i \in \mathbf{R}$, $i = 1, \dots, n$. Then, the boundary value problem (FS) has a unique solution in X .

Proof. We will use the Banach contraction principle to prove that the operator T has a fixed point. Using the properties of the function G_i , it yields

$$\begin{aligned} |T_i x(t) - T_i y(t)| &\leq \int_0^1 |G_i(t, s)| |g(s)| |f_i(x(s)) - f_i(y(s))| ds \\ &\leq \frac{L}{\Gamma(\alpha - 1)} \int_0^1 |g(s)| \sum_{i=1}^n |x_i(s) - y_i(s)| ds \\ &\leq \frac{L}{\Gamma(\alpha - 1)} \|g\|_{L^1[0,1]} \|x - y\|, \end{aligned}$$

then by taking the maximum over $t \in [0, 1]$, it follows

$$(3.13) \quad t \in [0, 1] \max |T_i x(t) - T_i y(t)| \leq \frac{L}{\Gamma(\alpha - 1)} \|g\|_{L^1[0,1]} \|x - y\|.$$

Summing the n inequalities in (3.13), it yields

$$\|Tx - Ty\| \leq \frac{nL \|g\|_{L^1[0,1]}}{\Gamma(\alpha - 1)} \|x - y\|.$$

Since $\frac{nL \|g\|_{L^1[0,1]}}{\Gamma(\alpha - 1)} < 1$, then T is a contraction. As a consequence of Banach fixed-point theorem, we deduce that T has a fixed point that is the unique solution of the (FS), this achieves the proof. $\square$

The second main statement of this work is an existence result for the boundary problem (FS).

Theorem 9. Assume that $f_i(0) \neq 0, i \in \{1, \dots, n\}$, there exist $\eta > 0$ and a nonnegative function $\Psi \in C(\mathbf{R}^n, (0, \infty))$ satisfying $\Psi(x_1, \dots, x_n) \leq \Psi(y_1, \dots, y_n)$ for $0 \leq x_i \leq y_i, i = 1, \dots, n$. If

$$(3.14) \quad |f_i(u)| \leq \Psi(|u|),$$

for all $t \in [0, 1]$ and all $u \in \mathbf{R}^n$ and

$$(3.15) \quad \frac{n}{\Gamma(\alpha - 1)} \Psi(\eta, \dots, \eta) \|g\|_{L^1[0,1]} \leq \eta,$$

then, the problem (FS) has at least one nontrivial solution $u^* \in X$.

For the proof of Theorem we need the nonlinear alternative of Leray-Schauder:

Lemma 10. Let F be a Banach space and Ω a bounded open subset of F , $0 \in \Omega$. Let $T : \Omega \rightarrow F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x \in \Omega$ of T .

Proof. of Theorem 9. The continuity of the operator T follows from the continuity of f . Set $B_\eta = \{u \in X : \|u\| \leq \eta\}$. Let us prove that $T : B_\eta \rightarrow X$ is a completely continuous operator. From (3.14), we have for each $t \in [0, 1]$

$$(3.16) \quad |T_i u(t)| \leq \int_0^1 |G_i(t, s)| |g(s)| |f_i(u(s))| ds$$

$$(3.17) \quad \leq \frac{1}{\Gamma(\alpha - 1)} \int_0^1 |g(s)| \Psi(|u(s)|) ds$$

$$(3.18) \quad = \frac{1}{\Gamma(\alpha-1)} \int_0^1 |g(s)| \Psi(|u_1(s)|, \dots, |u_2(s)|) ds$$

$$(3.19) \quad \leq \frac{1}{\Gamma(\alpha-1)} \Psi(\eta, \dots, \eta) \|g\|_{L^1[0,1]}$$

Taking the supremum over $[0, 1]$, then summing the obtained inequalities according to i from 1 to n , we get

$$(3.20) \quad \|Tu\| \leq \frac{n\Psi(\eta, \dots, \eta) \|g\|_{L^1[0,1]}}{\Gamma(\alpha-1)}$$

which implies that $T(B_\eta)$ is uniformly bounded.

Let us show that (Tu) is equicontinuous, $u \in B_\eta$. Let $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, then

$$\begin{aligned} |T_i u(t_1) - T_i u(t_2)| &\leq \int_0^1 |G_i(t_1, s) - G_i(t_2, s)| |g(s)| |f_i(u(s))| ds \\ &\leq \int_0^{t_1} |G_i(t_1, s) - G_i(t_2, s)| |g(s)| |f_i(u(s))| ds \\ &\quad + \int_{t_1}^{t_2} |G_i(t_1, s) - G_i(t_2, s)| |g(s)| |f_i(u(s))| ds \\ &\quad + \int_{t_2}^1 |G_i(t_1, s) - G_i(t_2, s)| |g(s)| |f_i(u(s))| ds \end{aligned}$$

