On the graded classical prime spectrum of a graded module

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Abstract

Let $G$ be a group with identity $e$. Let $R$ be a $G$-graded commutative ring and $M$ a graded $R$-module. In this paper, we introduce and study a new topology on $\text{Cl.Spec}_{g}(M)$, the collection of all graded classical prime submodules of $M$, called the Zariski-like topology. Then we investigate the relationship between algebraic properties of $M$ and topological properties of $\text{Cl.Spec}_{g}(M)$. Moreover, we study $\text{Cl.Spec}_{g}(M)$ from point of view of spectral space.


Keywords : Graded classical prime submodule, graded classical prime spectrum, Zariski topology.
1. Introduction and Preliminaries

Before we state some results, let us introduce some notations and terminologies. Let $G$ be a group with identity $e$ and $R$ be a commutative ring with identity $1_R$. Then $R$ is a $G$-graded ring if there exist additive subgroups $R_g$ of $R$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g,h \in G$. We denote this by $(R,G)$ (see [8]). The elements of $R_g$ are called homogeneous of degree $g$ where the $R_g$’s are additive subgroups of $R$ indexed by the elements $g \in G$. If $x \in R$, then $x$ can be written uniquely as $\sum_{g \in G} x_g$, where $x_g$ is the component of $x$ in $R_g$. Moreover, $h(R) = \bigcup_{g \in G} R_g$. Let $I$ be an ideal of $R$. Then $I$ is called a graded ideal of $(R,G)$ if $I = \bigoplus_{g \in G} (I \cap R_g)$. Thus, if $x \in I$, then $x = \sum_{g \in G} x_g$ with $x_g \in I$. An ideal of a $G$-graded ring need not be $G$-graded (see [8].)

Let $R$ be a $G$-graded ring and $M$ an $R$-module. We say that $M$ is a $G$-graded $R$-module (or graded $R$-module) if there exists a family of subgroups $\{M_g\}_{g \in G}$ of $M$ such that $M = g \in G \bigoplus M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g,h \in G$. Here, $R_g M_h$ denotes the additive subgroup of $M$ consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = g \in G \bigcup M_g$ and the elements of $h(M)$ are called homogeneous elements of $M$. Let $M = g \in G \bigoplus M_g$ be a graded $R$-module and $N$ a submodule of $M$. Then $N$ is called a graded submodule of $M$ if $N = g \in G \bigoplus N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, $N_g$ is called the $g$-component of $N$ (see [8].)

Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. A proper graded ideal $I$ of $R$ is said to be a graded prime ideal if whenever $rs \in I$, we have $r \in I$ or $s \in I$, where $r,s \in h(R)$. The graded radical of $I$, denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x^{n_g} \in I$. Note that, if $r$ is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in N$. Let $Spec_g(R)$ denote the set of all graded prime ideals of $R$ (see [11].)

A proper graded submodule $N$ of $M$ is said to be a graded prime submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $r \in (N:_R M) = \{ r \in R : rM \subseteq N \}$ or $m \in N$ (see [2].) It is shown in [2, Proposition 2.7] that if $N$ is a graded prime submodule of $M$, then $P := (N:_R M)$ is a graded prime ideal of $R$, and $N$ is called graded $P$-prime submodule. Let $Spec_g(M)$ denote the set of all graded prime submodules of $M$. Note that some graded $R$-modules $M$ have no graded prime submodules. We call such graded modules $g$-primeless. The graded radical of a graded submodule $N$ of $M$, denoted by $Gr_M(N)$, is defined to be the
intersection of all graded prime submodules of $M$ containing $N$. If $N$ is not contained in any graded prime submodule of $M$, then $\text{Gr}_M(N) = M$ (see [2, 9].)

A proper graded submodule $N$ of $M$ is called a graded classical prime submodule if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rm \in N$ or $sm \in N$ (see [1, 4].) Of course, every graded prime submodule is a graded classical prime submodule, but the converse is not true in general (see [1], Example 2.3.) Let $\text{Cl.Spec}_g(M)$ denote the set of all graded classical prime submodules of $M$. Obviously, some graded $R$-modules $M$ have no graded classical prime submodules; such modules are called $g$-Cl.primeless. The graded classical radical of a graded submodule $N$ of a graded $R$-module $M$, denoted by $\text{Gr}_M^c(N)$, is defined to be the intersection of all graded classical prime submodules of $M$ containing $N$. If $N$ is not contained in any graded classical prime submodule of $M$, then $\text{Gr}_M^c(N) = M$ (see [4].) We know that $\text{Spec}_g(M) \subseteq \text{Cl.Spec}_g(M)$. As it is mentioned in ([1], Example 2.3), it happens sometimes that this containment is strict. We call $M$ a graded compatible $R$-module if its graded classical prime submodules and graded prime submodules coincide, that is if $\text{Spec}_g(M) = \text{Cl.Spec}_g(M)$. If $R$ is a $G$-graded ring, then every graded classical prime ideal of $R$ is a graded prime ideal. So, if we consider $R$ as a graded $R$-module, it is graded compatible.

Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. For each graded ideal $I$ of $R$, the graded variety of $I$ is the set $V^g_R(I) = \{ P \in \text{Spec}_g(R) | I \subseteq P \}$. Then the set $\{ V^g_R(I) | I \text{ is a graded ideal of } R \}$ satisfies the axioms for the closed sets of a topology on $\text{Spec}_g(R)$, called the Zariski topology on $\text{Spec}_g(R)$ (see [7, 10].)

In [3], $\text{Spec}_g(M)$ has endowed with quasi-Zariski topology. For each graded submodule $N$ of $M$, let $V^g(N) = \{ P \in \text{Spec}_g(M) | N \subseteq P \}$. In this case, the set $\zeta^g(M) = \{ V^g(N) | N \text{ is a graded submodule of } M \}$ contains the empty set and $\text{Spec}_g(M)$, and it is closed under arbitrary intersections, but it is not necessarily closed under finite unions. The graded $R$-module $M$ is said to be a $g$-Top module if $\zeta^g(M)$ is closed under finite unions. In this case $\zeta^g(M)$ satisfies the axioms for the closed sets of a unique topology $\tau^g$ on $\text{Spec}_g(M)$. The topology $\tau^g(M)$ on $\text{Spec}_g(M)$ is called the quasi-Zariski topology.

In [4], $\text{Cl.Spec}_g(M)$ has endowed with quasi-Zariski topology. For each graded submodule $N$ of $M$, let $V^g(N) = \{ C \in \text{Cl.Spec}_g(M) | N \subseteq C \}$. In this case, the set $\eta^g(M) = \{ V^g(N) | N \text{ is a graded submodule of } M \}$ contains the empty set and $\text{Cl.Spec}_g(M)$, and it is closed under arbitrary
intersections, but it is not necessarily closed under finite unions. The graded
$R$-module $M$ is said to be a $g$-Cl. Top module module if $\eta^g(M)$ is closed
under finite unions. In this case $\eta^g(M)$ satisfies the axioms for the closed
sets of a unique topology $g^a$ on $\text{Cl.Spec}_g(M)$. In this case, the topology
$g^a(M)$ on $\text{Cl.Spec}_g(M)$ is called the quasi-Zariski topology.

In this article, we introduce and study a new topology on $\text{Cl.Spec}_g(M)$,
called the Zariski-like topology, which generalizes the Zariski topology of
graded rings to graded modules. Let $R$ be a $G$-graded ring and $M$ a
graded $R$-module. For each graded submodule $N$ of $M$, we define
$U_g^a(N) = \text{Cl.Spec}_g(M) - V^a_g(N)$ and put $B^{cl}(M) = \{U^a_g(N) : N$ is a graded submodule of $M\}$. Then we define $\tau^{cl}_g(M)$ to be the topology on $\text{Cl.Spec}_g(M)$ by
the sub-basis $B^{cl}(M)$. In fact $\tau^{cl}_g(M)$ to be the collection $U$ of all unions of
finite intersections of elements of $B^{cl}(M)$. We call this topology the Zariski-
like topology of $M$.

If $N$ is a graded submodule (respectively proper submodule) of a graded
module $M$ we write $N \leq_g M$ (respectively $N_g M$).

2. Topology on $\text{Cl.Spec}_g(M)$

Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. A graded submodule
$C$ of $M$ will be called a graded maximal classical prime if $C$ is a graded
classical prime submodule of $M$ and there is no graded classical prime
submodule $P$ of $M$ such that $C \subseteq P$. Let $\text{Cl.Spec}_g(M)$ be endowed with
the Zariski-like topology. For each subset $Y$ of $\text{Cl.Spec}_g(M)$, We will denote
the closure of $Y$ in $\text{Cl.Spec}_g(M)$ by $\text{cl}(Y)$.

Lemma 2.1. Let $R$ be a $G$-graded ring and $M$ a graded $R$-module.

i) If $Y$ is a nonempty subset of $\text{Cl.Spec}_g(M)$, then $\text{cl}(Y) = \bigcup_{C \in Y} V^a_g(C)$.

ii) If $Y$ is a closed subset of $\text{Cl.Spec}_g(M)$, then $Y = \bigcup_{C \in Y} V^a_g(C)$.

