Sequentially spaces and the finest locally K-convex of topologies having the same onvergent sequences

A. El Amrani

University Sidi Mohamed Ben Abdellah, Morocco
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Abstract

The present paper is concerned with the concept of sequentially topologies in non-archimedean analysis. We give characterizations of such topologies.

Keywords : Non-archimedean topological space, sequentially spaces, convergent sequence in non-archimedean space.

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1. Introduction

In 1962 Venkataramen, in [19], posed the following problem:
Characterize "the class of topological spaces which can be specified completely by the knowledge of their convergent sequences".
Several authors then agreed to provide a solution, based on the concept of sequential spaces, in particular: In [9] and [10] Franklin gave some properties of sequential spaces, examples, and a relationship with the Frechet spaces; after Snipes in [17], has studied a new class of spaces called $T-$sequential space and relationships with sequential spaces; in [2], Boone and Siwiec gave a characterization of sequential spaces by sequential quotient mappings; in [4], Cueva and Vinagre have studied the $K - c-$Sequential spaces and the $K - s-$bornological spaces and adapted the results established by Snipes using linear mappings; thereafter Katsaras and Benekas, in [13], starting with a topological vector space (t.v.s.) $(E, \tau)$, have built up, the finest of topologies on $E$ having the same convergent sequences as $\tau$; and the thinnest of topologies on $E$ having the same precompact as $\tau$; using the concept of String (this study is a generalization of the study led by Weeb [21], on 1968, in case of locally convex spaces l.c.s. ); in [8], Ferrer, Morales and Ruiz, have reproduced previous work by introducing the concept of maximal sequentially topology. Goreham, in [11], has conducted a study linking sequentiality and countable subsets in a topological space by considering the five classes of spaces: spaces of countable first case, sequential spaces, Frechet spaces, spaces of "C.T." type and perfect spaces.

In this work, we will study, in the non-archimedean ($n.a$) case, for a locally $K-$convex space $E$ the finest sequential locally $K-$convex topology on $E$ having the same convergent sequences as the original topology.

2. Preliminaries

Throughout this paper $K$ is a ($n.a$) non trivially valued complete field with the valuation $|.|$, and the valuation ring is $B(0,1): = \{\lambda \in K : |\lambda| \leq 1\}$. There exists $\theta \in \mathbb{R}$ such that $\theta > 1$ and for all $n \in \mathbb{Z}$ there exists $\lambda_n \in K$ verifying $|\lambda_n| = \theta^n$, see [18], p.251.
The field $K$ is spherically complete if any decreasing sequence of closed balls in $K$ has a non-empty intersection.
For the basic notions and properties concerning locally $K-$convex spaces we refer to [14] or [18] if $K$ is spherically complete and to [15] if $K$ is not
spherically complete. However we recall the following:

Let \( E \) be a \( K \)-vector space, a nonempty subset \( A \) of \( E \) is called \( K \)-convex if \( \lambda x + \mu y + \gamma z \in A \) whenever \( x, y, z \in A, \lambda, \mu, \gamma \in K, |\lambda| \leq 1, |\mu| \leq 1, |\gamma| \leq 1 \) and \( \lambda + \mu + \gamma = 1 \). \( A \) is said to be absolutely \( K \)-convex if \( \lambda x + \mu y \in A \) whenever \( x, y \in A, \lambda, \mu \in K, |\lambda| \leq 1, |\mu| \leq 1 \). For a nonempty set \( A \subset E \) its absolutely \( K \)-convex hull \( c_0 (A) \) is the smallest absolutely \( K \)-convex set that contains \( A \). If \( A \) is a finite set \( \{x_1, ..., x_n\} \) we sometimes write \( c_0 (x_1, ..., x_n) \) instead of \( c_0 (A) \).

A topology on a vector space \( E \) over \( K \) is said to be locally \( K \)-convex \((lKcs)\) if there exists in \( E \) a fundamental system of zero-neighbourhoods consisting of absolutely \( K \)-convex subsets of \( E \).

If \( E \) is a \( lKcs, E' \) and \( E^* \) denote its topological and algebraic dual, respectively, and \( \sigma (E, E') \) and \( \sigma (E^*, E) \) the weak topology of \( E \) and \( E' \), respectively.

If \( (E, \tau) \) is a locally \( K \)-convex space with topology \( \tau \) we denote by \( \mathcal{P}_E \), (or by \( \mathcal{P} \) if no confusion is possible) a family of semi-norms determining the topology \( \tau \). We always assume that \( (E, \tau) \) is a Hausdorff space.

If \( A \) is a subset of \( E \) we denote by \( [A] \) the vector space spanned by \( A \). Remark that, if \( A \) is absolutely \( K \)-convex \( [A] = KA \). For an absolutely \( K \)-convex subset \( A \) of \( E \) we denote by \( p_A \) the Minkowski functional on \( [A] \), i.e for \( x \in [A], p_A (x) = \inf \{ |x| : x \in \lambda A \} \). If \( A \) is bounded then \( p_A \) is a norm on \( [A] \). We then denote by \( E_A \) the space \( [A] \) normed by \( p_A \).