then

$$|T_i u(t_1) - T_i u(t_2)| \leq$$

$$\begin{aligned} &\frac{\Psi(\eta, \dots, \eta)}{\Gamma(\alpha)} \left[\int_0^{t_1} \left[(t_2^{\alpha-1} - t_1^{\alpha-1}) + (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] |g(s)| ds \right. \\ &+ \int_{t_1}^{t_2} \left[(t_2^{\alpha-1} - t_1^{\alpha-1}) + (t_2 - s)^{\alpha-1} \right] |g(s)| ds \\ &\left. + \int_{t_2}^1 \left[(t_2^{\alpha-1} - t_1^{\alpha-1}) \right] |g(s)| ds \right] \end{aligned}$$

$$\begin{aligned} &\leq \frac{\Psi(\eta, \dots, \eta)}{\Gamma(\alpha)} \left[\left[(t_2^{\alpha-1} - t_1^{\alpha-1}) + (t_2 - t_1)^{\alpha-1} \right] \int_0^{t_1} |g(s)| ds \right. \\ &\quad \left. + \left[(t_2^{\alpha-1} - t_1^{\alpha-1}) + (t_2 - t_1)^{\alpha-1} \right] \int_{t_1}^{t_2} |g(s)| ds \right] \end{aligned}$$

$$\begin{aligned}
& + \left[\left(t_2^{\alpha-1} - t_1^{\alpha-1} \right) \right] \int_{t_2}^1 |g(s)| ds \\
& \leq \frac{\Psi(\eta, \dots, \eta)}{\Gamma(\alpha)} \left[3 \left(t_2^{\alpha-1} - t_1^{\alpha-1} \right) + 2(t_2 - t_1)^{\alpha-1} \right] \int_0^1 |g(s)| ds.
\end{aligned}$$

As $t_1 \rightarrow t_2$, the right-hand side of the above inequality tends to zero. By Ascoli-Arzelà theorem, we conclude that the operator $T : X \rightarrow X$ is completely continuous.

Now we apply the nonlinear alternative of Leray-Schauder. Let $u \in \partial B_\eta$, such that $u = \lambda T u$ for some $0 < \lambda < 1$. We have

$$\begin{aligned}
u_i(t) &= \lambda T_i u(t) \leq t \in [0, 1] \max |T_i u(t)| \\
&\leq \frac{1}{\Gamma(\alpha - 1)} \Psi(\eta, \dots, \eta) \|g\|_{L^1[0,1]}.
\end{aligned}$$

Taking the supremum over $[0, 1]$, then summing the obtained inequalities according to i from 1 to n , we get

$$\|u\| \leq \frac{n}{\Gamma(\alpha - 1)} \Psi(\eta, \dots, \eta) \|g\|_{L^1[0,1]}.$$

taking into account (3.15) we conclude

$$(3.21) \quad \|u\| < \eta,$$

that contradicts the fact that $u \in \partial B_\eta$. So, we conclude that T has at least one fixed point $u^* \in B_\eta$ and then the (FS) has a nontrivial solution $u^* \in B_\eta$.

□

4. Examples

In this section, we give examples to illustrate the usefulness of our main results.

Example 1. Consider the following two-dimensional fractional order system

$$(4.1) \quad (S_i) = \begin{cases} D_{0+}^{\frac{5}{2}} u_1(t) = 2t \frac{e^{-(u_1^2 + u_2^2)}}{1 + u_1^2 + u_2^2}, & D_{0+}^{\frac{5}{2}} u_2(t) = 2t \frac{e^{-u_1^2}}{1 + u_1^2 + u_2^2}, \\ u_1(0) = 0, \quad u_1'(0) = 0, & u_2(0) = 0, \quad u_2'(0) = 0, \\ au_1(1) - bu_1'(0) = 0, & au_2(1) - bu_2'(0) = 0. \end{cases}$$

We have $\alpha = \frac{5}{2}$, $g(t) = 2t$, $f_1(u_1, u_2) = \frac{e^{-(u_1^2+u_2^2)}}{1+u_1^2+u_2^2}$, $f_2(u_1, u_2) = \frac{e^{-u_1^2}}{1+u_1^2+u_2^2}$, $f_i \in C(\mathbf{R}^2, \mathbf{R})$, $f_i(0) \neq 0$. If we choose $\Psi(u_1, u_2) = \frac{1}{1+u_1^2+u_2^2}$, then

$$|f_i(u_1, u_2)| \leq \frac{1}{1+u_1^2+u_2^2} = \Psi(|u_1|, |u_2|).$$

For $\eta = 2$, we get

$$\frac{n}{\Gamma(\alpha-1)} \Psi(\eta, \eta) \|g\|_{L^1[0,1]} \leq \frac{2}{\Gamma\left(\frac{3}{2}\right)(1+2\eta^2)} = 0.25075 \leq \eta.$$

Then, according to the Theorem 9, the boundary value problem (4.1) has at least one fixed point $u^* \in B_2$.