Proof.

i) Clearly, $\text{cl}(Y) \subseteq \bigcup_{C \in Y} V^a_g(C)$. Let $S$ be a closed subset of $\text{Cl.Spec}_g(M)$
containing $Y$. Thus, $S = \bigcap_{i \in I} (\bigcup_{j=1}^{n_i} V^a_g(N_{ij}))$, for some $N_{ij} \leq_g M$,
i $i \in I$ and $n_i \in \mathbb{N}$. Let $P \in \bigcup_{C \in Y} V^a_g(C)$. Then, there exists $C_0 \in Y$
such that $P \in \mathbf{V}^{g}_{s}(C_{0})$ and so $C_{0} \subseteq P$. Since $C_{0} \in S$, then for each
$i \in I$ there exists $j$, $1 \leq j \leq n$, such that $N_{ij} \subseteq C_{0}$, and hence
$N_{ij} \subseteq C_{0} \subseteq P$. It follows that $P \in S$. Therefore, $\bigcup_{C \in Y} \mathbf{V}^{g}_{s}(C) \subseteq S.$

ii) Clearly $Y \subseteq \bigcup_{C \in Y} \mathbf{V}^{g}_{s}(C)$. For each $C \in Y$ we have
$\mathbf{V}^{g}_{s}(C) = cl(\{C\}) \subseteq cl(Y) = Y$ by part(i). Hence $\bigcup_{C \in Y} \mathbf{V}^{g}_{s}(C) \subseteq Y$. Therefore, $Y = \bigcup_{C \in Y} \mathbf{V}^{g}_{s}(C)$.

Now the above lemma immediately yields the following result.

**Corollary 2.2.** Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. Then.

1. $cl(\{C\}) = \mathbf{V}^{g}_{s}(C)$, for all $C \in Cl.Spec_{g}(M)$.

2. $Q \in cl(\{C\})$ if and only if $C \subseteq Q$ if and only if $\mathbf{V}^{g}_{s}(Q) \subseteq \mathbf{V}^{g}_{s}(C)$.

3. The set $\{C\}$ is a closed in $Cl.Spec_{g}(M)$ if and only if $C$ is a graded
maximal classical prime submodule of $M$.

The following theorem shows that for any graded $R$-module $M$, $Cl.Spec_{g}(M)$
is always a $T_{0}$-space.

**Theorem 2.3.** Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. Then, $Cl.Spec_{g}(M)$
is a $T_{0}$-space.

**Proof.** Let $C_{1}, C_{2} \in Cl.Spec_{g}(M)$. By Corollary 2.2, $cl(\{C_{1}\}) = cl(\{C_{2}\})$ if and only if $\mathbf{V}^{g}_{s}(C_{1}) = \mathbf{V}^{g}_{s}(C_{2})$ if and only if $C_{1} = C_{2}$.

Now, by the fact that a topological space is a $T_{0}$-space if and only if
the closures of distinct points are distinct, we conclude that for any graded
$R$-module $M$, $Cl.Spec_{g}(M)$ is a $T_{0}$-space. □

Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. Let every graded
classical prime submodule of $M$ is contained in a graded maximal classical prime submodule. We define, by transfinite induction, sets $X_{\alpha}$ of graded
classical prime submodule of $M$. To start, let $X_{-1}$ be the empty set. Next, consider an ordinal $\alpha \geq 0$; if $X_{\beta}$ has been defined for all ordinals $\beta < \alpha$, then let $X_{\alpha}$ be the set of those graded classical prime submodules $C$ in $M$ such
that all graded classical prime submodules proper containing $C$ belong to $\cup_{\beta<\alpha} X_\beta$. In particular, $X_0$ is the set of graded maximal classical prime submodules of $M$. If some $X_\gamma$ contains all graded classical prime submodules of $M$, then we say that $dim^{cl}_g(M)$ exists, and we set $dim^{cl}_g(M)$ -the graded classical dimension of $M$ to be to the smallest such $\gamma$. We write $dim^{cl}_g(M) = \gamma$ as an abbreviation for the statement that $dim^{cl}_g(M)$ exists and equal $\gamma$. In fact, if $dim^{cl}_g(M) = \gamma < \infty$, then $dim^{cl}_g(M) = sup\{ht(C)\}C$ is graded classical prime submodule of $M\}$. Where $ht(C)$ is the greatest non-negative integer $n$ such that there exists a chain of graded classical prime submodules of $M$, $C_0 \subseteq C_1 \subseteq \ldots \subseteq C_n = C$, and $ht(C) = \infty$ if no such $n$ exists.

Let $X$ be a topological space and let $x_1$ and $x_2$ be two points in $X$. We say that $x_1$ and $x_2$ can be separated if each lies in an open set which does not contain the other point. $X$ is a $T_1$-space if any two distinct points in $X$ can be separated. A topological space $X$ is a $T_1$-space if and only if all points of $X$ are closed in $X$, (see [6].)

**Theorem 2.4.** Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. Then $Cl.Spec_g(M)$ is $T_1$-space if and only if $dim^{cl}_g(M) \leq 0$.