Let \( \langle \cdot, \cdot \rangle \) be a duality between \( E \) and \( F \) where \( E \) and \( F \) are two vectors spaces over \( K \) (see [1] for general results), if \( A \) is a subset of \( E \), the polar of \( A \) is a subset of \( F \) defined by \( A^\circ = \{ y \in F / \forall x \in A \langle x, y \rangle \leq 1 \} \).

We define also the polar of a subset \( B \) of \( F \) in the same way. A subset \( A \) of \( E \) is said to be a polar set if \( A^{\circ\circ} = A \) (\( A^{\circ\circ} \) is the bipolar of \( A \))

A continuous semi-norm \( p \) on \( E \) is called a polar seminorm if the corresponding zero-neighbourhood \( A = \{ x \in E : p(x) \leq 1 \} \) is a polar set. The space \( E \) is called strongly polar if every continuous semi-norm on \( E \) is polar, and it is called polar if \( \exists \mathcal{P}_E \) such that every \( p \in \mathcal{P}_E \) is polar. (see [15]). Obviously:

\[ E \text{ strongly polar } \implies E \text{ polar.} \]
If $E$ is a polar space then the weak topology $\sigma (E, E^\prime)$ is Hausdorff. ([15] prop. 5.6). In that case we have a dual pair $(E, E^\prime)$. The value of the bilinear form on $E \times E$ (and similarly on $E^\prime \times E$) is denoted by $\langle x, a \rangle$, $x \in E$, $a \in E^\prime$. If $E$ is a polar space and $p$ is a continuous semi-norm on $E$ we denote by $E_p$ the vector space $E/Ker(p)$ and by $\pi_p$ the canonical surjection $\pi_p : E \rightarrow E_p$. The space $E_p$ is normed by $\| \pi_p (x) \|= p(x)$. Its unit ball is $\pi_p (U)$, with $U = \{ x \in E : p (x) \leq 1 \}$. Its completion is denoted by $\bar{E}_p$.

3. Sequential spaces in non-Archimedean analysis

3.1. Definitions and properties

Definitions 1. 1. Let $E$ a locally $K$–convex space and $V$ a subset of $E$.

$V$ is called a sequential neighborhood ($S$–neighborhood ) of $0$ if every null sequence in $E$ lies eventually in $V$, that is to say:

$$\forall (x_n)_n \in C_0 \ (\exists N_0 \in \mathbb{N}) : (\forall n \geq N_0), x_n \in V.$$

2. A locally $K$–convex space $E$ is called sequential space if every convex sequential neighborhood of $0$ is a neighborhood of $0$.

Remark 1. Every sequential neighborhood of $0$ is absorbent and contains $0$.

Proposition 1. If $V$ is absolutely $K$–convex and absorbent subset of a locally $K$–convex space $E$, the following are equivalent:

(i) $V$ is a $S$–neighborhood of $0$;

(ii) $p_V$ is sequentially continuous. Where $p_V$ is the Minkowski functional associated to $V$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $V$ is a sequentially neighbourhood of $0$; and let $(x_n)_n \in C_0 (E)$, let us show that $P_V (x_n) \xrightarrow{n \rightarrow +\infty} 0$.

Let $\varepsilon > 0$. Let us consider $\lambda \in K$ such that $0 < |\lambda | \leq \varepsilon$, then $\left( \frac{x_n}{\lambda} \right)_n \in C_0 (E)$, from where there exists $N \in \mathbb{N}$ such that $(\forall n \geq N), \frac{x_n}{\lambda} \in V$. 


which implies that $\forall n \geq N$, $p_V \left( \frac{x_n}{k} \right) \leq 1$, or $(\forall n \geq N)$, $p_V (x_n) \leq |\lambda| \leq \varepsilon$. Thus the result follows.

Reciprocally, suppose that $p_V$ is sequentially continuous over $E$. Let $(x_n)_n \in C_0 (E)$, so $P_V (x_n) \xrightarrow{n \to \infty} 0$, therefore there exists $N \in \mathbb{N}$ such that $(\forall n \geq N)$, $p_V (x_n) < 1$, and so $(\forall n \geq N)$, $x_n \in V$. ■

**Proposition 2.** For a locally $K-$convex space $E$ the following are equivalent:

(i) $E$ is a sequential space;

(ii) Every sequentially continuous seminorm on $E$ is continuous;

(iii) For every locally $K-$convex space $F$, every sequentially continuous linear map from $E$ to $F$ is continuous;

(iv) For every Banach space $F$, every sequentially continuous linear map from $E$ to $F$ is continuous.

**Proof.** $(i) \Rightarrow (ii)$. Suppose that $E$ is sequential and let $p$ a seminorm sequentially continuous on $E$. Let: $V = \{ x \in E : p(x) \leq 1 \}$.