Example 2. Consider the following two-dimensional fractional order system

$$(4.2)(S_i) = \begin{cases} D_{0+}^{\frac{5}{2}} u_1(t) = \frac{e^{-t}}{10} (u_1 - u_2), & D_{0+}^{\frac{5}{2}} u_2(t) = \frac{e^{-t}}{10} (u_1 + 1), \\ u_1(0) = 0, u_1'(0) = 0, & u_2(0) = 0, u_2'(0) = 0, \\ au_1(1) - bu_1'(0) = 0, & au_2(1) - bu_2'(0) = 0. \end{cases}$$

We have $\alpha = \frac{5}{2}$, $g(t) = \frac{e^{-t}}{10}$, $f_1(u_1, u_2) = \frac{e^{-t}}{10} (u_1 - u_2)$, $f_2(u_1, u_2) = \frac{e^{-t}}{10} (u_1 + 1)$, $f_i \in C(\mathbf{R}^2, \mathbf{R})$, then

$$|f_i(x_1, x_2) - f_i(y_1, y_2)| \leq L \sum_{i=1}^2 |x_i - y_i|$$

with $L = 1$ and $K = \frac{2(1-e^{-1})}{10\Gamma\left(\frac{3}{2}\right)} = 0.14265 < 1$, then hypotheses of

Theorem 8 are satisfied. So, the boundary value problem (4.2) has a unique solution $u \in X$.

Acknowledgement 1. The authors are grateful to the anonymous referees for their valuable comments and suggestions that improved this paper.

References

- [1] R. P. Agarwal, D. O'Regan, S. Stanek, Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations, *J. Math. Anal. Appl.*, 371, pp. 57–68, (2010).
- [2] B. Ahmad, A. Alsaedi, Existence and uniqueness of solutions for coupled systems of higher-order nonlinear fractional differential equations, *Fixed Point Theory Appl.*, Article ID 364560 (2010).

- [3] B. Ahmad, J. J. Nieto, Riemann–Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions, *Bound. Value Probl.*, 36, pp. 1–9, (2011).
- [4] B. Ahmad, Nonlinear fractional differential equations with anti-periodic type fractional boundary conditions, *Differ. Equ. Dyn. Syst.*, 21(4), pp. 387–401, (2013)
- [5] M. Feng, X. Zhang, W. Ge, New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions, *Bound. Value Probl.*, 2011, Art ID 720-702, (2011).
- [6] A. Guezane-Lakoud, R. Khaldi and A. Kılıçman, Solvability of a boundary value problem at resonance, *SpringerPlus*, (2016) 5:1504.
- [7] A. Guezane-Lakoud, A. Ashyralyev, Positive Solutions for a System of Fractional Differential Equations with Nonlocal Integral Boundary Conditions, *Differ Equ Dyn Syst*, DOI 10.1007/s12591-015-0255-9.
- [8] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam, (2006).
- [9] V. Lakshmikantham, A. S. Vatsala, Basic theory of fractional differential equations. *Nonlinear Anal.*, 69, pp. 2677–2682, (2008).
- [10] K.Q. Lan, Positive solutions of semi-positone Hammerstein integral equations and applications, *Commun. Pure Appl. Anal.*, 6 (2), pp. 441–451, (2007).
- [11] K.Q. Lan, Eigenvalues of semi-positone Hammerstein integral equations and applications to boundary value problems, *Nonlinear Anal.*, 71, (2009)
- [12] S. K. Ntouyas, M. Obaid, A coupled system of fractional differential equations with nonlocal integral boundary conditions, *Adv. Differ. Equ.*, Article ID 130, (2012).
- [13] I. Podlubny, *Fractional Differential Equations Mathematics in Sciences and Engineering*, Academic Press, New York, (1999).
- [14] M. Rehman, R. A. Khan, A note on boundary value problems for a coupled system of fractional differential equations, *Comput. Math. Appl.*, 61, pp. 2630–2637, (2011).

- [15] H. Salem, On the existence of continuous solutions for a singular system of nonlinear fractional differential equations, *Appl. Math. Comput.*, 198, pp. 445–452, (2008).
- [16] X. Su, Existence of solution of boundary value problem for coupled system of fractional differential equations, *Eng. Math.*, 26, pp. 134–137, (2009).
- [17] X. Su, Boundary value problem for a coupled system of nonlinear fractional differential equations. *Appl. Math. Lett.*, 22, pp. 64–69, (2009).

A. Guezane-Lakoud

Laboratory of Advanced Materials
University of Badji Mokhtar Annaba
P. O. Box 12, 23000 Annaba,
Algeria
e-mail : a_guezane@yahoo.fr

G. Rebiai

Department of Mathematics
University of Guelma
Algeria
e-mail : rebeai-ghania@yahoo.fr,

and

R. Khaldi

Laboratory of Advanced Materials
University of Badji Mokhtar Annaba
P. O. Box 12, 23000 Annaba,
Algeria
e-mail : rkhadi@yahoo.fr