**Proof.** First assume that $Cl.Spec_g(M)$ is a $T_1$-space. If $Cl.Spec_g(M) = \phi$, then $dim^{cl}_g(M) = -1$. Also, if $Cl.Spec_g(M)$ has one element, clearly $dim^{cl}_g(M) = 0$. So we can assume that $Cl.Spec_g(M)$ has more than two elements. We show that every graded classical prime submodules of $M$ is a graded maximal classical prime submodule. To show this, let $C_1 \subseteq C_2$, where $C_1, C_2 \in Cl.Spec_g(M)$. Since $\{C_1\}$ is a closed set, $\{C_1\} = \bigcap_{i \in I} V^g_\gamma(N_{ij}))$, Where $N_{ij} \subseteq M$ and $I$ is an index set. So for each $i \in I$, $C_1 \in \bigcup_{i \in I} V^g_\gamma(N_{ij})$ so that there exists $1 \leq t_i \leq n_i$ such that $C_1 \in V^g_\gamma(N_{it})$. Since $C_1 \subseteq C_2$, $C_2 \in V^g_\gamma(N_{it})$ for all $i \in I$. This implies that $C_2 \in \bigcup_{i \in I} V^g_\gamma(N_{ij})$, for all $i \in I$. Therefore, $C_2 \in \bigcap_{i \in I} (\bigcup_{i \in I} V^g_\gamma(N_{ij})) \{C_1\}$ as desired.

Conversely, suppose that $dim^{cl}_g(M) \leq 0$. If $dim^{cl}_g(M) = -1$, then $Cl.Spec_g(M) = \phi$, and hence it is a $T_1$-space. Now let $dim^{cl}_g(M) = 0$. Then $Cl.Spec_g(M) \neq \phi$ and for every graded classical prime submodule of
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The cofinite topology is a topology which can be defined on every set
X. It has precisely the empty set and all cofinite subsets of X as open sets.
As a consequence, in the cofinite topology, the only closed subset are finite
sets, or the whole of X (see [6].)

Now we give a characterization for a graded module M for which Cl.Spec\(_g\)(M)
is the cofinite topology.

**Theorem 2.5.** Let R be a G-graded ring and M a graded R-module. Then
the following statements are equivalent :

i) Cl.Spec\(_g\)(M) is the cofinite topology.

ii) dim\(_\text{cl}\)(M) \leq 0 and for every graded submodule N of M
either \(V^g_*(N) = \text{Cl.Spec}_g(M)\) or \(V^g_*(N)\) is finite.

**Proof.** (i) \(\Rightarrow\) (ii). Assume that Cl.Spec\(_g\)(M) is the cofinite topol-
gy. Since every cofinite topology satisfies the T\(_1\) axiom, by Theorem
2.4, \(\text{dim}_{\text{cl}}(M) \leq 0\). Now assume that there exists a graded submodule
N of M such that \(|V^g_*(N)| = \infty\) and \(V^g_*(N) \neq \text{Cl.Spec}_g(M)\). Then
\(U^g_*(N) = \text{Cl.Spec}_g(M) - V^g_*(N)\) is an open set in \(\text{Cl.Spec}_g(M)\) with infinite
complement, a contradiction. (ii) \(\Rightarrow\) (i). Suppose that \(\text{dim}_{\text{cl}}(M) \leq 0\) and
for every graded submodule N of M, \(V^g_*(N) = \text{Cl.Spec}_g(M)\) or \(V^g_*(N)\) is finite. Thus every finite union \(\bigcup_{j=1}^n V^g_*(N_j)\) of graded submodules \(N_j \leq g\) M is also finite or \(\text{Cl.Spec}_g(M)\). Hence any intersection of finite
union \(\bigcap_{i \in I} \bigcup_{j=1}^n V^g_*(N_{ij})\) of graded submodules \(N_{ij} \leq g\) M is finite or
\(\text{Cl.Spec}_g(M)\). Hence every closed set in \(\text{Cl.Spec}_g(M)\) is either finite or
\(\text{Cl.Spec}_g(M)\). Therefore \(\text{Cl.Spec}_g(M)\) is the cofinite topology. □

Suppose that X is a topological space. Let \(x_1\) and \(x_2\) be points in X.
We say that \(x_1\) and \(x_2\) can be separated by neighborhoods if there exists
a neighborhood U of \(x_1\) and neighborhood V of \(x_2\) such that \(U \cap V = \phi\). X is a T\(_2\)-space if any two distinct points of X can be separated by
neighborhoods (see [6].) It is well-known that if X is a finite space, then
X is T\(_1\)-space if and only if X is the discrete space (see [6].) Thus we have
the following corollary.

**Corollary 2.6.** Let R be a G-graded ring and M a graded R-module such
that \(\text{Cl.Spec}_g(M)\) is finite. Then the following statements are equivalent:
Suppose $M$ is a graded $R$-module such that $M$ has ACC on intersection of graded classical prime submodules. Then, $\text{Cl.Spec}_g(M)$ is a quasi-compact space.