$V$ is a sequential neighborhood of 0 and so $V$ is a neighborhood of 0 and consequently $p$ is continuous: $\forall \varepsilon > 0$, let $\lambda \in K$ such that $0 < |\lambda| < \varepsilon$. Then:

$$U = \lambda V$$

is a neighborhood of 0 and we have $p(U) \subset B (0, \varepsilon)$.

$(ii) \Rightarrow (i)$. Let $V$ a convex subset of $E$ which is a sequential neighborhood of 0. $V$ is absorbent and contains 0, therefore it’s absolutely $K-$convex ($K-$convex and contains 0). Then, by Proposition 1, $p_V$ is sequentially continuous, then continuous, and so $V$ is a neighborhood of 0. Therefore $E$ is sequential.

$(ii) \Rightarrow (iii)$. Let $F$ a locally $K-$convex space and $f : E \rightarrow F$ a sequentially-continuous linear mapping.

Let $V$ a $K-$convex neighborhood of 0 in $F$, $f^{-1}(V)$ is a sequential $K-$convex neighborhood of 0 in $E$, and so $f^{-1}(V)$ is a neighborhood of 0 in $E$, $(E$ is sequential and $(ii) \Leftrightarrow (i))$. And then $f$ is continuous.
(iv) $\Rightarrow$ (ii). Let $p$ a sequentially-continuous semi-norm on $E$; consider the Banach space $\widehat{E}_p$ the completion of $E_p = E/Ker(p)$. The canonical mapping $\pi_p : E \rightarrow \widehat{E}_p$ is sequentially continuous, because: for all $(x_n)_n \subset E$ such that $x_n \xrightarrow{n \rightarrow +\infty} 0$, we have:

\[
x_n \xrightarrow{n \rightarrow +\infty} 0 \Rightarrow p(x_n) \xrightarrow{n \rightarrow +\infty} 0 \\
\Rightarrow \widehat{p}(x_n) \xrightarrow{n \rightarrow +\infty} 0 \\
\Rightarrow \pi_p(x_n) \xrightarrow{n \rightarrow +\infty} 0.
\]

Then $\pi_p$ is continuous, and so $p$ is continuous:

$$(\forall \varepsilon > 0) \ U = \pi_p^{-1}\left(B_{\widehat{p}}(0, \varepsilon)\right) \text{ is a neighborhood of } 0 \text{ in } E$$

and we have $p(U) \subset B(0, \varepsilon)$.

(iii) $\Rightarrow$ (iv) Obvious.

3.2. The sequential topology

Let $(E, \tau)$ a locally $K$–convex space. Consider $\mathcal{U}$ the set of all sequentially $K$–convex neighborhood of $0$ and let $\mathcal{P}_s$ the family of all sequentially $\tau$–continuous n.a. semi-norm on $E$.

- $\mathcal{U}$ is a base of neighborhood of $0$ for a locally $K$–convex topology on $E$ which is denoted $\tau^s$ [16, 1.2. p.14]. Since every neighborhood of $0$ is a sequential neighborhood of $0$, then $\tau \leq \tau^s$.

- $\mathcal{P}_s$ define a locally $K$–convex topology on $E$ which is denoted $T^s$. A base of neighborhood of $0$ for $T^s$ is formed by the balls $B_p(0, \varepsilon)$ where $\varepsilon > 0$ and $p$ is a n.a. sequentially $\tau$–continuous semi-norm. $B_p(0, \varepsilon)$ is sequential neighborhood of $0$, because for all sequence $(x_n)_n$ converging to zero in $(E, \tau)$, there exists $n_0 \in \mathbb{N}$ such that for all $(n \geq n_0), p(x_n) < \varepsilon \left(p(x_n) \xrightarrow{n \rightarrow +\infty} 0\right)$.

Remark 2. The topology $T^s$ is sequential.

Proposition 3. $\tau^s$ is the coarser of all sequential locally $K$–convex topologies on $E$ finer than $\tau$.

Proof. $\tau^s$ is sequential and $\tau^s \geq \tau$.

Let $\varrho$ a sequential locally $K$–convex topology on $E$ finer than $\tau$. Let $U \in \mathcal{U}$; $U$ sequential neighborhood of $0$ for $\tau$, and so $U$ is a sequential neighborhood of $0$ for $\varrho$ ($\varrho \geq \tau$) and then $U$ is a neighborhood of $0$ for $\varrho$ ($\varrho$ is sequential). Therefore $\varrho \geq \tau^s$. Which proves the proposition.
3.2.1. Characterization of sequential locally $K$–convex spaces

**Proposition 4.** $\tau$ is sequential if, and only if, $\tau = \tau^s$.

**Proof.** $\Leftarrow$ Obvious.

Suppose that $\tau$ is sequential and let $U \in \mathcal{U}$; $U$ is a sequential $K$–convex neighborhood of $0$ for $\tau$, so $U$ is a neighborhood of $0$ for $\tau$ and then $\tau \geq \tau^s$.

Finally $\tau = \tau^s$.