**Proof.** Suppose $M$ is a graded $R$-module such that $M$ has ACC on intersection of graded classical prime submodules. Let $\mathcal{W}$ be a family of open sets covering $\text{Cl.Spec}_g(M)$, and suppose that no finite subfamily of covers $\text{Cl.Spec}_g(M)$. Since $\mathcal{W}_g^2(0) = \text{Cl.Spec}_g(M)$, then we may use the ACC on the intersection of graded classical prime submodules to choose a graded submodule $N$ maximal with respect to the property that no finite subfamily of covers $\mathcal{W}_g^2(N)$. We claim that $N$ is a graded classical prime submodule of $M$, for if not, then there exist $m_\lambda \in h(M)$ and $r_g, s_h \in h(R)$, such that $r_g s_h m_\lambda \in N$, $r_g m_\lambda \notin N$ and $s_h m_\lambda \notin N$. Thus $NN + Rr_g m_\lambda$ and $NN + Rs_h m_\lambda$. Hence, without loss of generality, there must exist a finite subfamily of that covers both $\mathcal{W}_g^2(N + Rr_g m_\lambda)$ and $\mathcal{W}_g^2(N + Rs_h m_\lambda)$. Let $C \in \mathcal{W}_g^2(N)$. Since $r_g s_h m_\lambda \in N$, $r_g s_h m_\lambda \in C$ and since $C$ is graded classical prime, $r_g m_\lambda \in C$ or $s_h m_\lambda \in C$. Thus either $C \in \mathcal{W}_g^2(N + Rr_g m_\lambda)$ or $C \in \mathcal{W}_g^2(N + Rs_h m_\lambda)$, and hence $\mathcal{W}_g^2(N) \subseteq \mathcal{W}_g^2(N + Rr_g m_\lambda) \cup \mathcal{W}_g^2(N + Rs_h m_\lambda)$. Thus, $\mathcal{W}_g^2(N)$ is covered with the finite subfamily of a contradiction. Therefore, $N$ is a graded classical prime submodule of $M$.

Now, choose $W \in \mathcal{W}$ such that $N \in W$. Hence $N$ must have a neighborhood $\bigcap_{i=1}^n U_g^2(P_i)$, for some graded submodule $P_i$ of $M$ and $n \in \mathbb{N}$, such that $\bigcap_{i=1}^n U_g^2(P_i) \subseteq W$. We claim that for each $i$ $(1 \leq i \leq n)$, $N \in U_g^2(P_i + N) \subseteq U_g^2(P_i)$. To see this, assume that $C \in U_g^2(P_i + N)$, i.e., $P_i + NC$. So $P_i C$, i.e., $C \in U_g^2(P_i)$. On the other hand, $N \in U_g^2(P_i)$, i.e., $P_i N$. Therefore, $P_i + NC$, i.e., $C \in U_g^2(P_i + N)$. Consequently, $N \in \bigcap_{i=1}^n U_g^2(P_i + N) \subseteq \bigcap_{i=1}^n U_g^2(P_i) \subseteq W$. 

**Theorem 2.7.** Let $R$ be a $G$-graded ring and $M$ a graded $R$-module such that $M$ has ACC on intersection of graded classical prime submodules. Then, $\text{Cl.Spec}_g(M)$ is a quasi-compact space.
Hence $\bigcap_{i=1}^{n} U^g_i(P'_i)$, where $P'_i := P_i + N$, is a neighborhood of $N$ such that $\bigcap_{i=1}^{n} U^g_i(P'_i) \subseteq W$. Since for each $i$ ($1 \leq i \leq n$), then $NP'_i$, $V^g_i(P'_i)$ can be covered by some finite subfamily $'_i$ of . But, $V^g_i(N) \setminus [\bigcup_{i=1}^{n} V^g_i(P'_i)] = V^g_i(N) \setminus [\bigcup_{i=1}^{n} U^g_i(P'_i)] \cap V^g_i(N) \subseteq W$, and so $V^g_i(N)$ can be covered by $'_1 \cup '_2 \cup ... \cup '_n \cup \{W\}$, contrary to our choice of $N$. Thus, there must exist a finite subfamily of which covers $Cl.Spec_g(M)$. Therefore, $Cl.Spec_g(M)$ is a quasi-compact space. □

3. Graded modules whose Zariski-like topologies are spectral spaces

A topological space $X$ is called irreducible if $X \neq \phi$ and every finite intersection of non-empty open sets of $X$ is non-empty. A (non-empty) subset $Y$ of a topology space $X$ is called an irreducible set if the subspace $Y$ of $X$ is irreducible, equivalently if $Y_1$ and $Y_2$ are closed subset of $X$ and satisfy $Y \subseteq Y_1 \cup Y_2$, then $Y \subseteq Y_1$ or $Y \subseteq Y_2$ (see [6].)