**Lemma 1.** For all sequence $(x_n)_n$ of $(E, \tau)$ we have:

\[ (x_n \xrightarrow{n \to +\infty} 0 \text{ for } \tau) \iff (x_n \xrightarrow{n \to +\infty} 0 \text{ for } \tau^s). \]

**Proof.** $\Rightarrow$ Let $U \in \mathcal{U}$, there exists $N \in \mathbb{N}$ such that: $\forall n \geq N \ x_n \in U$, hence $x_n \xrightarrow{n \to +\infty} 0$ for $\tau^s$.

The converse follows by $\tau \leq \tau^s$.

**Lemma 2.** Let $\varrho$ a locally $K$–convex topology on $E$ such that for all null sequence for $\tau$ is a null sequence for $\varrho$. Then $\tau^s \geq \varrho$.

**Proof.** Consider $i : (E, \tau) \longrightarrow (E, \varrho)$ the canonical injection. Then for every sequence $(x_n)_n$ in $E$ we have:

\[ x_n \xrightarrow{\tau^s} 0 \xRightarrow{\text{Lemma 1}} x_n \xrightarrow{\tau} 0 \]
\[ \iff x_n \xrightarrow{\varrho} 0 \]

Then, $i$ is sequentially continuous, and since $(E, \tau^s)$ is sequential, $i$ is continuous (Proposition 2). Hence $\tau \leq \tau^s$.

**Proposition 5.** $\tau^s$ is the finest locally $K$–convex topology on $E$ having the same convergent sequences as $\tau$.

**Proof.** By Lemma 1 before, $\tau^s$ and $\tau$ has the same convergent sequences. Let $\varrho$ a locally $K$–convex topology on $E$ having the same convergent sequence as $\tau$ and let $(x_n)_n$ a sequence of $E$ converging to $0$ for $\tau$, then $x_n \longrightarrow 0$ for $\varrho$, hence, by Lemma 3, $\tau^s \geq \varrho$.

**Remark 3.** $\tau^s$ is also the finer topology on $E$ having the same null sequences as $\tau$.

**Lemma 3.** Let $(E, \tau)$ a locally $K$–convex space and $A$ a subset of $E$, then: $A$ is $\tau$–bounded if, and only if, for all null sequence $(\lambda_n)_n$ in $K$ and all sequence $(x_n)_n$ in $A$; the sequence $(\lambda_n x_n)_n$ converges to zero in $(E, \tau)$ that is to say $(\lambda_n x_n)_n$ is a null sequence in $(E, \tau)$.
Proof. Suppose that $A$ be bounded in $(E, \tau)$.
Let $(\lambda_n)_n \in C_0(K)$ and $(x_n)_n$ a sequence in $A$.
Let $V$ a $K$–convex neighborhood of zero in $E$, then there exists $\lambda \in K^*$ such that $\lambda A \subset V$ and there exists $N \in \mathbb{N}^*$ such that $(\forall n \geq N) \ |\lambda_n| \leq |\lambda|$; but
\[
(\forall n \geq N) \quad \lambda_n x_n = \frac{\lambda_n}{\lambda} \lambda x_n
\]
\[
\in \frac{\lambda}{\lambda} \lambda A
\]
\[
\subset \frac{\lambda}{\lambda} V
\]
\[
\subset V.
\]

Then the sequence $(\lambda_n x_n)_n$ converges to zero in $(E, \tau)$.
Reciprocally, if $A$ is no $\tau-$bounded, then there exists $U$ a $K-$convex neighborhood of zero such that $\forall n \in \mathbb{N} \ A \not\subset \frac{1}{\lambda_n} U$ where $(\lambda_n)_n$ is the sequence of general term $|\lambda_n| = \varrho^n$ and $\varrho$ is the real number defined in the preliminary. For all $n \in \mathbb{N}$, let $x_n$ the element of $A$ such that $x_n \not\in \frac{1}{\lambda_n} U$, then,
\[
(\forall n \in \mathbb{N}) \quad \lambda_n x_n \not\in U
\]
that is to say that the sequence $(\lambda_n x_n)_n$ does not converge to zero, and we have: $(x_n)_n \subset A$ and $(\lambda_n)_n \in C_0(K)$. 

Proposition 6. Let $(E, \tau)$ a locally $K-$convex space, then:
$\tau$ and $\tau^s$ have the same bounded subsets.

Proof. Let $A$ a subset of $E$.
If $A$ is $\tau^s-$bounded, $A$ is $\tau-$bounded, because $\tau^s \geq \tau$.
If $A$ is $\tau-$bounded, let $(x_n)_n \subset A$ and $(\lambda_n)_n \in C_0(K)$, then, according to the previous Lemma, the sequence $(\lambda_n x_n)_n$ converges to zero in $(E, \tau)$ and therefore it converges to zero in $(E, \tau^s)$ (Lemma 1). So $A$ is $\tau^s-$bounded.

Proposition 7. Let $\left(F, \tau^s\right)$ a locally $K-$convex space and $f : E \longrightarrow F$ a linear mapping, then:
f is $\tau^s-$continuous if, and only if, $f$ is sequentially $\tau-$continuous.

Proof. Suppose that $f$ be $\tau^s-$continuous, and let $(x_n)_n$ a converging sequence to zero in $(E, \tau)$ and let $V \in \mathcal{V}_F(0)$, there exists $U \in \mathcal{U}$ such that $f(U) \subset V$. $U$ being a sequential neighborhood of zero, so there exists $n_0 \in \mathbb{N}$ such that $(\forall n \geq n_0) \ x_n \in U$ and consequently $(\forall n \geq n_0) \ f(x_n) \in f(U) \ (\subset V)$. Therefore the sequence $(f(x_n))_n$ converges to zero in $F$. 

Conversely, suppose that $f$ is sequentially $\tau$–continuous; let us show that $f : (E, \tau^s) \to F$ is continuous. According to Proposition 2, it suffices to show that $f$ is sequentially $\tau^s$–continuous. Let then $(x_n)_n$ a converging sequence to zero in $(E, \tau^s)$, then it converges to zero in $(E, \tau)$ (Lemma 1) and consequently $(f(x_n))_n$ is converging to zero in $F$.

### 3.3. Comparison of topologies $\tau^s$ and $T^s$

**Lemma 4.** For every $U \in \mathcal{U}$, $p_U$ is a n.a. sequentially $\tau$–continuous seminorm.

**Proof.** Let $U \in \mathcal{U}$; then for all $(x_n)_n \in C_0(E)$, all $\varepsilon > 0$ and all $\lambda \in K^*$ such that $0 < |\lambda| \leq \varepsilon$ we have: $(\lambda^{-1}x_n)_n \in C_0(E)$ from where it exists $n_0 \in \mathbb{N}$ : $(\forall n \geq n_0) \lambda^{-1}x_n \in U$ and then:

$(\forall n \geq n_0) \ p_U(\lambda^{-1}x_n) \leq 1 \Rightarrow (\forall n \geq n_0) \ p_U(x_n) \leq |\lambda| \leq \varepsilon$. Therefore the sequence $(p_U(x_n))_n$ converges to zero in $\mathbb{R}^+$ and consequently $p_U$ is sequentially $\tau$–continuous.

**Proposition 8.** $\tau^s = T^s$

**Proof.** $T^s$ being a sequential locally $K$–convex topology (Remark 2), whence $\tau^s \geq T^s$.

For the other inequality, it suffices to show that $i : (E, T^s) \to (E, \tau^s)$ is continuous, and by Proposition 2, it suffices to show that the mapping $i$ is sequentially $T^s$–continuous.

Let $(x_n)_n$ a sequence that tends towards zero in $(E, T^s)$. Then for any $U \in \mathcal{U}$, $p_U$ is sequentially $\tau$–continuous, therefore the sequence $(p_U(x_n))_n$ converges to zero in $\mathbb{R}^+$, from where it exists $n_0 \in \mathbb{N}$ : $(\forall n \geq n_0) \ p_U(x_n) < 1$, or $(\forall n \geq n_0) \ x_n \in U$. Therefore the sequence $(x_n)_n$ tends to zero in $(E, \tau^s)$. From where $i$ is $T^s$–sequentially continuous. And consequently $T^s \geq \tau^s$.

**Remark 4.** We can show otherwise the previous Proposition: Since any n.a. $\tau$–continuous seminorm on $E$ is sequentially $\tau$–continuous, $T^s \geq \tau$. But $T^s$ is sequential and $\tau^s$ is the coarser sequential locally $K$–convex topology finer than $\tau$, then $T^s \geq \tau^s$.

### 3.4. The sequential polar topology

Let $\mathcal{V}$ the family of all $K$–convex, subsets $A$ of $E$ which are polar and sequential neighborhood of $0$ in $(E, \tau)$. $\mathcal{V}$ is a base of neighborhood of $0$ of
a locally $K$–convex topology on $E$ which we noted $\tau^{ps}$ \cite[1.2., p. 14]{16}. \(\tau^{ps}\) is a polar topology on $E$ and $\tau^s \geq \tau^{ps}$ \(\forall U \subset V\).

**Remark 5.** Since, if $V \in \mathcal{V}$, then $V^c \in \mathcal{V}$, we can suppose that all $V \in \mathcal{V}$, $V$ is $\tau$–closed.

**Lemma 5.** Suppose that $K$ is spherically complete, and let $A$ a subset of $E$ absolutely $K$–convex and $\tau$–closed, then:

1. If $K$ is discrete, $A^{oo} = A$.

2. If $K$ is dense, $\forall \alpha \in K : |\alpha| > 1 A^{oo} \subset \alpha A$.

Where $A^{oo}$ is the bipolar of $A$ with respect the duality $\langle E, E' \rangle$.