Let $Y$ be a closed subset of a topological space. An element $y \in Y$ is called a generic point of $Y$ if $Y = cl\{y\}$. Note that a generic point of the irreducible closed subset $Y$ of a topological space is unique if the topological space is a $T_0$-space (see [5].)

A spectral space is a topological space homomorphism to the prime spectrum of a commutative ring equipped with the Zariski topology. Spectral spaces have been characterized by Hochster [5] as the topological space $W$ which satisfy the following conditions:

i) $W$ is a $T_0$-space.

ii) $W$ is quasi-compact.

iii) the quasi-compact open subsets of $W$ are closed under finite intersections and form an open basis.

iv) each irreducible closed subset of $W$ has a generic point.

Let $M$ be a $G$-graded $R$-Module and $Y$ a subset of $Cl.Spec_g(M)$. We will denote $\bigcap_{C \in Y} C$ by $\exists(Y)$ (note that if $Y = \phi$, then $\exists(Y) = M$).

Lemma 3.1. Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. Then for each $C \in Cl.Spec_g(M)$, $V^g(C)$ is irreducible.
Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $Y \subseteq Cl.Spec_g(M)$.

i) If $Y$ is irreducible, then $\mathfrak{N}(Y)$ is a graded classical prime submodule.

ii) If $\mathfrak{N}(Y)$ is a graded classical prime submodule and $\mathfrak{N}(Y) \in cl(Y)$, then $Y$ is irreducible.

**Proof.** (i) Assume that $Y$ is an irreducible subset of $Cl.Spec_g(M)$. Clearly, $\mathfrak{N}(Y) = \bigcap_{C \in Y} C_gM$ and $Y \subseteq V_g^k(\mathfrak{N}(Y))$. Let $I$, $J$ be graded ideals of $R$ and $N$ be a graded submodule of $M$ such that $IJN \subseteq \mathfrak{N}(Y)$. It is easy to see that $Y \subseteq V_g^k(IJN) \subseteq V_g^k(IN) \cup V_g^k(JN)$. Since $Y$ is irreducible, either $Y \subseteq V_g^k(IN)$ or $Y \subseteq V_g^k(JN)$. If $Y \subseteq V_g^k(IN)$, then $IN \subseteq C$, for all $C \in Y$. Thus $IN \subseteq \mathfrak{N}(Y)$. If $Y \subseteq V_g^k(JN)$, then $JN \subseteq C$, for all $C \in Y$. Hence $JN \subseteq \mathfrak{N}(Y)$. Thus by [1, Theorem 2.1.], $\mathfrak{N}(Y)$ is a graded classical prime submodule of $M$. (ii) Assume that $C := \mathfrak{N}(Y)$ is a graded classical prime submodule of $M$ and $C \in cl(Y)$. It is easy to see that $cl(Y) = V_g^k(C)$. Now let $Y \subseteq Y_1 \cup Y_2$, where $Y_1$, $Y_2$ are closed sets. Then we have $V_g^k(C) = cl(Y) \subseteq Y_1 \cup Y_2$. Since $V_g^k(C) \subseteq Y_1 \cup Y_2$ and by Lemma 3.1, $V_g^k(C)$ is irreducible, $V_g^k(C) \subseteq Y_1$ or $V_g^k(C) \subseteq Y_2$. Hence either $Y \subseteq Y_1$ or $Y \subseteq Y_2$. Thus $Y$ is irreducible. □

**Corollary 3.3.** Let $R$ be a $G$-graded ring, $M$ a graded $R$-module and $N$ a graded submodule of $M$. Then the subset $V_g^k(N)$ of $Cl.Spec_g(M)$ is irreducible if and only if $Gr^d_M(N)$ is a graded classical prime submodule. Consequently, $Cl.Spec_g(M)$ is irreducible if and only if $Gr^d_M(M)$ is a graded classical prime submodule.
Proof. \((\Rightarrow)\) Let \(Y := V_2^g(N)\) be an irreducible subset of \(Cl.\ Spec_g(M)\). Then we have \(\mathcal{G}(Y) = Gr^cl_{M}(N)\) so that \(Gr^cl_{M}(N)\) is a graded classical prime submodule of \(M\) by Theorem 3.2(i).

\((\Leftarrow)\) By [4, Proposition 3.4(1)], for each graded submodule \(N\) of \(M\), \(V_2^g(N) = V_2^g(Gr^cl_{M}(N))\). Now let \(Gr^cl_{M}(N)\) is a graded classical prime submodule of \(M\). Then \(Gr^cl_{M}(N) \in V_2^g(N)\), and hence by Theorem 3.2 (ii), \(V_2^g(N)\) is irreducible. \(\Box\)

**Lemma 3.4.** Let \(R\) be a \(G\)-graded ring and \(M\) a graded \(R\)-module. Then

i) Every \(C \in Cl.\ Spec_g(M)\) is a generic point of the irreducible closed subset \(V_2(C)\).

ii) Every finite irreducible closed subset of \(Cl.\ Spec_g(M)\) has a generic point.