**Proof.** See \cite[Theorems 4.14, 4.15, p.280 – 281]{18}.

**Lemma 6.** If $K$ is spherically complete, then $\tau^{ps}$ is a sequential topology.

**Proof.** Let $U$ a subset of $E$ which is $K$–convex, $\tau$–closed and sequential neighborhood of 0 on $(E, \tau)$. Let us show that $U$ is a neighborhood of 0 of $\tau^{ps}$. By Lemma 5, before $U^{oo} \subset \alpha U$ for $\alpha = 1$ if $K$ is discrete and $|\alpha| > 1$ is $K$ dense. We pose $V = U^{oo}$, then $V$ is $K$–convex, polar and sequential neighborhood of 0 on $(E, \tau) (U \subset V)$, then $V$ is a neighborhood of 0 for $\tau^{ps}$ and therefore $U$ is a neighborhood of 0 for $\tau^{ps}$ \(\frac{1}{\alpha} V \subset U\).

Then $\tau^{ps}$ is sequential.

**Proposition 9.** If $K$ is spherically complete, then $\tau^{ps} \geq \tau$.

**Proof.** It is a matter of showing that $i : (E, \tau^{ps}) \longrightarrow (E, \tau)$ is continuous, and since $\tau^{ps}$ is sequential (Lemma 6), it suffices to show that $i$ is sequentially $\tau^{ps}$–continuous.

Let $(x_n)_n$ a sequence of $E$ which is converging to zero on $(E, \tau^{ps})$ and let $U$ an absolutely $K$–convex and $\tau$–closed neighborhood of zero on $(E, \tau)$, then $U^{oo} \subset \alpha U$ where $\alpha = 1$ if $K$ is discrete and $|\alpha| > 1$ if $K$ is dense (Lemma 5).

The sequence $(\alpha x_n)_n$ converges to zero on $(E, \tau^{ps})$ and $U^{oo} \in \mathcal{V}$ hence there exists $n_0 \in \mathbb{N} : (\forall n \geq n_0) \alpha x_n \in U^{oo}$ and so $(\forall n \geq n_0) x_n \in \frac{1}{\alpha} U^{oo} \subset U$.

Then the sequence $(x_n)_n$ converges to zero on $(E, \tau)$.
Remark 6. If $K$ is spherically complete, then $\tau^{ps} \geq \tau$; but $\tau^s$ is the coarsest of all sequential locally $K$-convex topologies finest than $\tau$ and since $\tau^{ps}$ is sequential, then $\tau^{ps} \geq \tau^s$, and so $\tau^{ps} = \tau^s$.

Proposition 10. Let $(E, \tau)$ a locally $K$-convex space. Then $\tau^{ps}$ is the finer of all polar locally $K$-convex topologies $\delta$ on $E$ such that all sequence on $E$ which is $\tau$-convergent is $\delta$-convergent.

Proof. $\tau^{ps}$ is a locally $K$-convex polar topology on $E$. Let $(x_n)_n$ a converging sequence to zero on $(E, \tau)$, then for all $V \in \mathcal{V}$, there exists $n_0 \in \mathbb{N}$ : $(\forall n \geq n_0) \ x_n \in V$, then $(x_n)_n$ converges to zero on $(E, \tau^{ps})$.

Let $\delta$ a locally $K$-convex polar topology on $E$ such that all sequence on $E$ which is $\tau$-convergent is $\delta$-convergent; showing that $\tau^{ps} \geq \delta$. Let $U$ a $K$-convex and polar neighborhood of zero for $\delta$, and let $(x_n)_n$ a sequence which converges to zero on $(E, \tau)$, then it’s convergent to zero for $\delta$; hence there exists $n_0 \in \mathbb{N}$ : $(\forall n \geq n_0) \ x_n \in U$. Then $U$ is a sequential neighborhood of zero and so $U \in \mathcal{V}$. And then $\tau^{ps} \geq \delta$.

Corollary 1. If $\tau$ is polar, then $\tau^{ps} \geq \tau$ and $\tau^{ps}$ and $\tau$ have the same convergent sequences.

Proof. $\tau^{ps} \geq \tau$ follows immediately of the proposition before and we have all $\tau^{ps}$-convergent sequence is $\tau$-convergent. And we have already all $\tau$-convergent sequence is $\tau^{ps}$-convergent; then $\tau^{ps}$ and $\tau$ have the same convergent sequences.

Or equivalently the two topologies have the same null sequences.

Lemma 7. Let $p$ a seminorm n.a. over $E$. And let:

\[ A = \{ x \in E : p(x) < 1 \} \quad \text{and} \quad B = \{ x \in E : p(x) \leq 1 \}. \]

Then $A^o = B^o$.

Proof. If $K$ is discrete, $A = B$, then we can suppose that $K$ is dense.

$A$ is a subset of $B$, then $B^o \subset A^o$. 
Let $f \in E^*$ such that $f \notin B^\circ$, then there exists $y \in B$ such that $|f(y)| > 1$. Suppose that $f \in A^\circ$, that is to say that $(\forall x \in A) \ |f(x)| \leq 1$; then, since $K$ is dense, there exists $\lambda \in K$ such that $1 < |\lambda| < |f(y)|$ so:

$$1 < |f(\frac{\lambda}{y})| \implies \frac{\lambda}{y} \notin A$$
$$\implies p(\frac{\lambda}{y}) \geq 1$$
$$\implies p(y) \geq |\lambda|$$
$$\implies p(y) > 1$$
$$\implies y \notin B$$

Which is absurd. $lacksquare$

**Proposition 11.** $\tau^{p^*}$ coincides with the locally $K-$convex topology generated by all n.a. polar and sequentially $\tau-$continuous semi-norms.

**Proof.** Let $T^{p^*}$ the locally $K-$convex topology generated by $S_p$ the family of all n.a. polar and sequentially $\tau-$continuous semi-norms. Then $T^{p^*}$ admits a basis $\mathcal{B}$ of neighborhoods of zero formed by polar balls $B_p(0, \varepsilon)$ where $p \in S_p$ and $\varepsilon > 0$.

Let us show that $i : (E, \tau^{p^*}) \longrightarrow (E, T^{p^*})$ is bicontinuous.

Let $V = B_p(0, \varepsilon)$ an element of $\mathcal{B}$, then $V$ is $K-$convex. Let $(x_n)_n$ a sequence of elements of $E$ which converges to zero in $(E, \tau)$, then

$$\lim_{n \to +\infty} p(x_n) = 0 \ (p \text{ is sequentially } \tau-\text{continuous})$$

hence there is $n_0 \in \mathbb{N} : (\forall n \geq n_0) \ p(x_n) < \varepsilon$, or $(\forall n \geq n_0) \ x_n \in V$ which implies that $V$ is a sequentially neighborhood of zero, hence $V \in \mathcal{V}$ and so $T^{p^*} \leq \tau^{p^*}$.

Conversely, either $V \in \mathcal{V}$, then it’s sequentially $K-$convex neighborhood of zero. We have:

$$\{x \in E : p_V(x) < 1\} \subset V \subset \{x \in E : p_V(x) \leq 1\}.$$

And by the previous Lemma 7:

$$A^\circ = B^\circ = V^\circ = V,$$

where $A = \{x \in E : p_V(x) < 1\}$ and $B = \{x \in E : p_V(x) \leq 1\}$; from where $B^\circ = B$, and consequently $p$ is polar or $p_V$ is polar [15, Proposition 3.4, p. 195]. Let us show that $p_V$ is sequentially $\tau-$continuous. Let $(x_n)_n$ a sequence of elements of $E$ which converges to zero in $(E, \tau)$ and let $\varepsilon > 0$; let us consider $\lambda \in K$ such that $0 < |\lambda| < \varepsilon$, then
the sequence \((\lambda^{-1}x_n)_n\) converges to zero in \((E, \tau)\) and \(V\) being a sequential neighborhood of 0, then there exists \(n_0 \in N\) : \((\forall n \geq n_0)\ \lambda^{-1}x_n \in V\), from where \((\forall n \geq n_0)\ \varphi (x_n) \leq 1\) or \((\forall n \geq n_0)\ \varphi (x_n) \leq |\lambda^{-1}| < \varepsilon\); from where \(\varphi\) is sequentially \(\tau\)-continuous and consequently \(\varphi \in S_\varphi\); then \(T^{p_s} \geq \tau^p\). So what \(T^{p_s} = \tau^p\). \[\square\]

**Proposition 12.** \(\tau^{p_s}\) is the finer of all polar locally \(K\)-convex topologies which are less fine than \(\tau^s\).

**Proof.** \(\tau^{p_s}\) is a polar locally \(K\)-convex topology and \(\tau^{p_s} \leq \tau^s\). Let \(\varrho\) a polar locally \(K\)-convex topology such that \(\varrho \leq \tau^s\), and let \(V\) a polar \(K\)-convex neighborhood of 0 for \(\varrho\), then there exists \(U \in \mathcal{U}\) such that \(U \subset V\) (\(\varrho \leq \tau^s\)), from where \(V\) is a sequential neighborhood of zero, consequently it is a sequential neighborhood of zero for \(\tau^{p_s}\). Therefore \(\tau^{p_s} \geq \varrho\). \[\square\]

### 3.4.1. Continuity of linear mappings

**Lemma 8.** Let \(E\) and \(F\) be two locally \(K\)-convex spaces and \(f : E \rightarrow F\) a continuous linear mapping, then for any subset \(V\) of \(F\).

If \(V\) is polar in \(F\), \(f^{-1} (V)\) is polar in \(E\).