Proof.

i) is clear by Corollary 2.2(i).

ii) Let \(Y\) be an irreducible closed subset of \(Cl.\ Spec_g(M)\) and \(Y = \{C_1, C_2, ..., C_n\}\), where \(C_i \in Cl.\ Spec_g(M)\), \(n \in \mathbb{N}\). By Lemma 2.1(i), \(Y = cl(Y) = V_2(C_1) \cup V_2(C_2) \cup ... \cup V_2(C_n)\). Since \(Y\) is irreducible, \(Y = V_2(C_i)\) for some \(i(1 \leq i \leq n)\). Now by (i), \(C_i\) is a generic point of \(Y\).

\(\Box\)

**Theorem 3.5.** Let \(R\) be a \(G\)-graded ring and \(M\) a graded \(R\)-module such that \(Cl.\ Spec_g(M)\) is finite. Then \(Cl.\ Spec_g(M)\) is a spectral space (with the Zariski-like topology). Consequently, for each finite graded \(R\)-module \(M\), \(Cl.\ Spec_g(M)\) is a spectral space.

Proof. Since \(Cl.\ Spec_g(M)\) is finite, every subset of \(Cl.\ Spec_g(M)\) is quasi-compact. Hence the quasi-compact open sets of \(Cl.\ Spec_g(M)\) are closed under finite intersection and form an open basis (note: this basis is \(\beta = \{U_2(N_1) \cap U_2(N_2) \cap ... \cap U_2(N_k) : N_i \leq g M, 1 \leq i \leq k, \text{for some } k \in \mathbb{N}\}\)). Also by Theorem 2.3, \(Cl.\ Spec_g(M)\) a \(T_0\)-space. Moreover, every
irreducible closed subset of \( \text{Cl.Spec}_g(M) \) has a generic point by Lemma 3.4. Therefore \( \text{Cl.Spec}_g(M) \) is a spectral space by Hochster’s characterization.

Let \( X \) be a topological space. By the patch topology on \( X \), we mean the topology which has as a sub-basis for its closed sets the closed sets and compact open sets of the original space. By a patch we mean a set closed in the patch topology. The patch topology associated to a spectral space is compact and \( T_2 \)-space (see [5].)

**Definition 3.6.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module, and let \( P^g_*(M) \) be the family of all subsets of \( \text{Cl.Spec}_g(M) \) of the form \( V^g_*(N) \cap U^g_*(K) \), where \( N, K \leq_g M \). Clearly \( P^g_*(M) \) contains both \( \text{Cl.Spec}_g(M) \) and \( \phi \) because \( \text{Cl.Spec}_g(M) = V^g_*(0) \cup U^g_*(M) \) and \( \phi = V^g_*(M) \cap U^g_*(0) \). Let \( T^g_*(M) \) be the collection of all unions of finite intersections of elements of \( P^g_*(M) \). Then, \( T^g_*(M) \) is a topology on \( \text{Cl.Spec}_g(M) \) and is called the patch-like topology of \( M \), in fact, \( P^g_*(M) \) is a sub-basis for the patch-like topology of \( M \).

**Theorem 3.7.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. Then, \( \text{Cl.Spec}_g(M) \) with the patch-like topology is a \( T_2 \)-space.

**Proof.** Suppose distinct points \( C_1, C_2 \in \text{Cl.Spec}_g(M) \). Since \( C_1 \neq C_2 \), then either \( C_1C_2 \) or \( C_2C_1 \). Assume that \( C_1C_2 \). By Definition 3.6, \( P_1 := U^g_*(M) \cap V^g_*(C_1) \) is a patch-like-neighborhood of \( C_1 \) and \( P_2 := U^g_*(C_1) \cap V^g_*(C_2) \) is a patch-like-neighborhood of \( C_2 \). Clearly, \( U^g_*(C_1) \cap V^g_*(C_1) = \phi \), and thus \( P_1 \cap P_2 = \phi \). Therefore, \( \text{Cl.Spec}_g(M) \) is a \( T_2 \)-space.

The proof of the next theorem is similar to the proof of Theorem 2.7.

**Theorem 3.8.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module such that \( M \) has ACC on intersection of graded classical prime submodules. Then \( \text{Cl.Spec}_g(M) \) with the patch-like topology is a compact space.

**Theorem 3.9.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module such that \( M \) has ACC on intersection of graded classical prime submodules. Then every irreducible closed subset of \( \text{Cl.Spec}_g(M) \) (with the Zariski-like topology) has a generic point.
Proof. Let $Y$ be an irreducible closed subset of $\text{Cl.Spec}_g(M)$. By Definition 3.6 for each $C \in Y$, $V^g_*(C)$ is an open subset of $\text{Cl.Spec}_g(M)$ with the patch-like topology. On the other hand since $Y \subseteq \text{Cl.Spec}_g(M)$ is closed with the Zariski-like topology, the complement of $Y$ is open by this topology. This yields that the complement of $Y$ is open with the patch-like topology. So $Y \subseteq \text{Cl.Spec}_g(M)$ is closed with the patch-like topology. Since $\text{Cl.Spec}_g(M)$ is a compact space in patch-like topology by Theorem 3.8 and $Y$ is closed in $\text{Cl.Spec}_g(M)$, we have $Y$ is compact space in patch-like topology. Now $Y = \bigcup_{C \in Y} V^g_*(C)$ by Lemma 2.1(ii) and each $V^g_*(C)$ is open in patch-like topology. Hence there exists a finite set $Y_1 \subseteq Y$ such that $Y = \bigcup_{C \in Y_1} V^g_*(C)$. Since $Y$ is irreducible, $Y = V^g_*(C) = cl(\{C\})$ for some $C \in Y$. Therefore, $C$ is a generic point for $Y$. □

We need the following evident lemma

Lemma 3.10. Assume $\tau_1$ and $\tau_2$ are two topologies on $X$ such that $\tau_1 \subseteq \tau_2$. If $X$ is quasi-compact in $\tau_2$, then $X$ is also quasi-compact in $\tau_1$.

Theorem 3.11. Let $R$ be a $G$-graded ring and $M$ a graded $R$-module such that $M$ has ACC on intersection of graded classical prime submodules. Then for each $n \in \mathbb{N}$, and graded submodules $N_i (1 \leq i \leq n)$ of $M$, $U^g_*(N_1) \cap U^g_*(N_2) \cap \ldots \cap U^g_*(N_n)$ is a quasi-compact subset of $\text{Cl.Spec}_g(M)$ with the Zariski-like topology.

Proof. Clearly, for each $n \in \mathbb{N}$, and each graded submodules $N_i (1 \leq i \leq n)$ of $M$, $U^g_*(N_1) \cap U^g_*(N_2) \cap \ldots \cap U^g_*(N_n)$ is a closed set in $\text{Cl.Spec}_g(M)$ with patch-like topology. By Theorem 3.8, $\text{Cl.Spec}_g(M)$ is a compact space with the patch-like topology and since every closed subset of a compact space is compact, $U^g_*(N_1) \cap U^g_*(N_2) \cap \ldots \cap U^g_*(N_n)$ is compact in $\text{Cl.Spec}_g(M)$ with patch-like topology and so by Lemma 3.10, it is quasi-compact in $\text{Cl.Spec}_g(M)$ with the Zariski-like topology. □

Corollary 3.12. Let $R$ be a $G$-graded ring and $M$ a graded $R$-module such that $M$ has ACC on intersection of graded classical prime submodules. Then Zariski-like quasi-compact open sets of $\text{Cl.Spec}_g(M)$ are closed under finite intersections.
Proof. It suffices to show that the intersection $Q = Q_1 \cap Q_2$ of two Zariski-like quasi-compact open sets $Q_1$ and $Q_2$ of $\text{Cl.Spec}_g(M)$ is Zariski-like quasi-compact set. Each $Q_i$, $i = 1, 2$, is a finite union of members of the open base $\beta = \{ U^g_g(N_1) \cap U^g_g(N_2) \cap \ldots \cap U^g_j(N_n) : N_i \leq_g M, 1 \leq i \leq n, \text{for some } n \in \mathbb{N} \}$. Hence $Q = \bigcup_{i=1}^{n_i} \bigcap_{j=1}^{n_j} U^g(N_j)$. Let $\Gamma$ be any open cover of $Q$. So $\Gamma$ also covers each $\bigcap_{j=1}^{n_j} U^g(N_j)$ which is Zariski-like quasi-compact by Theorem 3.11. Thus each $\bigcap_{j=1}^{n_j} U^g(N_j)$ has a finite subcover of $\Gamma$ and so does $Q$. □

Theorem 3.13. Let $R$ be a $G$-graded ring and $M$ a graded $R$-module such that $M$ has ACC on intersection of graded classical prime submodules. Then $\text{Cl.Spec}_g(M)$ (with the Zariski-like topology) is a spectral space.

Proof. By Theorem 2.3, $\text{Cl.Spec}_g(M)$ is a $T_0$-space. Also, by Theorem 3.11., $\text{Cl.Spec}_g(M)$ is quasi-compact and has a basis of quasi-compact open subsets. Moreover, by Corollary 3.12, the family of quasi-compact open subset of $\text{Cl.Spec}_g(M)$ is closed under finite intersections. Finally, every irreducible closed subset of $\text{Cl.Spec}_g(M)$ has generic point by Theorem 3.9. Thus $\text{Cl.Spec}_g(M)$ is spectral space by Hochster’s characterization. □

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