**Proof.** Let \(V \subset F\), putting \(U = f^{-1} (V)\). Suppose that \(V\) is sequential.

Let \(x \in U^\circ\); let us show that \(x \in U\). By absurd, suppose that \(x \notin U\), and let \(y = f (x)\) then \(y \notin V\) from where \(y \notin V^\circ\) \((V^\circ = V)\) then there exists \(\varphi \in V^\circ\) : \(|\varphi (y)| > 1\). But \(\forall t \in U\), \(f (t) \in V\) from where

\[\forall t \in U\ \ |\varphi (f (t))| \leq 1\text{ and consequently }\varphi \circ f \in U^\circ\text{ and so }|\varphi (f (x))| \leq 1, \text{ therefore }|\varphi (y)| \leq 1; \text{ which is absurd.} \]\[\square\]

**Proposition 13.** Let \((E, \tau)\) and \((F, \tau_1)\) two locally \(K\)-convex spaces.

If \(f : (E, \tau) \rightarrow (F, \tau_1)\) is a continuous linear mapping, then \(f\) is \((\tau^s, \tau_1^s)\)-continuous and \((\tau^{p_s}, \tau_1^{p_s})\)-continuous.

**Proof.** Let us show that \(f : (E, \tau^s) \rightarrow (F, \tau_1^s)\) is continuous. For this it suffices to show that for every sequential neighborhood \(V\) of zero for
\( \tau_1, f^{-1}(V) \) is a sequential neighborhood \( V \) of zero for \( \tau \).

Let \( V \) a sequential neighborhood of zero for \( \tau_1 \) and let \((x_n)_n\) a sequence of \( E \) which converges towards zero in \((E, \tau)\); then the sequence \((f(x_n))_n\) converges towards zero in \((F, \tau_1)\), from where there exists \( n_0 \in \mathbb{N} : (\forall n \geq n_0) f(x_n) \in V \), then \((\forall n \geq n_0) x_n \in f^{-1}(V) \).

Let us show that \( f : (E, \tau_{ps}) \to (F, \tau_{ps}^1) \) is continuous. For this it suffices to show that for every polar and sequential neighborhood \( V \) of zero for \( \tau_1 \), \( f^{-1}(V) \) is a polar and sequential neighborhood \( V \) of zero for \( \tau \).

Let \( V \) a polar and sequential neighborhood of zero for \( \tau_1 \), then by Lemma 8, \( f^{-1}(V) \) is polar for \( \tau \). In the other hand, for all sequence \((x_n)_n\) of \( E \) which converges towards zero in \((E, \tau)\), the sequence \((f(x_n))_n\) converges to zero in \((F, \tau_1)\), from where there exists \( n_0 \in \mathbb{N} : (\forall n \geq n_0) f(x_n) \in V \), therefore \((\forall n \geq n_0) x_n \in f^{-1}(V) \).

**Proposition 14.** Let \((E, \tau) = \prod_{k=1}^{n} (E_k, \tau_k)\), then:

(i) \( \tau^{s} = \prod_{k=1}^{n} \tau_k^{s} \),

(ii) \( \tau^{ps} = \prod_{k=1}^{n} \tau_k^{ps} \).

**Proof.** Let us show that \( i : (E, \tau^s) \to \left( E, \prod_{k=1}^{n} \tau_k^s \right) \) is continuous. Let us show that \( V \) is neighborhood of zero for \((E, \tau^s)\), where \( V = (U_k)_{1 \leq k \leq n} \) is a \( K \)-convex neighborhood of zero for the arrival space.

Let \((y_p)_p = \left( \prod_{k=1}^{n} x_k^p \right)_p\) a sequence of \( E \) which converges to zero in \((E, \tau)\), then for all \( k \in \mathbb{N}, 1 \leq k \leq n \), the sequence \((x_k^p)_p\) converges to zero in \((E_k, \tau_k)\), from where there exists \( p_k \in \mathbb{N} : (\forall p \geq p_k) x_k^p \in U_k \). Let \( p_0 = \max_p p_k \), so \((\forall p \geq p_0) \forall k \in \mathbb{N}, 1 \leq k \leq n, x_k^p \in U_k \), from where \((\forall p \geq p_0) y_p \in V \). Therefore \( V \) is a sequential neighborhood of zero in \((E, \tau)\); \( V \) being \( K \)-convex, therefore \( V \) is a neighborhood of zero in \((E, \tau^s)\).
Let us show that \( i : \left( E, \prod_{k=1}^{n} \tau^s_k \right) \rightarrow (E, \tau^s) \) is continuous. Let \( V \) a sequential \( K \)-convex neighborhood of zero in \( (E, \tau) \). For all \( k, 1 \leq k \leq n \), let \( j_k : E_k \rightarrow E \) the canonical injection and pose \( V_k = j_k^{-1}(V) \), so \( V_k \) is a sequential neighborhood of zero in \( (E_k, \tau_k) \), from where \( V_k \) is a neighborhood of zero in \( (E_k, \tau^s_k) \), and consequently \( U = \prod_{k=1}^{n} V_k \) is a neighborhood of zero in \( \left( E, \prod_{k=1}^{n} \tau^s_k \right) \). But \( U \subset V \) (\( V \) is absolutely \( K \)-convex); therefore \( V \) is a neighborhood of zero in \( \prod_{k=1}^{n} \tau^s_k \).

References


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